It's T-duality, but not as we know it

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Outline (Flavours of T-duality)

1. Abelian T-duality
2. Non-abelian T-duality
3. Non-isometric T-duality

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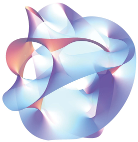
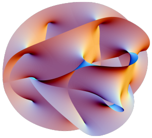
Context

I'm interested in *string backgrounds*:





They are used to describe strings moving in a fixed gravitational background.

Left image credit: Wikipedia en:User:Jbourjai

Right image credit: Wikipedia en:User:Lunch

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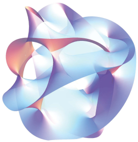

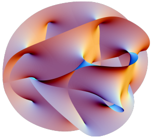

They are used to describe strings moving in a fixed gravitational background.

Note: gravitational = geometry

Left image credit: Wikipedia en:User:Jbourjai

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Context

String backgrounds are given by a (usually compact) manifold, together with:

- A Riemannian metric $g_{\mu\nu}$
- A two-form gauge field, $B_{\mu\nu}$
- Other fields (dilaton, RR-fields, superfields...) $\phi, C, \psi...$

satisfying the string analogue of the Einstein Field Equations from General Relativity

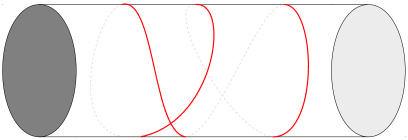
I study the geometry (and topology) of these string backgrounds.

A simple example

Strings moving on a cylinder provide a simple example of a string background. We can put a flat metric on the cylinder, and choose coordinates such that:

$$X^{25} \sim X^{25} + 2\pi m R$$

The integer m is called the **winding number**.



Duality

Using techniques from Calculus of Variations and Quantum Field Theory, one can study the properties of strings moving in such a background.

One important result is that the momentum, p^{25}, is quantised:

$$p^{25} = \frac{n}{R}$$

with $n \in \mathbb{Z}$.

Duality

A surprising feature of this system is that the spectrum of the closed string is invariant under the following transformation:

$$R \longleftrightarrow 1/R$$

$$n \longleftrightarrow m$$

This invariance is the most basic example of something known as (abelian) T-duality.

A few things...

- String theory compactified on small and large circles are equivalent!
- This is a symmetry of the entire interacting theory (not just the spectrum)
- This duality is not present with point particles!
- It's a duality

Abelian T-duality

Does this duality work for more complicated string backgrounds?

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Abelian T-duality

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If there is some notion of a 'circular' dimension, then the answer is yes!

More technically, if the metric $g_{\mu\nu}$ possesses a Killing vector, then there is a transformation of the fields which leaves the physics invariant*.

*We also want some restrictions on the B field

Buscher Rules

Explicitly, for coordinates $(x^\mu, x^\bullet)$, the transformation of the fields are given by the Buscher rules:

$$\hat{g}_{\bullet\bullet} = \frac{1}{g_{\bullet\bullet}}$$
$$\hat{g}_{\mu\bullet} = \frac{B_{\mu\bullet}}{g_{\bullet\bullet}}$$
$$\hat{g}_{\mu\nu} = g_{\mu\nu} - \frac{1}{g_{\bullet\bullet}}\left(g_{\mu\bullet}g_{\nu\bullet} - B_{\mu\bullet}B_{\nu\bullet}\right)$$
$$\hat{B}_{\mu\bullet} = \frac{g_{\mu\bullet}}{g_{\bullet\bullet}}$$
$$\hat{B}_{\mu\nu} = B_{\mu\nu} - \frac{1}{g_{\bullet\bullet}}\left(g_{\mu\bullet}B_{\nu\bullet} - g_{\nu\bullet}B_{\mu\bullet}\right)$$

A more interesting example

Consider the Lens space $L(1, k)$, with the following metric and B-field:

$$ds^2 = d\eta^2 + d\xi_1^2 + d\xi_2^2 - k \cdot 2\cos(2\eta)$$

$$B = j\cos(2\eta) d\xi_1 \wedge d\xi_2$$

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where $k, j \in \mathbb{Z}$. The T-dual is the manifold $L(1, j)$, with the following metric and B-field:

$$ds^2 = d\eta^2 + d\xi_1^2 + d\hat{\xi}_2^2 - j \cdot 2\cos(2\eta)$$

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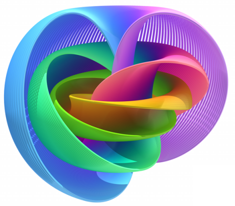
In particular...

A particularly simple case of this is $L(1,1) = S^3$ with $B = 0$.



Left image credit: Niles Johnson

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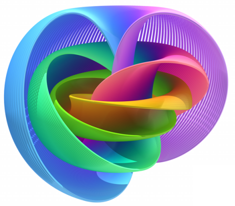

The T-dual is then $L(1,0) = S^2 \times S^1$ with one-unit of flux.

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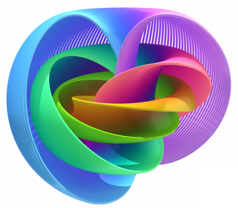

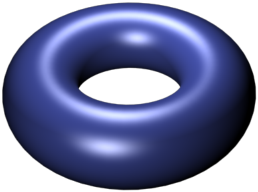

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Global statements

Remark (Multiple T-dualities)

Multiple T-dualities can be done simultaneously, if there are multiple *commuting* Killing vectors.

There are interesting global statements that can be made about (abelian) T-duality:

- Topology of the dual manifold
- Isomorphism of (twisted) K-theories
- Non-commutative string backgrounds
- Non-associative string backgrounds?

Outline (Flavours of T-duality)

Non-abelian T-duality

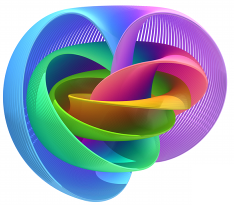
A natural generalisation of abelian T-duality considers Killing vectors which don't commute. These are the generators of a non-abelian group of isometries.

One can follow the same procedure for these non-commuting Killing vectors, and derive a set of transformations for the fields in the theory.

These define the non-abelian T-dual.

An example: S^3 again

Consider again S^3 with the round metric and $B = 0$.



This metric has an $SO(4)$ group of isometries, and we can find the non-abelian T-dual with respect to an $SU(2)$ subgroup of this.

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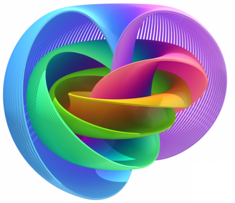

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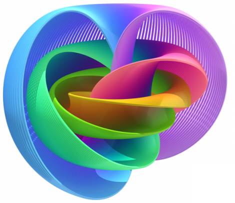

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Comments

Non-abelian T-duality has some big differences to abelian T-duality:

- The dual space typically has fewer isometries than the original space
  - Not a duality!
- May not be an equivalence of string theories
- A solution generating technique
- Topology of dual space not obvious

Outline (Flavours of T-duality)

Non-isometric T-duality

A very recent generalisation of non-abelian T-duality considers a set of vector fields which may not even be Killing vectors.

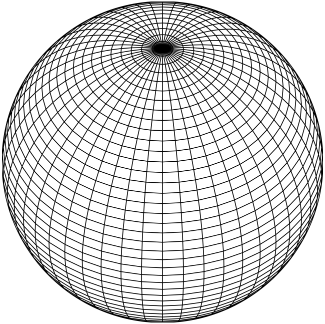
If the vector fields satisfy a (milder) set of constraints, one can again derive a set of transformation rules for the string background.

So far, no *new* examples have been found which realise this non-isometric duality.

A proposed example, and some weirdness

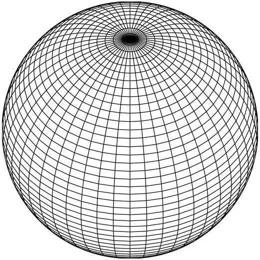
Consider S^2 with $B = 0$. We can choose local coorindates (u, v) for $S^2 \setminus \{equator\}$, and in these coordinates the round metric can be written as:

$$ds^2 = \frac{1 - v^2}{1 - u^2 - v^2} du^2 + \frac{2uv}{1 - u^2 - v^2} du\, dv + \frac{1 - u^2}{1 - u^2 - v^2} dv^2$$



A proposed example, and some weirdness

We can try to do non-isometric T-duality on this space, using the vector fields $\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\}$. We expect to get a dual metric and B field, depending on new coordinates (x, y).



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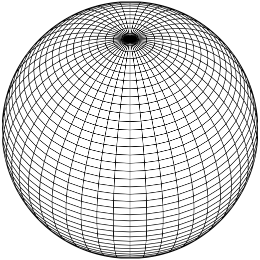

The metric we obtain depends on the dual coordinates (x, y), *but also on the original coordinates!*

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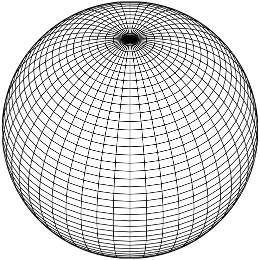

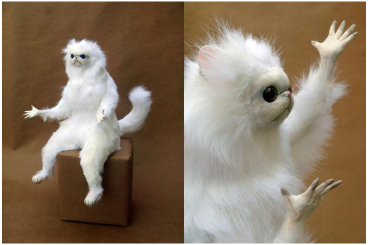

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Current and future work

- Understand the proposed example, and come up with new examples
- During the construction, multiple choices are made. How do these effect the dual space?
- What global/topological statements can we make about...
  - Non-abelian T-duality
  - Non-isometric T-duality
- Physics!
  - An equivalence of string theories, or just a technique to generate solutions?
  - How do the other fields, ϕ, C, ψ, ..., participate?
  - ...

Selected References

Abelian T-duality:

- T. Buscher, *A symmetry of the string background field equations*, Phys. Lett. **B194** (1987) 59
- P. Bouwknegt, J. Evslin, V. Mathai, *T-duality: Topology change from H-flux*, Commun. Math. Phys. **249** (2004) 383-415
- P. Bouwknegt, J. Evslin, V. Mathai, *On the topology and H-flux of T-dual manifolds*, Phys. Rev. Lett. **92** (2004) 181601

Non-isometric T-duality:

- A. Chatzistavrakidis, A. Deser, L. Jonke, *T-duality without isometry via extended gauge symmetries of 2D sigma models*, J. High Energy Phys. **Online** (2016)
- A. Chatzistavrakidis, *Non-isometric T-duality from gauged sigma models*, Proc. of Science, **Online**, (2016)

Image credits:

- Jbourjai, Image of Calabi-Yau manifold, http://en.wikipedia.org/wiki/File:Calabi_yau.jpg
- Lunch, Image of Calabi-Yau manifold, http://en.wikipedia.org/wiki/File:Calabi-Yau.png
- N. Johnson, Image of Hopf fibration, http://www.nilesjohnson.net/hopf.html