NL3433 **Integral transforms**

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Integral transforms have their genesis in nineteenth century work of Joseph Fourier and Oliver Heaviside, subsequently set into a general framework during the twentieth century. The fundamental idea is to represent a function $f(x)$ in terms of a *transform* $F(p)$, using an integral transform pair,

$$F(p) = \int K(p,x) f(x)\, dx, \qquad f(x) = \int L(x,p) F(p)\, dp. \tag{1}$$

The functions $K(p,x)$ and $L(x,p)$ are *kernels*. If either of $f(x)$, $F(p)$, is discontinuous, further interpretation must be supplied at those points.

Oliver Heaviside invented his operational calculus to solve differential equations, such as those which arise in the theory of electrical transmission lines. A simple example is the diffusion problem (Whittaker, 1928),

$$\partial^2 u/\partial x^2 = k\, \partial u/\partial t, \qquad x \geq 0, \qquad t \geq 0, \qquad u(0,t) = U_0. \tag{2}$$

Heaviside treated differentiation as a symbolic operation, $p = \partial/\partial t$, to write the operational solution

$$u(x,t) = \mathrm{e}^{-x\sqrt{kp}} \cdot U_0. \tag{3}$$

Using Euler's formula for differentiation,

$$\frac{d^n(x^p)}{dx^n} = \frac{p!}{(p-n)!}\, x^{p-n}, \tag{4}$$

with n *not* restricted to the integers, Heaviside was able to extract all the information he required from such an approach. However, his methods met with general disapproval, leading to life-long difficulties for him.

Heaviside's work was formalized, using the Laplace integral, to give the Laplace transform pair:

$$F(p) = \int_0^\infty \mathrm{e}^{-px} f(x)\, dx, \qquad f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathrm{e}^{+px} F(p)\, dp. \tag{5}$$

In the normal case, the transform integral converges for all p in a complex half-plane $\Re(p) > \alpha$, in which case the constant c in the inversion integral must be chosen to satisfy the restriction $c > \alpha$. Subjecting equation (2) to the Laplace transform leads to the same formula (3) for the *transform* $U(x,p)$. Everything may be extracted from this using complex variable techniques; the complex variable p is just a place-holder for Heaviside's operational symbol $p = \partial/\partial t$.

The Laplace transform has simple properties with respect to differentiation and convolutions (Schiff, 1999). For the former, if $F(p)$ is the transform of $f(t)$, then the transform of $f'(t)$ is $pF(p) - f(0)$. Again, if $h(t)$ is the convolution of $f(t)$ with $g(t)$, defined as

$$h(t) = \int_0^\infty f(s)g(t-s)\,ds, \tag{6}$$

then the transform is given by simple multiplication:

$$H(p) = F(p)\,G(p). \tag{7}$$

An important area of application is systems of ordinary differential equations with constant coefficients, such as (Barnett & Cameron, 1992)

$$\vec{x}' = A\vec{x} + B\vec{u}, \qquad \vec{y} = C\vec{x}. \tag{8}$$

Here $\vec{x}$ is a set of n internal state variables, $\vec{u}$ a set of r inputs, $\vec{y}$ a set of s outputs; A, B, and C are matrices. Taking the Laplace transform, and using elementary matrix algebra, the response of the system is given by

$$\vec{Y}(p) = G(p)\vec{U}(p), \qquad G(p) = C(pI - A)^{-1}B. \tag{9}$$

The transform being a product, the solution is a matrix of convolutions. However, the essential information about the system may be extracted directly, without recourse to inversion.

The Fourier transform represents functions as linear combinations of periodic functions, an idea pioneered by Joseph Fourier. A one-dimensional form is

$$F(\omega) = \int_{-\infty}^\infty f(t)\,\mathrm{e}^{i\omega t}dt, \qquad f(t) = \frac{1}{2\pi i}\int_{-\infty}^\infty F(k)\,\mathrm{e}^{-i\omega t}d\omega. \tag{10}$$

The most common uses are (i) with time as the original variable — transformation from the time domain to the frequency domain; (ii) with spatial position as the original variable — transformation from configuration space to momentum space. The Fourier transform also has simple properties with respect to differentiation and convolution (the integral in (6) now extends from $-\infty$ to $+\infty$), making it useful for investigation of partial differential equations, as well as many more complex systems.

Other important transforms are the Hankel transform, which uses Bessel functions for the kernels in the pair (1), alternative forms of the Fourier transform using sine or cosine functions, and many other lesser known pairs (Davies, 2002). In fact, every Sturm-Liouville problem generates integral transforms (Antimirov et. al., 1993). In addition, there are transforms which involve complex integration in an essential way, such as the Hilbert transform and Cauchy integrals, which use principal values.

The Mellin transform pair:

$$F(p) = \int_0^\infty x^{p-1}f(x)\,dx, \qquad f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-p}F(p)\,dp, \tag{11}$$

has great utility, particularly in asymptotics (Ninham et. al., 1992; Davies, 2002). For example, the Mellin transform representation of the exponential integral

$$\mathrm{Ei}(x) = \int_x^\infty \frac{\mathrm{e}^{-u}}{u}\,du, \tag{12}$$

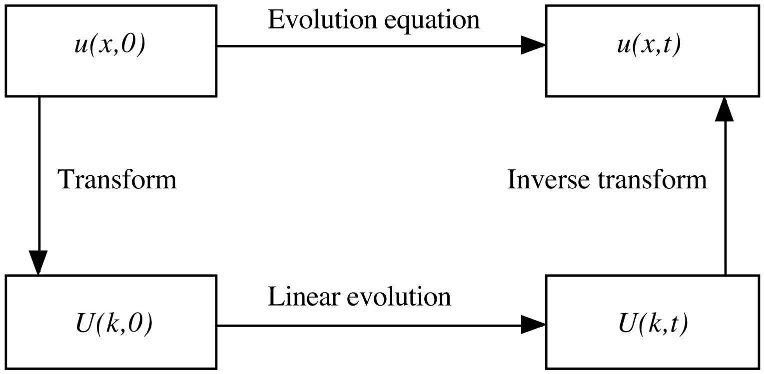

Figure 1. Mechanism: f and f_2 for $r = 2.8$ (left) and $r = 3.2$ (right).

is

$$\mathrm{Ei}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{(p-1)!x^{-p}}{p}\,dp, \qquad c > 0, \tag{13}$$

$$= -\ln x - \gamma - \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k!\,k}, \tag{14}$$

where the infinite sum is obtained by the standard theory of residues.

The Fourier transform is particularly well-suited to linear evolution problems. Typically, there is an initial configuration, $u(x,0)$, which is represented as a linear superposition of plane waves,

$$u(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(k,0)\,\mathrm{e}^{-ikt} dk. \tag{15}$$

The individual waves evolve according to a simple dispersion relation, $\omega = \omega(k)$, which means that the functions $\exp[-ikx + i\omega(k)t]$ satisfy the partial differential equation. As a consequence, the solution at later time is

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(k,0)\,\mathrm{e}^{-ikx+i\omega(k)t} dk. \tag{16}$$

The process can be visualized as in Figure **??**. The top arrow indicates the required evolution of the initial state $u(x,0)$ to the later state $u(x,t)$. The circuitous route has three steps: (i) map the initial state $u(x,0)$ to some wave data $U(k,0)$, (ii) evolve the wave data under the evolution equation to the later wave data $U(k,0)\mathrm{e}^{i\omega(k)t}$, (iii) use an inverse transform to reconstruct the new state from the new wave data.

Beginning in the 1970's, it was realized that a large number of nonlinear evolution equations may be solved using an analogous process (Dodd et. al., 1982). The method has come to be known as the *Inverse Scattering Method*. It is more complicated than the linear methods, but reduces to them in the limit of small nonlinearity. Briefly, the nonlinear equation is associated with a linear one, and there is a mechanism for mapping the data of the former onto the data of the latter. The method again proceeds according to the scheme of Figure **??**. However the inverse transform, which recovers the state from the *scattering data*, is not so simple, and the method derives its name from it. The first explicit use of this scheme was for the solution of the Korteweg-de Vries (KdV) equation, which is associated with a linear Schrödinger equation in which the function $u(x,t)$ plays the role of the potential. The evolution of the spectrum of the Schrdinger operator, in response to the nonlinear evolution of KdV equation, is linear; furthermore, in the limit of weak nonlinearity, the transform and its inverse reduce to the Fourier transform.

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See also Ablowitz-Kaup-Newell-Segur (AKNS) system; Burgers equation; Gel'fand-Levitan theory; Hirota's method; Inverse scattering method; Korteweg-de Vries equation; Nonlinear Schrödinger equation; Sine-Gordon equation; Solitons

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