Appendix B: Some Open Problems

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Denote by $r(k,n)$ the smallest integer for which every graph of $r(k,n)$ vertices either contains an independent set of k vertices or a complete graph of n vertices. Let us first assume $k = n$.

$$c_1 n 2^{n/2} < r(n,n) < \frac{\binom{2n}{n}}{n^\alpha} \qquad (1)$$

are the best current results. $r(5,5)$ is not known at present, the best known results are $43 \leq r(5,5) \leq 55$. The lower bound in (1) is an easy application of the probability method, only the value of c_1 has been improved. The first problem would be to prove

$$\frac{r(n,n)}{n 2^{n/2}} \to \infty \qquad (2)$$

Perhaps (2) will not be very difficult. The real problem is to compute

$$\lim_{n\to\infty} r(n,n)^{1/n} = c \qquad (3)$$

It is not even known if the limit in (3) exists. It is not at all certain that random methods will help here.

$$c_2 \frac{n^2}{(\log n)^2} < r(3,n) < c_3 \frac{n^2}{\log n} \qquad (4)$$

Both the upper and the lower bound in (4) are proved by random methods. It would be very nice to improve (4) and perhaps get an asymptotic formula for $r(3,n)$. This might be very difficult and perhaps probability methods will be of no help. I am fairly sure that for any fixed k

$$r(k,n) > n^{k-1-\epsilon} \qquad (5)$$

and believed that the proof for $k > 3$ would not be very difficult and that the difficulties are only technical. It is quite possible that I was wrong and

$$r(4,n) > n^{3-\epsilon}$$

is very hard and may require new methods and perhaps is not even true. The best known result is due to Spencer

$$r(4,n) > n^{5/2-\epsilon}$$

proved by the Lovász local lemma. One would expect

$$r(k,n) > \frac{n^{k-1}}{(\log n)^{c_k}}$$

The probabilistic methods of Ajtai-Komlós-Szemerédi gives

$$r(k,n) < c\frac{n^{k-1}}{(\log n)^{\alpha_k}}$$

An asymptotic formula for $r(k,n)$ seems out of reach at present.

Two related problems are: Let $G(n)$ be a trianglefree graph of n vertices, how large can its chromatic number be? $n^{1/2}/(\log n)^{\alpha}$ is a good upper and lower bound but the best value of α is not known. Let G_e be a trianglefree graph of e edges. How large can its chromatic number be? $e^{1/3}/(\log e)^{c}$ is roughly the correct order of magnitude.

One of the early triumphs of the probability method was to prove that there is a graph of arbitrarily large chromatic number and arbitrarily large girth, more precisely for every k there is a graph of n vertices, girth k, the largest independent set of which is $n^{1-\epsilon_k}$ where $\epsilon_k > c/k$. This holds for every k. Lovász gave a complicated construction for a graph of arbitrary large girth and chromatic number, later Nesetril and Rödl gave a much simpler construction. As observed by Alon, some of the known explicit constructions of expanders supply, for every k, explicit graphs of n vertices, with girth k, and with largest independent set $n^{1-\epsilon_k}$ where $\epsilon_k > c/k$. This essentially matches the result obtained by the probabilistic method.

I often asked the following question. Let G be a random graph of n vertices and cn edges. What can be said about the chromatic number of it? What is the largest $r = f(c)$ for which the probability that the chromatic number is r is positive? As far as I know this is open except for $r = 3$. Also the graph of chromatic number $r > 4$ will have size $c_r n$. Bollobás had the following idea. Let l be the largest integer for which our graph contains an induced subgraph each vertex of which has degree at least l. Is it true that the chromatic number of this graph is $l+1$? Or, if not, it should be at least l. I think this problem is still open. Its relevance to my problem is clear.

When Rényi and I developed our theory of random graphs we thought of extending our study for hypergraphs. We mistakenly thought that all (or most) of the extensions will be routine and we completely overlooked the following beautiful question of Shamir: Rényi and I proved that if n is even and we consider the random graph of $\frac{1}{2}n\log n + cn$ edges then G has a perfect matching with probability $f(c)$ where $f(c)$ is an exponential function in c. We used the theorem of Tutte. Now Shamir asked how many triples must one choose on $3n$ elements so that with probability bounded away from zero one should get n vertex disjoint triples. Shamir proved that $n^{3/2}$ triples suffice but the truth may very well be $n^{1+\epsilon}$ or even $cn\log n$. The reasons for the difficulty is that Tutte's theorems seem to have no analogy for triple systems or more generally for hypergraphs. Clearly many related questions can be asked for hypergraphs for all r-tuples, $r \geq 3$. But perhaps for random graphs there might be also unexplored questions, e.g., how many edges are needed in a graph of $3n$ vertices to be able to cover the vertices by n vertex disjoint triangles? The correct answer will be probably about $n^{4/3}$ edges but perhaps a little more will be needed. Also instead of a triangle one could ask for other graphs, e.g. the graph of six vertices and six edges with a triangle and a one factor. Again the lack of analogs to Tutte's theorem may cause serious trouble.

Has the following question of Spencer and myself been really settled? We have n vertices and draw edges at random one by one. We stop as soon as there is no isolated vertex. Is it true that then there already almost surely is a perfect matching if the number of vertices is even? If we stop when every vertex has degree 2 is it true that we already have a Hamiltonian circuit? Similarly if every vertex has degree r does the graph contain an r-chromatic (or $r+1$ chromatic) subgraph? Here of course surely an r chromatic subgraph might appear once there is a subgraph every vertex of which has degree $\geq r-1$. Also, if the first r-chromatic subgraph appears its size will be very large almost surely if $r > 2$. For $r = 2$ this is not true, the expected value is very large because of the large odd circuits. This has been computed by a Scandinavian mathematician and also by Knuth in a paper if I remember right dedicated to my 75-th birthday.

How accurately can one estimate the chromatic number of the random graph (with edge probability $\frac{1}{2}$)? Perhaps one can prove that the error is more (much more) than $O(1)$. (Shamir and Spencer have an $O(n^{1/2})$ upper bound.) Also, in the proof of Bollobás, is it true that every subgraph of size n^α will almost surely contain an independent set of size $c\log n$ with the right value of c?

As stated before, if a random graph of an even number of vertices has enough edges so that every vertex has degree ≥ 1 then there is with probability tending to one a perfect matching. Could this hold for other configurations too? I.e., if every vertex is contained in a triangle is there a set of vertex disjoint triangles covering every vertex? Many related questions can be asked.