NL2754 **Period Doubling**

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Like many terms used in the nonlinear sciences, period doubling has more than one meaning. Well-known is the response of a system at half the driving frequency, due to non-linear coupling. This was investigated already by Lord Rayleigh (Rayleigh, 1887). He wrote of an example:

> ... in which a fine string is maintained in transverse vibration by connecting one of its extremities with the vibrating prong of a massive tuning-fork, *the direction of motion of the point of attachment being parallel to the length of the string* ... the string may settle down into a state of permanent and vigorous vibration *whose period is double that of the point of attachment.*

A more everyday example is the fact that a child may set a swing into transverse motion by standing on the seat and moving up and down at twice the natural frequency. Such phenomena are in the province of *harmonic generation* and *parametric amplification*, and are not treated in this entry.

Period doubling (as discussed in this entry) is the most common of several *routes to chaos* for a nonlinear dynamical system.

The dynamics of natural processes, and the nonlinear equations used to model them, depend on externally set conditions, such as environmental or physical factors. These take fixed values over the development of the system in any particular instance, but vary from instance to instance. Expressed as numerical quantities, such factors are called parameters, and their role is vital. Often, increasing a parameter increases the nonlinearity. The simplest example is the logistic model (May, 1976)

$$x_{n+1} = rx_n(1 - x_n), \tag{1}$$

a discrete system with state variable x_n and parameter r; n is the generation. (If x represents population, then r characterises the underlying growth rate.)

For period doubling, it is sufficient to vary a single parameter. Over some range, the system may have a periodic attractor, that is, a periodic orbit which attracts neighbouring orbits. Typically, the stability (attraction) decreases as the parameter increases, changing to instability (repulsion) at a critical value. This is a bifurcation, or change of structure, of the orbit. In period doubling, this change is accompanied by the "birth" of a new attracting period-doubled orbit, in which the system alternates between two states. In biological systems, these are known as alternans (Glass & Mackey, 1988). In the period doubling route to chaos, each new periodic attractor loses its stability with increasing parameter value, whereupon the next period doubled attractor is born. If the original attractor was a fixed point, this generates orbits of period 1, 2, 2^2, 2^3, ..., called the main period doubling cascade.

	Logistic		Hénon	
n	r_n	$\left(\frac{r_{n+1}-r_n}{r_{n+2}-r_{n+1}}\right)$	a_n	$\left(\frac{a_{n+1}-a_n}{a_{n+2}-a_{n+1}}\right)$
1	3.0000000000		0.3675000000	
2	3.4494897428		0.9125000000	
3	3.5440903596	4.751446	1.0258554050	4.807887
4	3.5644072661	4.656251	1.0511256620	4.485724
5	3.5687594195	4.668242	1.0565637582	4.646894
6	3.5696916098	4.668739	1.0577308396	4.659569

Table 1. Estimates of the Feigenbaum constant δ, for two maps.

Experiment		Observed number of period doublings	estimated value of δ	estimated value of α
Hydrodynamic:	helium	4	3.5 ± 1.5	
	mercury	4	4.4 ± 0.1	
Electronic:	diode	5	4.3 ± 0.1	2.4 ± 0.1
	Josephson	4	4.5 ± 0.3	2.7 ± 0.2
Laser:	feedback	3	4.3 ± 0.3	
Acoustic:	helium	3	4.8 ± 0.6	

Table 2. Selected experimental data on period doubling.

A surprising *universality* was discovered by Mitchell Feigenbaum. In part, it relates to the sequence of parameter values at which successive period doubling occurs. Label the first by r_1, the second by r_2, and so on, and let the cascade end at the value r_∞. The differences $(r_\infty - r_n)$ decrease to zero; the surprising fact is that the *ratios* $(r_\infty - r_n)/(r_\infty - r_{n+1})$ converge to a universal constant δ, which takes the *same value* $\delta \approx 4.669202\ldots$ for all maps with a quadratic maximum (Feigenbaum, 1978). There is a second universal constant, $\alpha \approx 2.502908\ldots$, measuring the relative spatial scale of the orbits, which also becomes increasingly fine.

Numerical data for two main period doubling sequences is given in Table 1. Columns 2–3 display data for the logistic map (2), columns 4–5 for its two-dimensional cousin, the Hénon map:

$$x_{n+1} = 1 - ax_n^2 + y_n \qquad y_{n+1} = bx_n. \tag{2}$$

For it, the parameter b has been kept constant. Experimental data has also been obtained in quite a few systems (Cvitanović, 1989); a selection is shown in Table 2. To appreciate the experimental difficulty involved, remember that for each successive period doubling, the significance of errors increases five-fold while the complexity of the

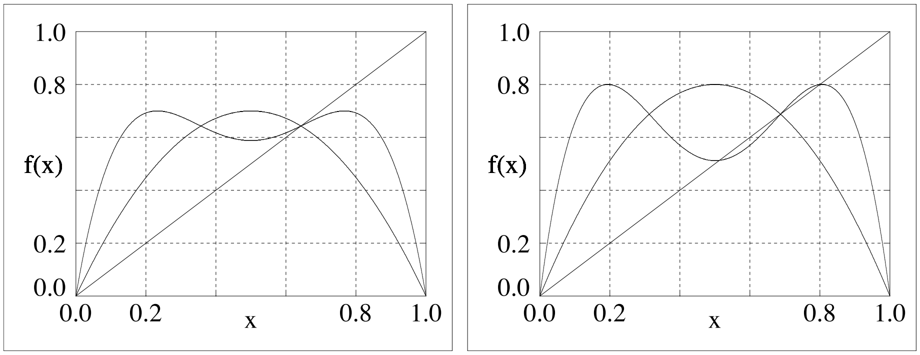

Figure 1. Mechanism: f and f_2 for $r = 2.8$ (left) and $r = 3.2$ (right).

dynamics doubles.

The mechanism for period doubling is already implicit in the fact that it is also known as a "flip bifurcation". Stability of a fixed point x^* of a smooth one-dimensional map f is the simplest case for theory. Consider a nearby point x that maps to x'. Linear approximation gives

$$(x' - x^*) \approx f'(x^*) \cdot (x - x^*). \tag{3}$$

This shows (and exact analysis confirms) that the fixed point is stable if $-1 < f'(x^*) < 1$, unstable if $f'(x^*) < -1$ or $f'(x^*) > 1$. Successive iterates flip from side to side for negative $f'(x^*)$, so period doubling occurs at $f'(x^*) = -1$. For higher-dimensional systems, stability is determined by by the eigenvalues of a matrix of partial derivatives; the occurrence of an eigenvalue $\lambda = -1$ leads to period doubling.

Since $x^{**}_{n+2} = x^{**}_n$ for the new orbit, period doubling is connected with double iteration, controlled by the second composition map $f_2(x) = f(f(x))$. Elementary calculus shows that, if f satisfies the two conditions

$$f(x^*) = x^*, \qquad f'(x^*) = -1, \tag{4}$$

then f_2 satisfies three conditions:

$$f_2(x^*) = x^*, \qquad f_2'(x^*) = +1, \qquad f_2''(x^*) = 0. \tag{5}$$

As a result, $x - f_2(x)$ must be approximated by a cubic near to the bifurcation. One of its zeros is the (now unstable) fixed point of f, the other two constitute the period doubled orbit, since they are not fixed points of f.

In the case of the logistic map, the first period doubling occurs at $r = 3$. Graphs of f and f_2 show the mechanism (Figure 1). As the cascade proceeds, the pictures and the analysis repeat, but with increasing complication. The common thread is that, at each step, conditions (4) for a composition f_n imply conditions (5) for f_{2n}.

Historical Note: Probably the earliest observation of period doubling was by Michael Faraday, in his investigations of shallow water waves (Faraday, 1831, arts. 98–101).

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See also Attractors; Bifurcations; Dripping faucet; Harmonic generation; Hénon map; Maps; One-dimensional maps; Parametric amplification; Routes to chaos; Stability; Universality

Further Reading

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