A Tour of T-duality

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Outline

1. Introduction
2. Local T-duality
3. Global T-duality

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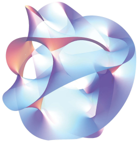
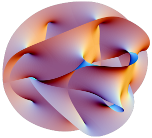
What is T-duality?

T-duality is an equivalence between two physical theories with different spacetime geometries.





These different geometries arise in string theory from the compactification of extra dimensions.

Left image credit: Wikipedia en:User:Jbourjai

Right image credit: Wikipedia en:User:Lunch

Why do we care?

T-duality provides a link between different types of string theories.

Type I

Heterotic-O
$\updownarrow$
Heterotic-E

Type II-A
$\updownarrow$
Type II-B

This led to the suggestion that the five distinct string theories are really just limits of one underlying theory, called **M-theory**.

In Mathematics

T-duality is also interesting from the mathematical point of view.

- Mirror Symmetry and the SYZ conjecture
- Isomorphisms in twisted K-theory
- Generalized Geometry
- ...

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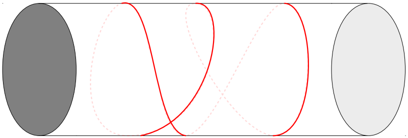
A toy model

The simplest example of T-duality considers a closed Bosonic string on a manifold where one dimension is curled up into a circle.

$$X^{25} \sim X^{25} + 2\pi m R$$

The integer m is called the **winding number**.



Equation of motion

The equation of motion for the compactified dimension can be obtained by extremising the Polyakov action. It is

$$\begin{aligned} X^{25}(\tau,\sigma) =& q^{25} + \alpha' p^{25}\tau + mR\sigma \\ & + i\sqrt{\frac{\alpha'}{2}} \sum_{k\neq 0} \left(\alpha_k^{25} e^{-ik(\tau+\sigma)} + \tilde{\alpha}_k^{25} e^{-ik(\tau-\sigma)} \right) \end{aligned}$$

where $p^{25} = \frac{n}{R}$.

Duality

Quantising this, we find that the spectrum of the closed string is invariant under the following transformation:

$$R \longleftrightarrow \alpha'/R$$

$$n \longleftrightarrow m$$

This invariance is known as T-duality.

A few things...

- String theory compactified on small and large circles are equivalent!
- This is a symmetry of the entire interacting theory (not just the spectrum)
- This duality is not present with point particles!

The B-field

The natural generalisation of this example considers strings coupled to the so-called B-field, in a curved spacetime background.

The B-field is the string equivalent to the electromagnetic potential. It is a locally defined 2-form gauge field.

Buscher Rules

Given the non-linear sigma model action

$$S[X] = \int d\sigma d\tau \Big(\sqrt{h} h^{\alpha\beta} g_{MN}(X) \partial_\alpha X^M \partial_\beta X^N \\ + \varepsilon^{\alpha\beta} B_{MN}(X) \partial_\alpha X^M \partial_\beta X^N \Big)$$

and a $U(1)$ isometry, the Buscher rules give a transformation of the g_{MN} and B_{MN} fields which leave this action invariant.

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Buscher Rules cont.

Explicitly, we have

$$\hat{g}_{\bullet\bullet} = \frac{1}{g_{\bullet\bullet}}$$

$$\hat{g}_{\mu\bullet} = \frac{B_{\mu\bullet}}{g_{\bullet\bullet}}$$

$$\hat{g}_{\mu\nu} = g_{\mu\nu} - \frac{1}{g_{\bullet\bullet}}\left(g_{\mu\bullet}g_{\nu\bullet} - B_{\mu\bullet}B_{\nu\bullet}\right)$$

$$\hat{B}_{\mu\bullet} = \frac{g_{\mu\bullet}}{g_{\bullet\bullet}}$$

$$\hat{B}_{\mu\nu} = B_{\mu\nu} - \frac{1}{g_{\bullet\bullet}}\left(g_{\mu\bullet}B_{\nu\bullet} - g_{\nu\bullet}B_{\mu\bullet}\right)$$

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Local vs. Global

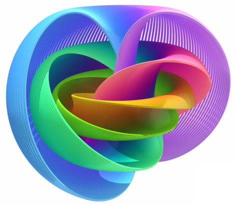
The Buscher rules give us a description of T-duality locally, that is, in coordinate patches.

What can we say about T-duality globally?

The global picture

We describe spacetime as a principal S^1-bundle

$$\begin{array}{ccc} S^1 & \longrightarrow & E \\ & & \downarrow \pi \\ & & M \end{array}$$



where E is the total spacetime, and M is the uncompactified part of spacetime.

Image credit: Niles Johnson

F and H

Circle bundles are classified by the first Chern class $F \in H^2(M, \mathbb{Z})$ of the associated line bundle $L_E = E \times_{S^1} \mathbb{C}$.

Additionally, we include a H-flux, which is a class $H \in H^3(E, \mathbb{Z})$.

We have

$$F \in H^2(M, \mathbb{Z})$$
$$H \in H^3(E, \mathbb{Z})$$

The Buscher rules suggest that T-duality corresponds to interchanging F and H in some sense.

Gysin Sequence

Theorem (Gysin)

Let $\pi : E \to M$ be a principal S^1 bundle. Then we have the following long exact sequence

$$\cdots \longrightarrow H^k(M) \xrightarrow{\pi^*} H^k(E) \xrightarrow{\pi_*} H^{k-1}(M) \xrightarrow{\wedge F} H^{k+1}(M) \longrightarrow \cdots$$

Defining the dual bundle

For $k = 3$ we have

$$\cdots \longrightarrow H^3(M) \xrightarrow{\pi^*} H^3(E) \xrightarrow{\pi_*} H^2(M) \xrightarrow{\wedge F} H^4(M) \longrightarrow \cdots$$

H

Defining the dual bundle

For $k = 3$ we have

$$\cdots \longrightarrow H^3(M) \xrightarrow{\pi^*} H^3(E) \xrightarrow{\pi_*} H^2(M) \xrightarrow{\wedge F} H^4(M) \longrightarrow \cdots$$

$$H \longmapsto \hat{F}$$

Defining the dual bundle

For $k = 3$ we have

$$\cdots \longrightarrow H^3(M) \xrightarrow{\pi^*} H^3(E) \xrightarrow{\pi_*} H^2(M) \xrightarrow{\wedge F} H^4(M) \longrightarrow \cdots$$

$$H \longmapsto \hat{F} \longmapsto 0$$

Defining the dual bundle

The pushforward of H defines an element $\hat{F} \in H^2(M, \mathbb{Z})$, and therefore a new circle bundle

$$\begin{array}{ccc} S^1 & \longrightarrow & \hat{E} \\ & & \big\downarrow \hat{\pi} \\ & & M \end{array}$$

What about the H-flux?

The dual H-flux

To get the dual H-flux, we note that $\hat{F} \wedge F = F \wedge \hat{F} = 0.$

$$\cdots \longrightarrow H^3(M) \xrightarrow{\hat{\pi}^*} H^3(\hat{E}) \xrightarrow{\hat{\pi}_*} H^2(M) \xrightarrow{\wedge \hat{F}} H^4(M) \longrightarrow \cdots$$

F

Exactness of the Gysin sequence for the dual bundle then says that there exists some $\hat{H} \in H^3(\hat{E}, \mathbb{Z})$ such that

$$F = \hat{\pi}_* \hat{H}$$

The dual H-flux

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$$\cdots \longrightarrow H^3(M) \xrightarrow{\hat{\pi}^*} H^3(\hat{E}) \xrightarrow{\hat{\pi}_*} H^2(M) \xrightarrow{\wedge \hat{F}} H^4(M) \longrightarrow \cdots$$

$$F \longmapsto 0$$

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$$\cdots \longrightarrow H^3(M) \xrightarrow{\hat{\pi}^*} H^3(\hat{E}) \xrightarrow{\hat{\pi}_*} H^2(M) \xrightarrow{\wedge \hat{F}} H^4(M) \longrightarrow \cdots$$

$$\hat{H} \longmapsto F \longmapsto 0$$

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Summary

Theorem (Bouwknegt, Evslin, Mathai)

Given a pair (F, H) consisting of a principal circle bundle $\pi : E \to M$ and a H-flux, the T-dual is a pair $(\hat{F}, \hat{H})$ consisting of a principal circle bundle $\hat{\pi} : \hat{E} \to M$ with H-flux.

$$\begin{array}{ccc} S^1 & \longrightarrow & E \\ & & \downarrow \pi \\ & & M \end{array} \qquad \begin{array}{ccc} S^1 & \longrightarrow & \hat{E} \\ & & \downarrow \hat{\pi} \\ & & M \end{array}$$

We have

$$\hat{F} = \pi_* H$$
$$F = \hat{\pi}_* \hat{H}$$

Beyond the basics

- Isomorphim of twisted K-theory
- T-duality for torus bundles
  - Classical duals
  - Non-commutative duals
  - Non-associative duals
- Spherical T-duality
- Generalised geometry
- Constructing SUGRA solutions

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