# ALGORITHMS FOR THE MULTIPLICATION TABLE PROBLEM

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ABSTRACT. We present several algorithms for computing M(n), the function that counts the number of distinct products in an  $n \times n$  multiplication table. In particular, we consider their run-times and space bounds for single evaluation, tabulation, and Monte-Carlo estimation. We give the result of exact computations up to  $n = 2^{30} - 1$  and Monte Carlo computations for larger n.

#### 1. INTRODUCTION

Even though a multiplication table is something that is understood by a typical student in elementary school, there remains much that we do not know. In 1955, Erdős studied the problem of counting the number M(n) of distinct products in an  $n \times n$  multiplication table. That is, we define  $M(n) = |\{ij : 1 \leq i, j \leq n\}|$ . The initial work [10] showed  $M(n) = o(n^2)$ . Five years later, Erdős [11] improved this to

(1) 
$$M(n) = \frac{n^2}{(\ln n)^{c+o(1)}} \text{ as } n \to \infty,$$

where (here and below)  $c = 1 - (1 + \ln \ln 2) / \ln 2 = 0.086071...$  Tenenbaum [24] improved on (1) by clarifying the o(1) term. In 2008, Ford [12, 13] got the correct order of magnitude as

(2) 
$$M(n) \asymp n^2 / \Phi(n),$$

where

(3) 
$$\Phi(n) := (\ln n)^c (\ln \ln n)^{3/2}$$

is a slowly-growing function. (Here  $f \simeq g$  means  $f \ll g$  and  $g \ll f$ .)

Exact evaluation of M(n) goes back to Brent and Kung [8] in 1981. They considered how much area and time are needed to perform an *n*-bit binary multiplication on a VLSI chip. For this, they needed a lower bound on M(n). They computed<sup>1</sup>  $M(2^n - 1)$  for  $1 \le n \le 17$ . In 2012, the first two authors revisited the problem, extending the computation through n = 25, and exploring Monte-Carlo estimates for larger n. We present two different algorithms for confirmation, extending the computation to  $M(2^{30} - 1)$ .

The paper is organized as follows. Section 2 starts with an overview of the sieve of Eratosthenes as a precursor to the various ways of evaluating M(n) exactly. We start with the method used in [8], then show how to compute M(n) in time  $O(M(n) \ln n)$  using an incremental algorithm. The incremental algorithm may be modified to compute M(n) in time  $o(n^2)$ . Section 3 explains two Monte-Carlo

<sup>&</sup>lt;sup>1</sup> Brent and Kung actually computed  $M(2^n - 1) + 1 = |\{ij : 0 \le i, j < 2^n\}|$ . For consistency in the exposition, we translate their results to the definition of M(n) stated above.

algorithms for estimating M(n). We conclude with numerical results and some comments on implementations.

## 2. EXACT EVALUATION OF M(n)

2.1. Sieve of Eratosthenes. The methods of evaluating M(n) bear a striking resemblance to the sieve of Eratosthenes, the simplest implementation of which involves for each  $1 < k \le n^{1/2}$ , removing the multiples of k from (k, n]. This naive implementation uses  $O(n \ln n)$  time and O(n) space and will find all primes up to n. There is a large body of literature, both practical and theoretical, dealing with improvements and variations to this sieve. We refer the reader to Helfgott [14] for a summary of the literature. Here, we highlight the aspects that are relevant for us. In practice, we are limited by the space constraint; lowering the space constraint may turn otherwise infeasible computations into feasible ones. Instead of marking off multiples of k in (k, n] we may segment this interval into subintervals. The asymptotic run-time remains unchanged so long as the "marking off" process is not doing empty work. That is, the space bound may be  $O(n^{1/2})$  with straightforward segmenting of the intervals. As the intervals decrease in size below that bound, time will increase (associated with the empty work). Helfgott allows the space bound to drop much lower by using Diophantine approximation to predict which integers less than  $n^{1/2}$  will divide a given subinterval. This prediction process is more difficult to implement but allows sieving on intervals of size  $O(n^{1/3}(\ln n)^{5/3})$ (Main Theorem in [14]) at no asymptotic cost in time.

2.2. **Computing Directly.** In what follows, it is helpful to think of the multiplication table as lying in the first quadrant of the Cartesian plane. We associate a unit square in the plane to a position in the multiplication table. The run-times of the algorithms that follow are then proportional to area in the Cartesian plane.

We can explicitly construct each product in a multiplication table and count the number of distinct products, as in Algorithm 1.

<b>Algorithm 1:</b> Computing $M(n)$ directly
<b>Input</b> : An integer $n$ .
<b>Output:</b> $M(n)$
1 Initialize a bit vector A of length $n^2$ to 0.
2 for $1 \le i \le n$ do
3 for $i \leq j \leq n$ do
$4  \bigsqcup  \text{Set} \ A[ij] = 1$
5 return Hamming weight of A

**Theorem 2.1.** Algorithm 1 correctly computes M(n) in time  $O(n^2)$  and space  $O(n^2)$ .

*Proof.* Obvious. We note that the area of the table is  $n^2$ .

This algorithm functions in a similar way to the sieve of Eratosthenes and many of the tricks one uses to speed up that sieving problem may be used here. The key difference is the stopping point for marking off multiples of i; Algorithm 1 only marks off through the *n*th multiple of i. Because of this early stopping point,

Algorithm 1 does not have the corresponding  $\ln n$  term seen in the run-time of the sieve of Eratosthenes. The space bound is stated in its naive form. As discussed above, standard segmenting allows subintervals of size  $O((n^2)^{1/2}) = O(n)$ , or by using Diophantine approximation to predict which integers divide a subinterval the space bound could be reduced to  $O(n^{2/3}(\ln n)^{5/3})$ .

Storing all  $n^2$  bits is often impractical. We note two theoretical consequences that are relevant. If the entire bit vector A may be stored then computing M(n)given the computation of M(n-1) may be done in O(n) time. Simply count how many bits are unset in  $S = [A[n], A[2n], \ldots, A[n^2]]$  and increment M(n-1) by that amount. Let the number of set bits in S be denoted by  $\delta(n)$  and the number of unset bits by D(n). We can compute  $\delta(n)$  or D(n) in O(n) time and  $O(n^2)$  space. Thus, we may compute M(n) using

$$M(n) = M(n-1) + (n-\delta(n)) = \sum_{k=1}^{n} (k-\delta(k)) = \frac{n^2+n}{2} - \sum_{k=1}^{n} \delta(k).$$

The next section shows how to compute  $\delta(n)$  almost as quickly but requiring only O(n) space. Second, tabulation and evaluation are the same problem if the bits of A are always accessible. If we use a segmenting strategy to minimize memory, tabulation has to be accomplished by evaluating M(k) for all  $k \leq n$ . Since individual evaluation is  $O(k^2)$ , tabulation is  $O(n^3)$ .

2.3. Computing via Differences. We compute M(n) by computing its change. In particular, we compute  $\delta(k)$  for all  $k \leq n$  where  $\delta(n)$  counts the elements in  $\{n, 2n, 3n, \dots, n^2\} = \{mn : 1 \le m \le n\}$  that appear in the  $(n-1) \times (n-1)$ multiplication table. If mn appears in the smaller multiplication table then it may be factored so that each factor is strictly less than n. Letting ij = m and qh = n, we require ih < n and jg < n. This gives i < g and j < h for any factorization gh = n. This corresponds to counting distinct products in the  $(q-1) \times (h-1)$  rectangle in the multiplication table. By counting the distinct products in the shape formed by all rectangles whose boundaries are set by the divisors of n, we may compute  $\delta(n)$ .

Algorithm 2: Computing  $\delta(n)$ 

**Input** :  $D = [[d_0 = 1, n], \dots, [d_{\ell-1}, n/d_{\ell-1}]]$ , containing the ordered divisors of n where  $d_{\ell-1}$  is the largest divisor in  $[1, \sqrt{n}]$ . **Output:**  $\delta(n)$ 

- **1** Initialize counters i = 1 and k = 0
- **2** Initialize a bit vector A of length n to 0.
- **3 for**  $i < D[\ell 1][0]$  **do**
- if i == D[k][0] then  $\mathbf{4}$
- $\mathbf{5}$ Increment k
- for  $i \leq j < D[k][1]$  do | Set A[ij] = 16
- 7
- **8 return** Hamming weight of A

**Remark.** The algorithm, as stated, exploits the symmetry of the multiplication table to minimize the computation time. In line 6 the restriction  $i \leq j$  may be  $1 \leq j$ . The restriction that  $D[\ell - 1][0] \leq \sqrt{n}$  may be removed but the divisors must still be ordered. Removing these restrictions does not change the asymptotic running time; after all, together both represent a constant factor of two in time.

**Remark.** If the input to Algorithm 2 is missing some divisor pair, then the Hamming weight of A is a lower bound on  $\delta(n)$ .

**Example 2.2.** In Figure 1, we can see the three rectangles corresponding to the divisor pairs  $2 \cdot 21$ ,  $3 \cdot 14$ , and  $6 \cdot 7$ . The grey area corresponds to the products Algorithm 2 constructs.

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6	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114	120	126
5	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100	105
4	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80	84
3	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60	63
2	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21

FIGURE 1. The shape for computing  $\delta(42)$ 

Algorithm 2 runs in time proportional to the area under consideration. One upper bound is  $O(n \ln n)$ , obtained by realizing that no product in a rectangle is ever larger than n. We may count all products less than n, which lie under a hyperbola and this gives the claim. Another upper bound is  $O(n\tau(n))$ , where  $\tau(n)$ counts the divisors of n. This comes from the fact that for each divisor of n, we construct a rectangle of area less than n. Both of these functions over-count. The former over-counts when n is not smooth and the latter over-counts when n is smooth. Either of these bounds may be used to show that M(n) may be tabulated in time  $O(n^2 \ln n)$ ; together they show that M(n) is tabulated in time  $o(n^2 \ln n)$ .

Let  $\tau(n; y, z)$  be the number of divisors d of n which satisfy  $y < d \leq z$  and  $\tau^+(n) = |\{k \in \mathbb{Z} : \tau(n, 2^k, 2^{k+1}) \geq 1\}|.$ 

**Theorem 2.3.** Algorithm 2 correctly computes  $\delta(n)$  in space O(n) and in time  $O(n\tau^+(n))$ .

*Proof.* Claim: Algorithm 2 is correct. By the above discussion, the algorithm is correct. As i increases, the counter k keeps track of which rectangle boundary to use. The counter j is then bounded above by the appropriate divisor of n.

Claim: Algorithm 2 uses  $O(n\tau^+(n))$  time. Recall that the run-time is proportiaonal to the area. Let the area be  $\mathcal{A}$ . For each k, consider all the divisors of n in the interval  $(2^k, 2^{k+1}]$ . They all have the same bottom left corner, namely, the origin, and shapes range from  $2^k \times n/2^k$  to  $2^{k+1} \times n/2^{k+1}$ . Hence they are all enclosed by a rectangle of shape  $2^{k+1} \times n/2^k$  which has area 2n. Thus we get an upper bound  $\mathcal{A} \leq 2n\tau^+(n)$ .

Claim: Algorithm 2 uses O(n) space. The vector A uses n bits.

**Theorem 2.4.** Algorithm 2 may used to tabulate M(n) in time

$$O\left(\frac{n^2\ln n}{\Phi(n)}\right).$$

*Proof.* We compute M(n) by evaluating  $\delta(k)$  for  $1 \le k \le n$ . The total run-time is

$$O\left(\sum_{k \le n} k\tau^+(k)\right) = O\left(n^2 \left(\frac{1}{n} \sum_{k \le n} \tau^+(k)\right)\right) = O\left(\frac{n^2(\ln n)^{1-c}}{(\ln \ln n)^{3/2}}\right).$$

The last equality is Corollary 5 of [12] for the average value of  $\tau^+(n)$ . The result follows from the definition (3) of  $\Phi(n)$ .

The space bound in Theorem 2.3 is stated for a naive implementation. It is not difficult to see that the storage can drop to  $O(n^{1/2})$  with straightforward segmenting, or even to  $O(n^{1/3}(\ln n)^{5/3})$  with Diophantine approximation as in [14]. Algorithm 2 represents an improvement by a factor of n in the naive storage cost and a significant improvement in run-time if the problem is tabulation. For single evaluation, this algorithm is arguably worse. In practice, it seems to be competitive and may be made even more competitive (see comments below). A contribution comes from a much smaller implied constant which big-O notation suppresses. Another contribution comes from the straightforward algorithm having larger memory requirements which can cause deviation from the expected quadratic run time.

2.4. Working modulo w. We may generate products in a multiplication table in specific residue classes; this is akin to sieving with a wheel. This has two advantages. First, if w is the modulus, then the vector used in Algorithm 2 may be declared to be of size  $\lfloor n/w \rfloor$  and unique products may be counted by residue class. Second, the zero index of the array may be declared to be the least integer in that residue class larger than the last consecutive element. The consecutive elements that are  $i \pmod{w}$  may be found in a row that is a divisor of gcd(i, w). By not explicitly constructing small consecutive products and simply counting them, we get a faster algorithm.

2.4.1. Working modulo 1. If n is not prime, then the first row of the table contains the consecutive integers less than the largest nontrivial divisor of n. Store the number of consecutive integers and initialize the bit vector A so that the zero index is associated with the largest divisor of n. Iterate through each row of the multiplication table starting at the first entry greater than or equal largest divisor. Figure 2 shows the area that is considered in computing  $\delta(42)$ . The first row has 20 distinct products and we add to that the number of distinct products in the shaded area to get  $\delta(42) = 25$ . This improvement reduces both the time and space requirements by a factor of (1 - 1/p), where p is the smallest prime factor of n. As a consequence, it is trivial to see that  $\delta(2p) = p - 1$ 

24 30 36 42 48 54 60 66 72 78 84 12 18 90 96 102 108 114 120 126 100 105 20 24 15 18 21 24 27 30 33 36 39 42 22 24 26 14 16 18 10 11 12 13 16 17 18 19 10 11 12 13 14 15 16 18 19 20 21

FIGURE 2. The shape for computing  $\delta(42)$  working modulo 1.

2.4.2. Working modulo 2. If n is composite and not of the form 2p for p a prime, then its shape will have nontrivial entries in the first two rows. Create a bit vector associated with odd numbers. The first row contains consecutive odd numbers up to some bound. Either the first row or the second row will contain the bound for the consecutive even numbers that are stored. For rows associated with an odd multiplier, start with the lower bound associated with the odd vector and iterate through the table creating only the odd entries. For the even vector, consider even rows and the even numbers in the odd rows. This reduces the time by reducing area although the overhead in setting up the loops to iterate through the table in the specified manner is higher. More importantly, it reduces the memory. By splitting the products into residue classes modulo 2, we require half the storage. The above discussion also makes it easy to see that  $\delta(3p) = p - 1 + \lfloor (p-1)/2 \rfloor$ .

Figure 3 shows the area that is considered in computing  $\delta(75)$ . The bit vector storing even numbers starts at 50, records 52, 56, and knows there are 24 + 2 = 26 unique even products. The bit vector storing odd numbers starts at 25, stores 27, 33, 39, and knows there are 12 + 3 = 15 unique odd products. So,  $\delta(75) = 26 + 15 = 41$ .

FIGURE 3. The shape for computing  $\delta(75)$  working modulo 2.

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5	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100	105	110	115	120	125
4	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80	84	88	92	96	100
3	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60	63	66	69	72	75
2	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25

2.4.3. Working modulo 6. A naive invocation of Algorithm 2 to compute  $\delta(377)$  requires the construction of 270 products. By constructing products in residue classes modulo 6 only 120 products need to be constructed. In Figure 4, we see that the sixth row tells us there are 28 consecutive multiples of 6. Therefore, we only need to construct products 0 (mod 6) that are greater than 168. Similarly, the third row tells us that there are 14 consecutive numbers 3 (mod 6). Therefore, we only construct products 3 (mod 6) that are greater than 84. The second row tells us that we only need to construct products greater than 56 when we deal with 2, 4 (mod 6) cases. Finally, the first row tells us that we need to construct products greater than 28 for the 1, 5 (mod 6) cases.

It is possible to create rules of the form  $\delta(mp)$  via counting arguments. The argument is accomplished by counting consecutive products in residue classes modulo  $w = \text{lcm}(1, 2, 3, \dots, m-1)$ . The third author created a website that may be used to count the products constructed and display the shape associated with a  $\delta(n)$  computation when using Algorithm 2 naively or with a modulus of w = 1, 2, 6, 12, 60, or 120 [22].

2.5. Towards Subquadratic Tabulation. Recall that if all  $n^2$  bits of A can be held at once in Algorithm 1, then tabulation and evaluation are the same problem. We apply this idea to computing  $\delta(n)$ . Consider the use of Algorithm 2 in computing

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13	13	26	39	52	65	78	91	104	117	130	143	156	169	182	195	208	221	234	247	260	273	286	299	312	325	338	351	364	377
12	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228	240	252	264	276	288	300	312	324	336	348
11	11	22	33	44	55	66	77	88	99	110	121	132	143	154	165	176	187	198	209	220	231	242	253	264	275	286	297	308	319
10	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200	210	220	230	240	250	260	270	280	290
9	9	18	27	36	45	54	63	72	81	90	99	108	117	126	135	144	153	162	171	180	189	198	207	216	225	234	243	252	261
8	8	16	24	32	40	48	56	64	72	80	88	96	104	112	120	128	136	144	152	160	168	176	184	192	200	208	216	224	232
7	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105	112	119	126	133	140	147	154	161	168	175	182	189	196	203
6	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114	120	126	132	138	144	150	156	162	168	174
5	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100	105	110	115	120	125	130	135	140	145
4	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80	84	88	92	96	100	104	108	112	116
3	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60	63	66	69	72	75	78	81	84	87
2	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50	52	54	56	58
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29

FIGURE 4. The shape for computing  $\delta(377)$  working modulo 6

 $\delta(6 \cdot 7)$ ,  $\delta(6 \cdot 9)$ ,  $\delta(6 \cdot 11)$ , and  $\delta(6 \cdot 13)$ . The divisor list for each of these is of the form  $[1, 6 \cdot k], [2, 3 \cdot k], [3, 2 \cdot k]$ , and [6, k] for k = 7, 9, 11, 13. One shape is always a subset of the next shape and so the set of distinct products in each shape is a subset of the next one. Rather than think of four independent computations, we consider the one computation of  $\delta(6 \cdot 13)$ . Unlike in Algorithm 2 where the bit vector storing distinct products is populated by a row of the multiplication table, we will populate the bit vector by incrementally shifting the end-points of the rectangles over. While computing  $\delta(6 \cdot 13)$  we will learn  $\delta(6 \cdot 7), \delta(6 \cdot 9)$ , and  $\delta(6 \cdot 11)$  along the way. Rather than computing  $\delta(6 \cdot 9)$  from the beginning, we use the computation of  $\delta(6 \cdot 7)$  and only account for the new products that may arise in the new area.

In general, this requires that we tabulate  $\delta(n)$ , for those *n* that have similar shapes. For a fixed *m*, the divisor list of *mp* and *mq* are very similar. In particular, if both primes are larger than *m* the first entries in the divisor lists correspond only to the divisors of *m*. We may re-use the bit vector in computing  $\delta(mp)$  to compute  $\delta(mq)$ . All we need to account for are the (relatively few) new products that appear as the corresponding rectangles shift.

**Remark 2.5.** The restriction that p > m may be lifted but doing so requires a more careful accounting of the divisor lists for these cases. The thing of key importance is that the first entries in both lists are only the divisors of m. So long as we have the first entries in the divisor lists be the divisors of m then we may also lift the restriction that p and q are prime. In which case the output will be a lower bound on  $\delta(mq)$  and will be equality if q is prime.

**Theorem 2.6.** Algorithm 3 computes  $\delta(nq)$  in time  $O(nd(q) \ln n)$  where d(q) is q - p.

*Proof.* The run-time is, like Algorithm 2, proportional to the area. The difference between consecutive primes is d(q). Also, there are  $O(n \ln n)$  individual products to check per unit shift.

Algorithm 2 would compute  $\delta(nq)$  in time  $O(nq \ln nq)$  which is significantly more costly than  $O(nd(q) \ln n)$ . The real benefit is while computing  $\delta(mq)$ , we learn

**Algorithm 3:** Computing  $\delta(mq)$  given  $\delta(mp)$  for q > p > m.

**Input** : A bit vector A of length mq with weight w containing the products from computing  $\delta(mp)$ . The divisor lists for mp and mq:  $D_p = [[d_0 = 1, mp], [d_1, mp/d_1], \ldots]$  and  $D_q = [[d_0 = 1, mq], [d_1, mq/d_1], \ldots]$  both of length  $\ell$ . **Output:**  $\delta(mq)$ **1** Initialize counters i = 1 and k = 0**2** for  $i < D_p[\ell - 1][0]$  do if  $i == D_p[k][0]$  then 3 4 Increment kfor  $D_p[k][1] \le j < D_q[k][1]$  do 5 if A[ij] == 0 then 6 Set A[ij] = 17 Increment w8 9 return w

 $\delta(mp)$  for all p < q. In computing M(n), we may compute  $\delta(mq)$  at a cost of  $O(n \ln n)$  but in the process we will learn  $\delta(mq)$  for all p < q for no additional cost.

**Theorem 2.7.** Algorithms 2 and 3 may be used in conjunction to compute M(n) in time  $O(n^2/L^{\sqrt{2}/2+o(1)})$ , where  $L = L(n) := \exp(\sqrt{\ln n \ln \ln n})$ .

Proof. We consider all k < n in two classes. The first is when k is smooth. Let  $P(k) < L^{\gamma}$  and there are about  $n/L^{1/2\gamma+o(1)}$  such numbers. These have  $\delta(k)$  computed using Algorithm 2. Those k < n may be computed in time  $O(n^2/L^{1/2\gamma+o(1)})$ . The second is when k is not smooth; let k = mq, where  $q = P(k) > L^{\gamma}$ . Now we consider the numbers m that can arise, they are all smaller than  $n/L^{\gamma}$ . For each m, take the largest prime q such that mq < n and compute  $\delta(mq)$  using Algorithm 3 and learn  $\delta(mp)$  for all primes p < q. The run-time for this is  $O(n^2/L^{\gamma+o(1)})$ . These two computations are balanced when  $\gamma = \sqrt{2}/2$ .

Since Algorithm 3 computes a lower bound  $\delta(mk)$  when k is composite, it is possible to write a correction algorithm. It takes the lower bound, the incomplete divisor list, the complete divisor list and counts the new distinct products from the part of the shape that Algorithm 3 missed. Care must be taken that the missing products are not recorded in the bit vector used by Algorithm 3. The third and fourth authors implemented this correction algorithm and used a set data structure to store the new products.

## 3. Monte-Carlo Estimations

If n is too large for the exact computation of M(n) to be feasible, we can resort to Monte Carlo estimation of M(n). In the following we describe two different Monte Carlo methods, which we call the *Bernoulli* and *product* methods. In the descriptions of these two methods, we assume that n is fixed.

3.1. The Bernoulli Method. We perform a sequence of  $T \ge 2$  trials, where each trial involves choosing a pseudo-random integer  $z \in [1, n^2]$ . The integers z are

assumed to be independent and uniformly distributed. For each z, we count a success if z appears in the  $n \times n$  multiplication table, i.e. if z can be written as z = xy, where  $1 \le x \le y \le n$ . Let S be the number of successes after T trials. Since we are performing a sequence of T Bernoulli trials with probability of success  $p = M(n)/n^2$ , the expected number of successes is  $\mathbb{E}(S) = pT$ , and the variance is  $\mathbb{V}(S) = \mathbb{E}((S-pT)^2) = p(1-p)T$ , see for example [20]. Thus, an unbiased estimate of  $M(n)/n^2$  is given by  $\hat{p} = S/T$ , and the variance of this estimate is p(1-p)/T. For large T the error  $M(n)/n^2 - S/T$  is asymptotically normally distributed. By the "law of the iterated logarithm" [16], this error is almost surely  $O((T^{-1} \log \log T)^{1/2})$  as  $T \to \infty$ .

In a practical computation, p is unknown, but an unbiased estimate of the variance of the error is  $\hat{p}(1-\hat{p})/(T-1)$ , where the denominator T-1 takes into account the loss of one degree of freedom in using the sample mean  $\hat{p}$  instead of the population mean p (this is known as *Bessel's correction*).

3.2. The Product Method. In this method, each trial takes z = xy, where x and y are independently and uniformly distributed integers in [1, n]. Thus, z is guaranteed to appear in the  $n \times n$  multiplication table. Let  $\nu = \nu(z) \ge 1$  denote the number of times that z appears in the table. The probability that a trial samples z is  $\nu(z)/n^2$ . Thus,  $\mathbb{E}(1/\nu) = M(n)/n^2 = p$  (where p is as in the Bernoulli method). Consider a sequence of T independent trials, giving values  $\nu = \nu_1, \ldots, \nu_T$ . An unbiased estimate of  $M(n)/n^2$  is given by  $E := T^{-1} \sum_{1 \le j \le T} 1/\nu_j$ , and the variance of this estimate is  $V := T^{-1}\mathbb{E}((\nu^{-1} - p)^2)$ . Lemma 3.1 shows that, for the same values of T and n, the variance in the estimate of  $M(n)/n^2$  given by the product method.

**Lemma 3.1.** If V is the variance of the estimate E after T trials of the product method, then  $V \le p(1-p)/T$ .

*Proof.* Using  $p = \mathbb{E}(\nu^{-1})$ , we have

 $V = T^{-1}\mathbb{E}((\nu^{-1} - p)^2) = T^{-1}(\mathbb{E}(\nu^{-2}) - p^2).$ 

Since  $\nu$  is a positive integer,  $\nu^{-2} \leq \nu^{-1}$ , and  $\mathbb{E}(\nu^{-2}) \leq \mathbb{E}(\nu^{-1}) = p$ . It follows that  $V \leq T^{-1}(p-p^2)$ , as desired.

**Remark 3.2.** It is easy to see that equality holds in Lemma 3.1 only in the trivial case n = 1. From Ford's result (2), we have  $TV = O(1/\Phi(n))$  as  $n \to \infty$ .

An unbiased estimate of the variance of the error for the product method in terms of computed quantities is  $\sum_{1 \le j \le T} (\nu_j^{-1} - E)^2 / (T(T-1))$  (see the comment above on Bessel's correction).

3.3. Avoiding Factorization via Bach/Kalai. For the Bernoulli method, we have to determine if an integer  $z \in [1, n^2]$  occurs in the  $n \times n$  multiplication table. Equivalently, we have to check if z has a divisor d satisfying  $z/n \leq d \leq n$ . A straightforward algorithm for this would first find the prime power factorization of z, then attempt to construct a divisor d in the interval [z/n, n], using products of the prime factors of z.

Similarly, for the product method, we have to count the number of divisors d of xy in the interval [xy/n, n]. A straightforward algorithm for this would first find the prime power factorizations of x and y.

To avoid having to factor the random integers z (or x and y) occurring in the Bernoulli (or product) methods, we can generate random integers along with their prime power factorizations, using the algorithms of Bach [2] or Kalai [15]. This is much more efficient, on average, than generating random integers and then attempting to factor them, since the integer factorization problem is not known to be solvable in polynomial time.

The algorithm described by Bach, specifically his "Process R", returns an integer x uniformly distributed in the interval (N/2, N], together with the prime power factorization of x. Using Bach's algorithm, which we call "procedure R", it is easy to give a recursive procedure B which returns x uniformly distributed in the interval [1, N], together with the prime power factorization of x. For details see Algorithm 4. The following comments on the complexity of Bach's algorithm also apply to procedure B.

### Algorithm 4: Modification of Bach's algorithm

```
1 procedure R(N)
   Input : A positive integer N
   Output: A random integer x \in (N/2, N] and its prime power factorization
2 Details omitted: see Bach [2, "Process R", pg. 184]
\mathbf{s} end procedure R
4 procedure B(N)
   Input : A positive integer N
   Output: A random integer x \in [1, N] and its prime power factorization
5 if N == 1 then
   return 1
6
 7 generate random real u uniformly distributed in [0, 1)
 s if u < |N/2|/N then
   return B(\lfloor N/2 \rfloor)
9
10 else
   return R(N)
11
12 end procedure B
```

The expected running time of Bach's algorithm is dominated by the time for primality tests.<sup>2</sup> Bach's algorithm requires, on average,  $O(\log N)$  primality tests. The AKS deterministic primality test [1] requires  $(\log N)^{O(1)}$  bit-operations, so overall Bach's algorithm has average-time complexity  $(\log N)^{O(1)}$ . In our implementation, we replaced the AKS primality test by the Miller-Rabin Monte Carlo test [9, 17, 18, 23], which is much faster, at the cost of a small probability of error.<sup>3</sup>

<sup>&</sup>lt;sup>2</sup>More precisely, Bach's algorithm requires prime power tests, but it is relatively easy to check if an integer is a perfect power (see Bernstein [4]), so primality tests and prime power tests have (on average) almost the same complexity. Also, it is possible to modify Bach's algorithm so that only primality (not prime power) tests are required. Thus, we ignore the distinction between primality tests and prime power tests.

<sup>&</sup>lt;sup>3</sup>The probability of error can be reduced to  $\leq 4^{-k}$  by repeating the test k times with independent random inputs, see [23].

A small probability of an error (falsely claiming that a composite integer is prime) is acceptable when the overall computation is a Monte Carlo estimation. One such error will have a negligible effect on the final result, assuming that the number of trials is large.

Kalai [15] gave an algorithm with the same inputs and outputs as our modification (procedure B) of Bach's algorithm, but much simpler and easier to implement. The only disadvantage of Kalai's algorithm is that it is asymptotically slower than Bach's, by a factor of order log N. More precisely, Kalai's algorithm requires, on average, of order  $(\log N)^2$  primality tests, whereas procedure B requires of order log N prime power tests. We implemented both algorithms using Magma [5], and found that, as expected, Kalai's algorithm was slower than procedure B for N sufficiently large. With our implementations<sup>4</sup>, the crossover point was  $N \approx 2^{45}$ . For  $N = 2^{100}$ , procedure B was faster by a factor of about 2.2. This is in agreement with the theoretical prediction that Kalai's algorithm should be slower by a factor of order log N.

#### 4. Implementations and Results

There were several independent implementations of Algorithm 1, and three independent implementations of Algorithm 2 in three different languages: C, C++, and Sage <sup>5</sup>. The last published exact computations from [8] were of the form  $M(2^n - 1)$  for  $1 \le n \le 17$ . In Table 1, we include the next  $18 \le n \le 30$ . The entries in Table 1 were computed independently using both Algorithm 1 and Algorithm 2. No discrepancies were found.<sup>6</sup> Timing comparisons are difficult because different (time-shared) computer systems were used, but we estimate that Algorithm 2 was about three times faster than Algorithm 1 for n = 30.

k	$M(2^k - 1)$	k	$M(2^k - 1)$
18	14081089287	25	209962593513291
19	55439171530	26	830751566970326
20	218457593222	27	3288580294256952
21	861617935050	28	13023772682665848
22	3400917861267	29	51598848881797343
23	13433148229638	30	204505763483830092
24	53092686926154		

TABLE 1. Extension of the Brent-Kung computation

A table of  $M(k \cdot 2^{10})$  for  $1 \le k \le 2^{20}$  was computed by the third and fourth authors [21]. This implementation used a wheel modulus approach as described in Subsection 2.4 with w = 60. The computation took about 7 weeks on Butler University's BigDawg cluster which has 32 Intel Xeon E5-2630 (192 total cores) processors. Table 2 shows the time (in seconds) to compute  $\delta(n)$  for all  $n \in (10^8, 10^8 + 10^3]$  on an Intel i7-4700 with 16GB RAM.

<sup>&</sup>lt;sup>4</sup>Further details concerning our implementations, and approximations/optimizations valid for very large N, may be found in [6, 7].

 $<sup>^5\</sup>mathrm{We}$  thank Paul Zimmerman for verifying some of these calculations in his implementation in Sage.

<sup>&</sup>lt;sup>6</sup>The entries given in OEIS A027417 differ by one because they include the zero product.

Algorithm	time $(s)$
Algorithm 2	909
$\pmod{1}$	302
$\pmod{2}$	184
$\pmod{6}$	106
$\pmod{12}$	85
$\pmod{60}$	59

TABLE 2. Runtime comparison

Algorithm 1, implemented in C, ran mainly on the ARCS computer system at the University of Newcastle, Australia. The computer nodes used were a mixture of 2.2 GHz Intel Xeon 3 and 2.6GHz Intel Xeon 4.

The third and fourth authors implemented Algorithm 3 and the correction algorithm but were unable to make it computationally competitive.

To illustrate Monte Carlo, consider the case  $N = 2^{30} - 1$ , for which we know the exact value M(N) = 204505763483830092 from our deterministic computations. Taking  $T = 10^6$  trials of the "product" Monte Carlo method, we estimate  $M(N)/N^2 = 0.17750$ , whereas the correct value (to 5D) is  $M(N)/N^2 = 0.17738$ . The variance estimate here is  $V = 2.873 \times 10^{-8}$ , so  $\sigma := V^{1/2} \approx 0.00017$ . Thus, the Monte Carlo estimate is as accurate as predicted from the standard deviation  $\sigma$ . The same number of trials with the Bernoulli method gives variance  $1.459 \times 10^{-7}$ , larger by a factor of about five. Thus, the product method is more efficient, (other things being equal), as predicted by Lemma 3.1. In practice the comparison is not so straightforward, because the product method requires checking more divisors (on average) than the Bernoulli method, and hence has a larger space requirement.

The results of some Monte Carlo computations are given to 4D in Table 3. We used the product method (mainly for  $n < 10^6$ ) and the Bernoulli method (mainly for  $n \ge 10^6$ ), combined with Bach's algorithm (described in Section 3). Kalai's algorithm was used for confirmation (mainly for  $n \le 100$ ). The second column gives an estimate of  $M(N)/N^2$ , and the last column gives the normalized value  $(N^2/M(N))/\Phi(N)$ . By Ford's result (2), this should be  $\approx 1$ . The third column gives  $10^4\sigma$ , where  $\sigma^2$  is an estimate of the variance of the corresponding entry in the second column. Because of the factor  $10^4$ , this corresponds to units in the last place (ulps) for the second column. Since the entries in the third column are bounded by 0.12, the entries in the second column are unlikely to be in error by more than 0.7 ulp. Similarly, the entries in the last column of the table are unlikely to be in error by more than 1 ulp. The entries for  $n \le 30$  may be verified (up to the predicted accuracy) using the exact results of Table 1.

As described in [6, 7], we can extrapolate the last column of Table 3 and conjecture that

(4) 
$$\lim_{N \to \infty} \frac{N^2/M(N)}{\Phi(N)} \approx 0.12.$$

It has not been proved that the limit in (4) exists; Ford's result (2) only shows that the lim sup and limit are finite and positive. In some similar problems the corresponding limit does not exist. For example, let S(x) be the number of  $n \leq x$  with  $\tau(n) \geq \ln x$ . Norton showed [19] there are positive constants  $c_1, c_2$  with

n	$M(N)/N^2$	$10^4\sigma$	trials	$N^2/M(N)$
	$(N=2^n-1)$		$10^{8}$	$\Phi(N)$
20	0.1987	0.12	2	0.9414
30	0.1774	0.02	100	0.8213
40	0.1644	0.02	100	0.7549
50	0.1552	0.02	100	0.7112
$10^{2}$	0.1311	0.02	100	0.6068
$10^{3}$	0.0798	0.02	100	0.4264
$10^4$	0.0517	0.01	100	0.3435
$10^{5}$	0.0348	0.06	2	0.2958
$10^{6}$	0.0240	0.05	10	0.2652
$10^{7}$	0.0170	0.05	6.7	0.2432
$10^{8}$	0.0121	0.10	1.32	0.227

TABLE 3. Monte Carlo computations

 $c_1 < R(x) < c_2$  for x sufficiently large, where  $R(x) = S(x)x^{-1}(\ln x)^c(\ln \ln x)^{1/2}$ . Later, Balazard, *et al.* [3] showed that  $\lim_{x\to\infty} R(x)$  does not exist.

# References

- Manindra Agrawal, Neeraj Kayal and Nitin Saxena, *PRIMES is in P*, Ann. of Math. (2) 160 (2004), 781–793.
- [2] Eric Bach, How to generate factored random numbers, SIAM J. Comput. 17 (1988), 179–193.
- [3] M. Balazard, J. L. Nicolas, C. Pomerance, and G. Tenenbaum, Grandex deviations pour certaines fonctions arithmétiques, J. of Number Theory 40 (1992), 146–164.
- [4] Daniel J. Bernstein, Detecting perfect powers in essentially linear time, Math. Comp. 67 (1998), 1253–1283.
- [5] Wieb Bosma, John Cannon, and Catherine Playoust, The Magma algebra system. I. The user language, J. Symbolic Comput. 24 (1997), 235–265.
- [6] Richard P. Brent, The multiplication table, and random factored integers, slides from a seminar given at Hong Kong Baptist University, Hong Kong, Feb. 6, 2015. https://maths-people. anu.edu.au/~brent/pd/multiplication-HK.pdf.
- [7] Richard P. Brent, Algorithms for the multiplication table, slides from a seminar given at CARMA, Newcastle, NSW, May 29, 2018. https://maths-people.anu.edu.au/~brent/pd/ multiplication-CARMA.pdf.
- [8] Richard P. Brent and Hsiang T. Kung, The area-time complexity of binary multiplication, J. Assoc. Comput. Mach. 28 (1981), 521–534. Corrigendum: ibid 29 (1982), 904. MR 624731
- [9] Ronald J. Burthe, Jr., Further investigations with the strong probable prime test, Math. Comp. 65 (1996), 373–381.
- [10] Paul Erdős, Some remarks on number theory, Riveon Lematematika 9 (1955), 45–48 (in Hebrew).
- [11] \_\_\_\_\_, An asymptotic inequality in the theory of numbers, Vestnik Leningrad. Univ. 15 (1960), no. 13, 41–49 (in Russian).
- [12] Kevin Ford, The distribution of integers with a divisor in a given interval, Ann. of Math. (2) 168 (2008), no. 2, 367–433.
- [13] \_\_\_\_\_, Integers with a divisor in (y, 2y], Anatomy of integers, CRM Proc. Lecture Notes, vol. 46, Amer. Math. Soc., Providence, RI, 2008, pp. 65–80.
- [14] Harald A. Helfgott, An improved sieve of Eratosthenes, Math. Comp. online April 23, 2019.
- [15] Adam Kalai, Generating random factored numbers, easily, J. Cryptology 16 (2003), 287–289.
- [16] Aleksandr Y. Khinchin, Über einen Satz der Wahrscheinlichkeitsrechnung, Fundamenta Mathematicae 6 (1924), 9–20.
- [17] Jared D. Lichtman and C. Pomerance, Improved error bounds for the Fermat primality test on random inputs, Math. Comp. 87 (2018), 2871–2890. https://doi.org/10.1090/mcom/3314.

- [18] Gary L. Miller, Riemann's hypothesis and tests for primality, J. Comp. System Sci. 13 (1976), 300–317.
- [19] K. K. Norton On the number of restricted prime factors of an integer, I, Illinois J. Math. 20 (1976), 681–705.
- [20] Athanasios Papoulis, Probability, Random Variables, and Stochastic Processes, McGraw-Hill, New York, USA, first edition, 1965.
- [21] David Purdum and Jonathan Webster, http://blue.butler.edu/~jewebste/Mn2pow30.txt
- [22] David Purdum, https://rutrum.github.io/multiplication-table/
- [23] Michael O. Rabin, Probabilistic algorithm for testing primality, J. of Number Theory 12 (1980), 128–138.
- [24] Gérald Tenenbaum, Sur la probabilité qu'un entier possède un diviseur dans un intervalle donné, Seminar on Number Theory, Paris 1981–82 (Paris, 1981/1982), Progr. Math., vol. 38, Birkhäuser Boston, Boston, MA, 1983, pp. 303–312 (in French).