T-duality for Higher Rank Torus Bundles, Revisited

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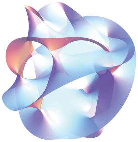
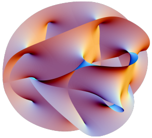
What is T-duality?

T-duality is an equivalence between two physical theories with different spacetime geometries.





These different geometries arise in string theory from the compactification of extra dimensions.

Left image credit: Wikipedia en:User:Jbourjai

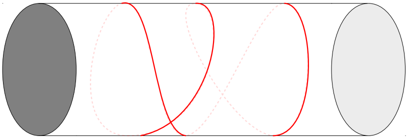
Right image credit: Wikipedia en:User:Lunch

A toy model

The simplest example of T-duality considers a closed Bosonic string on a manifold where one dimension is curled up into a circle.

$$X^{25} \sim X^{25} + 2\pi mR$$

The integer m is called the **winding number**.



Equation of motion

The equation of motion for the compactified dimension can be obtained by extremising the Polyakov action. It is

$$X^{25}(\tau,\sigma) = q^{25} + \alpha' p^{25}\tau + mR\sigma \\ + i\sqrt{\frac{\alpha'}{2}}\sum_{k\neq 0}\left(\alpha_k^{25} e^{-ik(\tau+\sigma)} + \tilde{\alpha}_k^{25} e^{-ik(\tau-\sigma)}\right)$$

where $p^{25} = \frac{n}{R}$.

Duality

Quantising this, we find that the spectrum of the closed string is invariant under the following transformation:

$$R \longleftrightarrow \alpha'/R$$

$$n \longleftrightarrow m$$

This invariance is known as T-duality.

A few things...

- String theory compactified on small and large circles are equivalent!
- This duality is not present with point particles!

Buscher Rules

Given the non-linear sigma model action

$$S[X] = \int d\sigma d\tau \Big(\sqrt{h} h^{\alpha\beta} g_{MN}(X) \partial_\alpha X^M \partial_\beta X^N + \varepsilon^{\alpha\beta} B_{MN}(X) \partial_\alpha X^M \partial_\beta X^N \Big)$$

and a $U(1)$ isometry, the Buscher rules give a transformation of the g_{MN} and B_{MN} fields which leave this action invariant.

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Buscher Rules cont.

Explicitly, we have

$$\hat{g}_{\bullet\bullet} = \frac{1}{g_{\bullet\bullet}}$$

$$\hat{g}_{\mu\bullet} = \frac{B_{\mu\bullet}}{g_{\bullet\bullet}}$$

$$\hat{g}_{\mu\nu} = g_{\mu\nu} - \frac{1}{g_{\bullet\bullet}}\left(g_{\mu\bullet}g_{\nu\bullet} - B_{\mu\bullet}B_{\nu\bullet}\right)$$

$$\hat{B}_{\mu\bullet} = \frac{g_{\mu\bullet}}{g_{\bullet\bullet}}$$

$$\hat{B}_{\mu\nu} = B_{\mu\nu} - \frac{1}{g_{\bullet\bullet}}\left(g_{\mu\bullet}B_{\nu\bullet} - g_{\nu\bullet}B_{\mu\bullet}\right)$$

Local vs. Global

The Buscher rules give us a description of T-duality locally, that is, in coordinate patches.

What can we say about T-duality globally?

The global picture

We describe spacetime as a principal S^1-bundle

$$\begin{array}{ccc} S^1 & \longrightarrow & E \\ & & \downarrow \pi \\ & & M \end{array}$$

where E is the total spacetime, and M is the uncompactified part of spacetime.

F and H

Circle bundles are classified by the first Chern class $F \in H^2(M, \mathbb{Z})$ of the associated line bundle $L_E = E \times_{S^1} \mathbb{C}$.

Additionally, we include a H-flux, which is a class $H \in H^3(E, \mathbb{Z})$.

We have

$$F \in H^2(M, \mathbb{Z})$$
$$H \in H^3(E, \mathbb{Z})$$

The Buscher rules suggest that T-duality corresponds to interchanging F and H in some sense.

Summary

Theorem (Bouwknegt, Evslin, Mathai)

Given a pair (F, H) *consisting of a principal circle bundle* $\pi : E \to M$ *and a H-flux, there exists a pair* $(\hat{F}, \hat{H})$ *consisting of a principal circle bundle* $\hat{\pi} : \hat{E} \to M$ *with H-flux*

$$\begin{array}{ccc} S^1 & \longrightarrow & E \\ & & \downarrow \pi \\ & & M \end{array} \qquad\qquad \begin{array}{ccc} S^1 & \longrightarrow & \hat{E} \\ & & \downarrow \hat{\pi} \\ & & M \end{array}$$

such that

$$\hat{F} = \pi_* H$$
$$F = \hat{\pi}_* \hat{H}$$

Examples

A simple example is a trivial bundle over the two-sphere

$$\pi : S^1 \times S^2 \to S^2$$

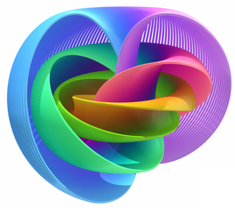
equipped with the (normalised) volume form as a H-flux.

$$F = 0$$

$$H = 1$$

Examples

The T-dual is the Hopf fibration



$$\hat{\pi} : S^3 \to S^2$$

with trivial H-flux.

$$\hat{F} = 1$$

$$\hat{H} = 0$$

Image credit: Niles Johnson

Examples

More generally, the Lens space $S^3/\mathbb{Z}_k$ with j units of H-flux is dual to the Lens space $S^3/\mathbb{Z}_j$ with k units of H-flux.

$$F = k \qquad \hat{F} = j$$

$$H = j \qquad \hat{H} = k$$

Higher Rank Torus Bundles

The Buscher rules can be extended to N circle isometries.

When can we make sense of higher rank T-duality in a global sense?

Examples

Consider the three torus $\mathbb{T}^3 = S^1 \times S^1 \times S^1$ equipped with k units of H-flux.

As a trivial circle bundle $\pi : \mathbb{T}^2 \times S^1 \to \mathbb{T}^2$, the T-dual E is the so-called "Nilmanifold", or "Twisted Torus" with no flux.

T-dualising again on one of the two base circles no longer makes sense, because these circles are no longer globally defined on E.

In other words, for a trivial torus bundle $\pi : \mathbb{T}^3 = \mathbb{T}^2 \times S^1 \to S^1$ with k units of H-flux, there is no T-dual which is a principal torus bundle.

Classical Flux

By comparison, the trivial circle bundle $\pi : \mathbb{T}^2 \times S^1 \to \mathbb{T}^2$ with no flux is self-dual, so T-dualising a second time makes sense, and so $\mathbb{T}^2 \times S^1 \to S^1$ with no flux is a (self-dual) principal torus bundle.

In particular, the T-dual is a principal torus bundle.

Definition

For a principal torus bundle $\pi : E \to M$, a H-flux $H \in H^3(E, \mathbb{Z})$ is called **Classical** if the T-dual $\hat{\pi} : \hat{E} \to M$ is a principal torus bundle.

Theorem (Mathai, Rosenberg)

Let (E, H) be a principal $\mathbb{T}^n$ bundle over M with H-flux. Then H is classical iff

$$\pi_* H = 0 \text{ in } H^1(M, H^2(\mathbb{T}^n))$$

Example

Consider the trivial 2-torus bundle $\pi : \mathbb{T}^3 \to S^1$ with $H = k\ dx \wedge dy \wedge dz$. Then

$$\int_{\mathbb{T}^2} H = \int_{y,z} k\ dx \wedge dy \wedge dz = k\ dx \neq 0$$

So the torus bundle $\mathbb{T}^3 \to S^1$ has a classical T-dual iff the H-flux is trivial, which we saw earlier.

Current Problematic Example

Consider the torus bundle

$$\pi : SU(3) \to SU(3)/\mathbb{T}^2$$

The third cohomology of $SU(3)$ is isomorphic to $\mathbb{Z}$, so fluxes, and in particular, classical fluxes, are classified by integers.

On the other hand, dimensional reduction arguments from [ref] suggest that classical fluxes are classified by pairs of integers.

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