Comments on non-isometric T-duality

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Based on [1705.09254]
with **P. Bouwknegt**, **C. Klimčík**, and **K. Wright**

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Outline

Setting up notation

Consider a non-linear sigma model $X : \Sigma \to M$ described by the following action:

$$S = \int_\Sigma g_{ij}\, dX^i \wedge \star dX^j + \int_\Sigma B_{ij}\, dX^i \wedge dX^j$$

In this talk we will ignore the dilaton, and assume that both g and B are globally defined fields on M.

Gauging isometries

Suppose now that there are vector fields generating the following global symmetry:

$$\delta_\epsilon X^i = v_a^i \epsilon^a$$

for ϵ^a constant. The sigma model action is invariant under this transformation if

$$\mathcal{L}_{v_a} g = 0 \qquad \mathcal{L}_{v_a} B = 0$$

If this is the case, we can gauge the model by promoting the global symmetry to a local one.

The gauged action

Introducing gauge fields A^a and Lagrange multipliers η_a, the gauged action is

$$S_G = \int_\Sigma g_{ij}\, DX^i \wedge \star DX^j + \int_\Sigma B_{ij}\, DX^i \wedge DX^j + \int_\Sigma \eta_a F^a$$

where

- $F = dA + A \wedge A$ is the standard Yang-Mills field strength
- $DX^i = dX^i - v^i_a A^a$ are the gauge covariant derivatives.

Gauge invariance

The gauged action is invariant with respect to the following (local) gauge transformations:

$$\delta_\epsilon X^i = v_a^i \epsilon^a$$
$$\delta_\epsilon A^a = d\epsilon^a + C^a_{bc} A^b \epsilon^c$$
$$\delta_\epsilon \eta_a = -C^c_{ab} \epsilon^b \eta_c$$

T-duality

$$\begin{tikzcd}
 & S_G[X, A, \eta] \\
S[X] \arrow[ur, bend right, "\text{gauge isometries}"'] &
\end{tikzcd}$$

T-duality

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Varying the Lagrange multipliers forces the field strength F to vanish. If we then fix the gauge $A = 0$ we recover the original model.

T-duality

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Varying the Lagrange multipliers forces the field strength F to vanish. If we then fix the gauge $A = 0$ we recover the original model.

On the other hand, we can eliminate the non-dynamical gauge fields A, obtaining the dual sigma model.

Can we do it without isometries?

The existence of global symmetries is a very stringent requirement. A generic metric will not have any Killing vectors.

Question

Is it possible to follow the same procedure when the vector fields are not Killing vectors?

Gauging without isometry

Kotov and Strobl[1] generalised the standard gauging using Lie algebroids (see Kyle's talk).
Chatzistavrakidis, Deser, and Jonke[2] applied this non-isometric gauging to the Buscher procedure we just reviewed.

They introduce a matrix-valued one-form ω_a^b satisfying

$$\mathcal{L}_{v_a} g = \omega_a^b \vee \iota_{v_b} g$$
$$\mathcal{L}_{v_a} B = \omega_a^b \wedge \iota_{v_b} B$$

[1] [1403.8119]
[2] [1509.01829] and [1604.03739]

The gauged action

The gauged action is almost the same:

$$S_G^{\omega} = \int_{\Sigma} g_{ij}\, DX^i \wedge \star DX^j + \int_{\Sigma} B_{ij}\, DX^i \wedge DX^j + \int_{\Sigma} \eta_a F_{\omega}^a$$

where the curvature is now given by

$$F_{\omega}^a = dA^a + \frac{1}{2} C_{bc}^a(X) A^b \wedge A^c - \omega_{bi}^a A^b \wedge DX^i$$

Modified gauge invariance

The modified gauge transformations are now

$$\delta_\epsilon X^i = v_a^i \epsilon^a$$
$$\delta_\epsilon A^a = d\epsilon^a + C_{bc}^a A^b \epsilon^c + \omega_{bi}^a \epsilon^b DX^i$$
$$\delta_\epsilon \eta_a = -C_{ab}^c \epsilon^b \eta_c + v_a^i \omega_{bi}^c \epsilon^b \eta_c$$

T-duality

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In principle, we could use this to construct T-duals of spaces which have no isometries.

The problem?

This proposal is equivalent to non-abelian T-duality.[3]

That is, if we can find a set of vector fields and ω_a^b which give a non-isometric T-dual, then there exists a set of Killing vectors for the model. The T-dual with respect to these Killing vectors is the same as the non-isometric T-dual.

[3] [1705:09254] P. Bouwknegt, M.B., C. Klimčík, K. Wright

A necessary condition for gauge invariance

Gauge invariance of the action requires the structure functions to be constant, as well as the vanishing of the following variation:

$$\delta_\epsilon(\eta_a F^a_\omega) = \eta_a(d\omega^a_b + \omega^a_c \wedge \omega^c_b)\epsilon^b + \mathcal{O}(A) + \mathcal{O}(A^2).$$

We therefore require that ω^b_a is flat:

$$R^b_a = d\omega^b_a + \omega^b_c \wedge \omega^c_a = 0,$$

and this tells us that ω^b_a is of the form $K^{-1}dK$ for some $K^b_a(X)$.

A field redefinition

Using this K, we can perform the following field redefinitions:

$$\widehat{A}^a = K^a_b A^b$$
$$\widehat{\eta}^a = \eta_b (K^{-1})^b_a$$
$$\widehat{v}_a = v^i_b (K^{-1})^b_a$$

The non-abelian action!

The gauged action can now be rewritten in terms of the new fields $(X^i, \widehat{A}^a, \widehat{\eta}_a)$.

$$
\begin{aligned}
S_G^\omega[X, \widehat{A}, \widehat{\eta}] &= \int_\Sigma g_{ij}\, \widehat{DX}^i \wedge \star \widehat{DX}^j + \int_\Sigma B_{ij}\, \widehat{DX}^i \wedge \widehat{DX}^j + \int_\Sigma \widehat{\eta}_a \widehat{F}^a \\
&= S_G[X, \widehat{A}, \widehat{\eta}]
\end{aligned}
$$

where

$$
\widehat{F}^a = d\widehat{A}^a + \frac{1}{2}\widehat{C}^a_{bc}\widehat{A}^b \wedge \widehat{A}^c
$$

The gauge transformations become the usual non-abelian gauge transformations, and a short computation reveals

$$\mathcal{L}_{\widehat{v}_a} g = 0 \qquad \mathcal{L}_{\widehat{v}_a} B = 0$$

Conclusion

This proposal is equivalent, via a field redefinition, to the standard non-abelian T-duality

Examples!

First example

Consider the 3D Heisenberg Nilmanifold, or twisted torus. It has a metric given by

$$ds^2 = dx^2 + (dy - x\,dz)^2 + dz^2$$

The non-abelian T-dual of this space is given by

$$\widehat{ds^2} = dY^2 + \frac{1}{1+Y^2}\left(dX^2 + dZ^2\right)$$

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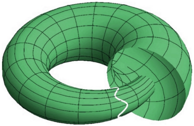

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First example

We can gain a better understanding of the geometry by writing the manifold as a group:

$$Heis := \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

(left-invariant) MC forms $= (dx, dy - xdz, dz)$
(right-invariant) vector fields $= (\partial_x + z\partial_y, \partial_y, \partial_z)$

First example

We could instead try to gauge this space non-isometrically using the left-invariant vector fields: $\{\partial_x, \partial_y, x\partial_y + \partial_z\}$.

These are not all isometries:

$$\begin{aligned}
\mathcal{L}_{v_1} g &= -dy \otimes dx - dz \otimes dy + 2x dz \otimes dz \\
\mathcal{L}_{v_2} g &= 0 \\
\mathcal{L}_{v_3} g &= dx \otimes dy + dy \otimes dx - x dx \otimes dz - x dz \otimes dx
\end{aligned}$$

and they don't commute:

$$[v_1, v_3] = v_2,$$

however...

First example[4]

If we take $\omega_3^2 = dx$ and $\omega_1^2 = -dz$, with other components vanishing, the non-isometric gauging constraints are satisfied and we can calculate the non-isometric T-dual model.

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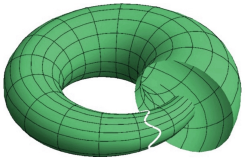

Unsurprisingly, it is also the T-fold.

[4]Gauged non-isometrically in [1509:01829]
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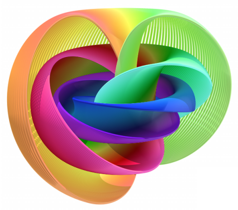
Second example

Consider S^3 with the round metric and $B = 0$.



This metric has an $SO(4)$ group of isometries, and we can find the non-abelian T-dual with respect to an $SU(2)$ subgroup of this.

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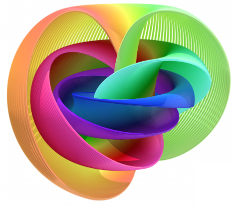

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The non-abelian T-dual is well-known. The metric is the 'cigar' metric, and there is also a non-zero B-field.

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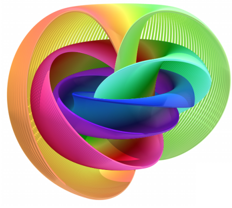

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Second example

We can write the round metric as

$$g = \lambda^1 \otimes \lambda^1 + \lambda^2 \otimes \lambda^2 + \lambda^3 \otimes \lambda^3$$

where the λ^i are the left-invariant Maurer-Cartan forms.

The right-invariant vector fields are isometries of this metric, so let's try gauging with respect to the left-invariant vector fields[5].

[5]These also happen to be isometries of the metric, but let's try to gauge them non-isometrically

Second example

The Lie derivatives of the metric with respect to the left-invariant vector fields, L_a are

$$\mathcal{L}_{L_a} g = -\sum_b C^b_{ac} \lambda^c \vee \lambda^b$$
$$= -C^b_{ac} \lambda^c \vee \iota_{L_b} g$$

We can do non-isometric T-duality by taking $\omega_a^b = -C^b_{ac}\lambda^c$.

Second example

The remaining gauging constraints are satisfied, and we can calculate the non-isometric T-dual. It is the 'cigar' metric, as expected.

Comments

- The equivalence of non-isometric and non-abelian T-duality remains valid for non-exact H
- Geometric interpretation of ω_a^b as a connection on a Lie algebroid
- There are proposals for alternate gauging. Unknown how to incorporate into T-duality
    - non-flat ω_a^b
    - include a term $\phi_{ai}^b \epsilon^b \star DX^i$ into $\delta_\epsilon A^a$

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Thanks!