Higher homotopies in commutative algebra
Project description
Jesse Burke

1 Introduction

Differential homological algebra, a branch of math developed by topologists, has been applied to great effect in commutative algebra. This project proposes to continue this transfer of techniques from topology to algebra by applying generalized twisting cochains, which codify higher homotopies or A_∞-structures, to study modules over commutative rings.

Free resolutions with a differential graded (dg) algebra or module structures are an important tool in commutative algebra. They were introduced by Tate in [44], they are at the center of work Buchsbaum and Eisenbud [15] on the structure of codimension 3 Gorenstein ideals, and form one of the main themes of the book by Avramov on infinite free resolutions [4]. The existence of an algebra or module structure rigidifies a free resolution while opening it up to new tools of study. A central example of the power of these structures comes from the Buchsbaum-Eisenbud-Horrocks rank conjecture, which would follow if for a given a finite free resolution, one could find an appropriate Koszul complex over which it is a dg-module.

While not every resolution of an algebra is a dg-algebra, it is an algebra up to homotopy, or an A_∞-algebra. This fact has not received much attention in commutative algebra. The main goal of this project is to exploit this structure to study free resolutions and homological algebra over commutative rings. Until recently, much of the machinery of generalized twisting cochains was only defined over augmented objects, and the objects one studies in commutative algebra are usually not augmented. For example, any resolution of a quotient ring over the base ring will never be augmented. Positselski has shown how to add a "curvature element" to overcome a lack of augmentation.The first objective of this project is to generalize his machinery, especially the curved Koszul-Moore duality arising from a twisting cochain, from a field to the case of an arbitrary commutative ring. This will provide the foundation for the rest of the work in the project.

A large part of this project is concerned with complete intersection (CI) rings. Over such a ring, the Koszul-Moore duality may be interpreted as a generalization of the classical BGG-correspondence [12]. The exterior algebra is replaced by the Koszul complex and the symmetric algebra is generalized to include a curvature element (this curvature is behind the phenomenon of matrix factorizations). This gives both theoretical insight into, and computational tools for, the study of free resolutions over CI rings. If $R = Q/I$ is such a ring, and M is an R-module, the higher homotopies on a Q-free resolution of M are well known and have been studied by several authors. However, there are higher homotopies on the R-free resolution of M, and from these one can recover the Q-free resolution. This has not been studied previously. The duality also gives a new approach to study the lattice of thick subcategories and the dimension of the derived category of R. Baranovsky [8, 9] has studied a similar setup for graded complete intersections defined over a field. But he considers A_∞-structures over the field, and he doesn't include applications for his results. In particular, his results do not imply those contained here and the machinery he uses does not apply to our setup.

Another focus of the project is on Golod rings. These are in a sense orthogonal to CI rings: a ring is both CI and Golod if and only if it is a hypersurface, and while complete intersections are characterized by polynomial growth of free resolutions, Golod rings are characterized by a strong exponential growth of free resolutions. For Golod rings, the Koszul-Moore duality gives a way to construct the minimal free resolution of any module in a fixed finite number of steps. Studying these minimal R-free resolutions, how they relate to Q-free resolutions (if $R = Q/I$ is a Golod ring with Q regular), what form the A_∞-structures take, and what the global structure of the derived category of R is, are all objectives of this project related to Golod rings.

In [4] Avramov lists eleven questions on infinite free resolutions over local rings. These are all known for CI rings and many are known for Golod rings. For rings outside of these classes, all of the questions are open. Finding "small" acyclic twisting cochains for classes of rings that are neither CI nor Golod will shed light on free resolutions over these rings and give an approach to some of Avramov's questions. This will be the third major focus of the project. The first class to consider will be codimension 3 Gorenstein rings (every codimension 2 ring is CI or Golod). If $R = Q/I$ is such a ring, the Q-free resolution of R is known by Buchsbaum and Eisenbud [15]. The defining feature of a Gorenstein ring is a symmetry in this Q-free resolution. Finding a way to account for this symmetry in the context of higher homotopies will be key.

The last part of this project is a proposal to generalize Boij-Soederberg theory from polynomial rings to certain graded hypersurface rings. While not directly related to higher homotopies, it is similar in spirit to the other objectives considered here.

Finally, let us mention that all of the machinery and much of the motivation of this project extends to non-commutative rings. I don't plan to pursue this in the immediate future, but a long term goal will be to examine the application of this to non-commutative rings; a starting point will be restricted Lie algebras, whose homological algebra has much in common with CI rings.

2 Twisting cochains and A_∞-structures

2.1 Twisting cochains

Let A be a differential graded (dg) algebra and C a dg coalgebra, both defined over a commutative ring k. A *twisting cochain* between C and A is a degree -1 k-linear map that allows one to define an adjoint pair of functors between the homotopy category of dg C-comodules and dg A-modules.

$$\underline{\mathrm{comod}}(C) \underset{C\otimes^\tau -}{\overset{A\otimes^\tau -}{\rightleftarrows}} \underline{\mathrm{mod}}(A) \tag{2.1}$$

See e.g. [36] for the full definition. The notation (and terminology) is due to the fact that τ "twists" the tensor product differential on $C \otimes M$ to make it a dg C-comodule, and similarly for $A \otimes^\tau N$.

These were introduced by Brown in [14] in the context that $C = C_*(X, k)$ is the chains of a toplogical space X and $A = C^*(\Omega X, k)$ is algebra of cochains of the Moore path space of X (the Moore path space is defined so that this is a dg-algebra under composition). Brown used the machinery to compute the homology of the total space of a fibration of simply connected CW-complexes.

Twisting cochains always exist: given a dg-algebra A, one can form a dg-coalgebra $\mathrm{Bar}(A)$, called the bar complex, and a twisting cochain $\tau_A : \mathrm{Bar}(A) \to A$. Similarly, if C is a dg-coalgebra, one can form the cobar complex and a twisting cochain $\tau_C : C \to \mathrm{Cob}(C)$. Any twisting cochain $C \to A$ induces a map of dg-coalgebras $C \to \mathrm{Bar}(A)$ and a map of dg-algebras $\mathrm{Cob}(C) \to A$. For details, see again [36].

It is important and interesting to know when the functors in (2.1) are inverse equivalences up to "weak equivalence." Lefévre-Hasegawa studied this in [33] where he defined the coderived category $\mathsf{D}^{\mathsf{co}}(C)$ of dg C-comodules and he showed that (2.1) descends to an adjoint pair between (co)derived categories.

$$\mathsf{D}^{\mathsf{co}}(C) \underset{C\otimes^{\tau}-}{\overset{A\otimes^{\tau}-}{\rightleftarrows}} \mathsf{D}(A) \tag{2.2}$$

Definition 2.3. A twisting cochain is *acyclic* if the functors above are an equivalence.

In [33], six equivalent conditions are given for τ to be acyclic. They include:

2.4. *Acyclicity conditions.*

(1) the co-unit $A \otimes^{\tau} C \otimes^{\tau} M \to M$ of (2.2) is an isomorphism for all M in $\mathsf{D}(A)$;
(2) the co-unit $A \otimes^{\tau} C \otimes^{\tau} A \to A$ of (2.2) is an isomorphism;
(3) the map $C \to \mathrm{Bar}(A)$ is a weak-equivalence.

Much of this goes back to the "constructions" of Cartan and Moore [1]. In [32] Keller shows how this machinery generalizes and removes boundedness conditions from classical Koszul duality, as in e.g. [10]. For these reasons (perhaps) this setup has been called "Koszul-Moore" duality.

An acyclic twisting cochain can be a powerful way to study a dg-algebra. For instance it gives a standard resolution of every A-module via 2.4.(1), as well as a possibly smaller model of the bar complex via 2.4.(3). This can make working with higher homotopies, or A_∞-structures, much easier (I'll say more about this in the next subsection). However, for subtle reasons that are still slightly opaque to me, the machinery as presented above only works when A and C are (co)augmented. (Here, A is augmented if there is an algebra map $A \to k$ that splits the unit; coaugmented is defined dually.) However, the dg-algebras that I plan to study are not augmented. The über-example is the Koszul complex A on an element $f \in k$. Then

$$A = 0 \to k \xrightarrow{f} k \to 0,$$

where the nonzero modules are in degrees 1 and 0. There is never a map of chain complexes $A \to k$, unless $f = 0$. Thus A is only augmented when $f = 0$.

Positselski presents a program for working without an augmentation in [39]. He showed that one should replace dg-coalgebras with *curved* dg-coalgebras. He then defined a twisting cochain $\tau : C \to A$ to a non-augmented dg-algebra A, and defined curved C-comodules and showed that they form a replacement for dg-comodules in (2.1). Let us emphasize that curved coalgebras and curved modules are objects whose differential *does* *not square to zero.* In particular one cannot take homology and thus there are no quasi-isomorphisms to invert. Positselski found a way around this and defined the *coderived category of curved comodules.* He used this to give an analogue of the functors (2.2).

The material on coalgebras in [39] assumes that k is a field. Additionally, there are no analogues of the equivalent conditions for a twisting cochain to be acyclic as in 2.4. For the applications I have in mind, it is important that this machinery applies to the case when k is an arbitrary Noetherian commutative ring and to have an analogue of 2.4 in the curved setting. Thus the first objective of this project is:

Objective 1. *Generalize the machinery of coderived categories of curved comodules from a field to a commutative ring; generalize the functors of 2.2 to this context and investigate when they are equivalences, especially investigate which of the conditions of 2.4 still hold in this setting.*

This should be fairly straight-forward, however coalgebras and comodules over arbitrary rings can exhibit some pathological behavior.

2.2 A_∞-structures and twisting cochains

Twisting cochains are intimately connected to A_∞-structures. Recall that if A is a complex over k, an *A_∞-algebra* structure on A means that A is an algebra "up-to-homotopy", and this is encoded in maps $m_n : A^{\otimes n} \to A$ for $n \geq 1$. Positselski has shown in [39, §7], that putting a *strictly unital* A_∞-structure on A is equivalent to finding a coderivation of the coalgebra $\mathrm{Bar}(A) = \bigoplus_{i\geq 0}(A/k\cdot 1_A)^{\otimes i}$ that makes it a curved dg-coalgebra. He then defines a generalized twisting cochain

$$\tau : C \to A$$

for C a curved dg coalgebra and A a strictly unital A_∞-algebra.

Objective 2. *Generalize the functors* (2.2) *and the conditions* (2.4) *to the setting of strictly unital A_∞-algebras.*

If A is an A_∞-algebra, a complex M is an A_∞ A-module if there is a coderivation on $\mathrm{Bar}(A)\otimes M$ that makes it a curved $\mathrm{Bar}(A)$-comodule. The following, in the non-curved context, is a folklore result I learned from [30], which credits the idea to Stasheff and Halperin [42].

Proposition 2.5. *Let A be an A_∞algebra, C a curved dg coalgebra and $\tau : C \to A$ an acyclic twisting cochain. Let M be a complex of k-modules that is a free k-module in every degree. Then the following are equivalent:*

(1) an A_∞ A-module structure on M;
(2) a coderivation on $C \otimes M$ that makes this a curved dg C-comodule.

For instance if $C = \mathrm{Bar}(A)$, the result above holds by definition. However, if C is smaller than $\mathrm{Bar}(A)$, this can make working with A_∞-structures considerably easier.

The dual of the statement should also hold: if C is a curved A_∞ *coalgebra* and N is a complex of free k-modules, then an A_∞ C-comodule structure on N is equivalent to a differential on $A \otimes N$ that makes it a dg A-module.

Objective 3. *Write a detailed proof of the above statements in the generality of an arbitrary commutative ring k and an acyclic twisting cochain $C \to A$, with C a curved dg-coalgebra over k and A a strictly unital A_∞-algebra over k.*

2.3 Applications to commutative rings

Let Q be a commutative ring and let $R = Q/I$ for an ideal I in Q. We let $k = Q$ in the above notation. I've recently proven the following result. This is the first step in applying generalized twisting cochains to homological algebra over commutative rings.

Proposition 2.6. *Let $R = Q/I$ be a commutative ring, M an R-module, and let $A \xrightarrow{\cong} R$ and $G \xrightarrow{\cong} M$ be Q-free resolutions. Then A has the structure of an A_∞-algebra over Q and G an A_∞-module over A.*

This generalizes a result of Buchsbaum and Eisenbud [15] who noted that A is an algebra up to homotopy, but they didn't pursue the higher homotopies. The proof of the proposition uses the obstruction theory of [33, Appendix 1].

Let us fix A and G as above, and let $\tau : C \to A$ an acyclic twisting cochain (we know that $C = \mathrm{Bar}(A) \to A$ is at least one such). This gives the following tools and approaches to study the homological algebra of R relative to Q.

2.7. **Tools from twisting cochains.**

(1) the equivalences of derived categories

$$\begin{tikzcd} \mathsf{D}^{\mathsf{co}}(C) \arrow[r, shift left, "A\otimes^\tau -"] \arrow[r, phantom, "\cong"] & \mathsf{D}(A) \arrow[l, shift left, "C\otimes^\tau -"] \arrow[r, shift left] \arrow[r, phantom, "\cong"] & \mathsf{D}(R) \arrow[l, shift left] \end{tikzcd}$$

from (2.2) and the quasi-isomorphism $A \xrightarrow{\cong} R$;

(2) the resolution of A-modules via 2.4.(1);

(3) a convenient encoding of the higher homotopies on the A_∞ A-module G in the C-comodule $C \otimes G$, via Proposition 2.5;

(4) intuition from the classical BGG correspondence.

The following is the over-arching objective of this project:

Objective 4. *Use the tools of 2.7 to study the homological algebra of $R = Q/I$ relative to Q, where Q is a Noetherian commutative ring.*

To show that this will not be entirely fruitless, let me state a recent theorem I've proven which generalizes a result of Iyengar [31]. Let M be an R-module. The theorem takes *any* Q-free resolution of R and *any* Q-free resolution of M and produces an R-free resolution of M. The resolution depends on a choice of acyclic twisting cochain.

Theorem 2.8. *Let $R = Q/I$ be a commutative ring, M an R-module, and let $A \xrightarrow{\cong} R$ and $G \xrightarrow{\cong} M$ be Q-free resolutions, with A_∞-structures. Let $\tau : C \to A$ be an acyclic twisting cochain. Then, there is a natural map*

$$R \otimes C \otimes G \to M$$

which is an R-free resolution.

Iyengar assumed that C was the bar complex of A, and required that both A and G have dg-structures (as opposed to A_∞). In the case R is a Golod ring, the theorem gives a free resolution that hits the defining bound of Poincare series, which was not previously known to exist (see Section 4 for definitions). In the case of a complete intersection ring, where A is the Koszul complex and C is a symmetric coalgebra, this recovers the *standard resolution* of [19]. The proof of the theorem follows almost immediately from 2.4.(1).

3 Complete intersection rings

In this section we concentrate on the special case that $R = Q/I$, where I is generated by a Q-regular sequence $\mathbf{f} = (f_1, \ldots, f_c)$. Let A be the Koszul complex on $\mathbf{f}$. Since $\mathbf{f}$ is Q-regular, the natural map $A \to H_0(A) = R$ is a quasi-isomorphism.

Let U be a rank c free Q-module in homological degree 2, with basis $X_1, \ldots, X_c$. Let C be the curved dg-coalgebra with underlying coalgebra the symmetric coalgebra on U, zero differential, and curvature projection onto U followed by the map $f_1X_1^* + \ldots + f_cX_c^* : U \to Q$. We have the following:

Proposition 3.1. *The degree -1 map*

$$\tau : C \to A \qquad X_i \mapsto \xi_i$$

is an acyclic twisting cochain, where $\xi_1, \ldots, \xi_c$ is a basis of A_1 with $d(\xi_i) = f_i$.

The coalgebra C is much smaller than the bar construction of A and this is in some way behind most of the work in this section.

3.1 Generalized BGG duality

Applying (2.2) to the acyclic twisting cochain of 3.1 we have an equivalence

$$\mathsf{D}^{\mathsf{co}}(C) \underset{C\otimes^\tau -}{\overset{A\otimes^\tau -}{\rightleftarrows}} \mathsf{D}(A)\ . \tag{3.2}$$

The Bernstein-Gelfand-Gelfand (BGG) correspondence [12] for dg-modules over the symmetric and exterior algebras, as formulated in e.g. [24, 5] was shown in [23] to come from an acyclic twisting cochain between the symmetric coalgebra and exterior algebra. Thus 3.1 is a direct generalization: by adding a differential to the right hand side, we compensate by adding a curvature term to the left hand side. The BGG correspondence is a fundamental tool to study modules over the symmetric algebra and I think it has the potential to be very useful in the study of complete intersections.

The duality of (3.2) gives a duality between the Q-free resolution and R-free resolution of an R-module M in the following way. Let $G \to M$ be a Q-free resolution and $F \to M$ be an R-free resolution. Then G has the structure of an A_∞ A-module by 2.6, and this makes $C \otimes G$ into a curved dg C-module by 2.5. This structure is exactly the higher homotopies introduced in [19].

It was also shown in [19] that there is an "up-to-homotopy" $R[X_1, \ldots, X_c]$-comodule structure on F. However, the homotopies have not been pursued. I think they will be an integral part of the duality. Let us write $F = \widetilde{F} \otimes_Q R$, where $\widetilde{F}$ is a graded free Q-module. By [26, Theorem 2.2], there is a differential on $A \otimes \widetilde{F}$ that makes it a dg A-module and this dg-module is quasi-isomorphic to M. By the dual of 2.5, this makes

$\widetilde{F}$ a curved A_∞ C-comodule (applying $-\otimes_Q R$ we get the classical "up-to-homotopy" $R[X_1,\dots,X_c]$-comodule structure on F).

We now have the following:

$$C\otimes G\in \mathsf{D}^{\mathsf{co}}(C) \underset{C\otimes^\tau -}{\overset{A\otimes^\tau -}{\rightleftarrows}} \mathsf{D}(A)\ni A\otimes \widetilde{F} \tag{3.3}$$

and since G, F are resolutions of M, it's not hard to see that there are isomorphisms in the derived categories

$$C\otimes^\tau (A\otimes \widetilde{F})\cong C\otimes G \qquad A\otimes \widetilde{F}\cong A\otimes^\tau (C\otimes G).$$

Hence the "duality" between F and G.

Objective 5. *Relate a free Q-resolution G with its higher homotopies to a free R-resolution F with its higher homotopies, and vice versa, especially in the case that R is local and G, F are minimal resolutions.*

This has been a dominant theme of study for free resolutions on complete intersections since the topic was first studied by Tate in [44]. However, there has been no previous indication that the higher homotopies on F and G are linked as above.

Often in a BGG situation like this, computing a quantity on one side of the equivalence becomes easier after transferring to the other side. Recent work of Eisenbud and Peeva [21] on minimal resolutions over local complete intersection rings has shown that the regularity of $\mathrm{Ext}^*_R(M,k)$ as a graded module over the ring of polynomial operators (the action of these can be seen in the current setup as the algebra $\mathrm{Hom}_Q(C,k)$ acting on $\mathrm{Hom}_Q(C\otimes G,k)$) is key to understanding the minimal R-free resolution. A part of the above objective is to compute the regularity of $\mathrm{Ext}^*_R(M,k)$ (and other invariants, such as the depth) directly from the Q-free resolution G and its higher homotopies.

Let me also note that in everything above, we have not had to pick generators for the ideal I. This may be helpful in approaching some aspects of Eisenbud and Peeva's work, where they have to pick generators for the ideal I depending on the module in question.

3.2 Classification of thick subcategories

It is an interesting question to try to classify the thick subcategories of the finite derived category of a ring. This was done for zero-dimensional CI rings in [18]. Using the duality of (3.2), we can apply the main steps of the argument of [18] to the larger class of rings considered in this section.

Applying $\mathrm{Hom}_Q(-,R)$ to 3.1, we have an acyclic twisting cochain

$$E^* := \wedge^* (A_1\otimes_Q R)\to \mathrm{Sym}(U\otimes_Q R) =: S \tag{3.4}$$

where E^* is the exterior coalgebra and S the symmetric algebra. Let E be the dg R-algebra $\mathrm{Hom}_R(E^*,R)$. Since E is a finitely generated free R-module, every E^* comodule is an E-module and vice versa [1]. Applying (2.2) to (3.4) and restricting to bounded

[1] Given M an E^*-comodule, the E-action on M is given by $E\otimes M \xrightarrow{1\cong\Delta_M} E\otimes E^*\otimes M\to M$.

derived categories gives

$$\begin{tikzcd} \mathsf{D}^{\mathsf{f}}(R) \arrow[r, shift left, "\beta"] & \mathsf{D}^{\mathsf{f}}(E) \arrow[l, shift left, "\alpha"] \arrow[r, shift left, "\phi"] \arrow[r, phantom, "\cong"] & \mathsf{D}^{\mathsf{f}}(S) \arrow[l, shift left, "\psi"] \end{tikzcd}$$

where ϕ and ψ are induced by the twisting cochain, α is restriction along the unit map $R \to E$, and $\beta = - \otimes_R E$.

In [18, 3.2], thick subcategories of perfect objects over any dg-algebra with zero differential are classified in terms of support. In particular the result applies to perfect objects in $\mathsf{D}^{\mathsf{f}}(S)$. Also, as pointed out by Iyengar, if M is in $\mathsf{D}^{\mathsf{f}}(R)$, then $\alpha(\beta(M)) = M \otimes_R E$ and M generate the same thick subcategory. Thus, thick subcategories of R-modules that are perfect over Q are classified by the support of their images in $\mathsf{D}^{\mathsf{f}}(S)$.

To finish the classification of thick subcategories that are perfect over Q, we have to settle the following:

Objective 6. *Determine which closed subsets of homogeneous primes of S arise as the support of the images of objects of $\mathsf{D}^{\mathsf{f}}(R)$.*

In [43], Stevenson classified the thick subcategories of $\mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(R) = \mathsf{D}^{\mathsf{f}}(R)/\operatorname{perf}(R)$ and showed that the classification depends on the singularities of $\operatorname{Spec} R$ in $\operatorname{Spec} Q$. If successful, the above objective would presumably recover his result, while extending it to the case that Q is not necessarily regular.

Another novelty of this approach is that the support of $\phi(\beta(M))$ can be computed using Fitting ideals of the higher homotopies on a Q-free resolution of M. We could thus determine, using e.g. Macaulay2 [25], when one module is in the thick subcategory of another.

3.3 Obstructions to dg-(co)module structures

Let Q be local, and M an R-module that has finite projective dimension over Q. Let G and $F = \widetilde{F} \otimes_Q R$ be minimal Q and R-free resolutions of M. We know G is an A_∞ A-module and $\widetilde{F}$ an A_∞ C-comodule. It is very interesting to know when they are actually dg (co)modules (that is, when the higher homotopies are zero).

Objective 7.

(1) find obstructions that characterize when G is a dg A-module;
(2) find obstructions that characterize when $\widetilde{F}$ is a strict C-comodule.

This has strong consequences and has been studied by several authors. Buchsbaum and Eisenbud showed that if G is a dg A-module, then the rank of G is at least the rank of A [15]. This led to their famous rank conjecture, and led them to ask whether G is always a dg-module. Avramov developed an obstruction theory in [6] and gave an example of a module with non-vanishing obstruction, showing that G need not be an A-module. However, he noted that his obstructions do not necessarily *characterize* when G is a dg-module, and that such a characterization would be a large step towards solving the Buchsbaum-Eisenbud rank conjecture (and would solve it if one could always find regular sequences for which the obstructions vanished).

In [19], Eisenbud asked whether $\widetilde{F}$ is always a dg-comodule. However, in [7], the authors developed a spectral sequence which degenerates on the E_2 page if $\widetilde{F}$ is a dg-comodule and then found an example where this is not the case. This example did not rule out the possibility that a high truncation of $\widetilde{F}$ is a dg-comodule. The construction of minimal free resolutions of high syzygies by Eisenbud and Peeva [21] would be much simplified if we knew that a high truncation of $\widetilde{F}$ were always a dg-comodule.

The machinery of this project gives several new approaches to this problem. First, it creates a new obstruction theory, although we cannot compute these obstructions yet. Second, it shows an unsuspected relation between the two obstruction theories described above.

The first approach is the most naive. As noted in 3.1, we know $A \otimes \widetilde{F}$ is a dg A-module which is quasi-isomorphic to M. In particular, $A \otimes \widetilde{F}$ is a Q-free resolution of M. Since G is minimal, there is a surjection of Q-complexes

$$\phi : A \otimes \widetilde{F} \to G.$$

If we can pick ϕ such that $\ker \phi$ is a dg A-*submodule* of $A \otimes \widetilde{F}$, then G must also be a dg A-module. Since G is bounded there are only finitely many degrees to check. We can dualize the idea: there is an injective map $\widetilde{F} \to C \otimes G$. If the cokernel is a curved dg C-comodule, then $\widetilde{F}$ is as well.

The obstruction of Avramov to G being a dg A-module [6] is

$$\frac{\ker\left(\operatorname{Tor}^Q_*(M,k) \to \operatorname{Tor}^R_*(M,k)\right)}{(\operatorname{Tor}^+_Q(R,k))(\operatorname{Tor}^*_Q(M,k))}$$

where k is the residue field (he also shows the numerator is exactly the submodule of decomposable matric Massey products of $\operatorname{Tor}^Q_*(M,k)$, which should be connected to A_∞-structures, but I don't yet see how). We can compute the map $\operatorname{Tor}^Q_*(M,k) \to \operatorname{Tor}^R_*(M,k)$ in two different ways. We know G is a Q-free resolution of M and by the twisting cochain machinery $C \otimes G \otimes R$ is an R-resolution of M. Therefore, we have

$$\operatorname{Tor}^Q_*(M,k) = G \otimes k \to H_*(C \otimes G \otimes k) = \operatorname{Tor}^R_*(M,k).$$

But also, we know that $A \otimes \widetilde{F}$ is quasi-isomorphic to M and a complex of free Q-modules; thus we have

$$\operatorname{Tor}^Q_*(M,k) = H_*(A \otimes \widetilde{F} \otimes k) \to F \otimes_R k = \operatorname{Tor}^R_*(M,k).$$

It is very intriguing that the higher homotopies on $\widetilde{F}$ and G are connected in this way. Moreover, these maps are very explicit and perhaps computable.

In [7, §4], the following spectral sequence is constructed:

$${}^2E^q_p = H_p\left(K(\operatorname{Ext}^*_R(M,k))\right)^q \Rightarrow \operatorname{Ext}^{q-p}_Q(M,k)$$

where $K(\operatorname{Ext}^*_R(M,k))$ is the Koszul complex on $\operatorname{Ext}^*_R(M,k)$ over the polynomial ring of cohomology operators. Using that $C \otimes G$ is a dg C-comodule, I can show that this is an Eilenberg-Moore spectral sequence, and thus the differentials may be computed in terms of matric Massey products using the theory of [26]. This is worth investigation and may give some new insight on the spectral sequence and the obstruction theory it represents.

3.4 Morphisms between higher homotopies

A corollary of the setup of twisting cochains is that we get a notion of morphism between higher homotopies. Let G, H be Q-free resolutions of R-modules M, N. We know that $C \otimes G$ and $C \otimes H$ are curved dg C-comodules, and we may consider

$$\operatorname{Hom}^{\infty}_{A}(G, H) := \operatorname{Hom}_C(C \otimes G, C \otimes H),$$

the A_∞-maps between G, H. We can then mimic Buchsbaum and Eisenbud's original approach to the rank conjecture. Let us assume for simplicity that $M = R/I$ is cyclic, but everything extends to the general case.

In [15], the authors showed that if $\phi : A \to G$ is a map of complexes lifting the map $R \to R/I$ of Q-modules, then ϕ_c is injective (where $c = \dim Q - \dim R$). If G is a dg A-module, then there is a unique map of dg A-modules $A \to G$ lifting $R \to R/I$. If the A-linear map $A \to G$ has a nonzero kernel, it must contain A_n, as this is the socle of the algebra. However, ϕ_n being injective makes this impossible. Thus $A \to G$ is injective and G has *Betti numbers bounded below by those of the Koszul complex* A.

If G does not have the structure of dg A-module, we can lift a map of complexes $\phi : A \to G$ to a map of higher homotopies

$$\widetilde{\phi} : C \otimes A \to C \otimes G.$$

Objective 8. *Show that* $\ker \widetilde{\phi} = 0$.

This would give the asked for bounds on G.

3.5 Graded matrix factorizations

Matrix factorizations were introduced by Eisenbud in [19] to study free resolutions over hypersurface rings $Q/(f)$. They have been a useful tool in commutative algebra since their introduction, and have recently gained a wider audience due to their rediscovery in string theory[2].

In [17] Mark Walker and I generalized Eisenbud's result on hypersurfaces to complete intersections in the following way.

Theorem 3.5 (Burke-Walker, [17])**.** *If $R = Q/(f_1, \ldots, f_c)$ is a complete intersection, there is an equivalence*

$$[\operatorname{mf}(\mathbb{P}^{c-1}_Q, \mathcal{O}(1), W)] \xrightarrow{\cong} \mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(R),$$

where $\operatorname{mf}(\mathbb{P}^{c-1}_Q, \mathcal{O}(1), W)$ *is the category of "matrix factorizations" of locally free sheaves on* $\mathbb{P}^{c-1}_Q$ *of the element* $W = f_1T_1 + \ldots + f_cT_c \in \Gamma(\mathbb{P}^{c-1}_Q, \mathcal{O}(1))$*, and* $\mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(R) := \mathsf{D}^{\mathsf{b}}(R)/\operatorname{perf}(R)$ *is the* singularity category *of R.*

The impetus of this entire project was to generalize the above theorem to graded matrix factorizations over a ring, instead of locally free matrix factorizations (in the hope of computing examples). And the machinery does this. By substituting *contramodules*, see [39], for comodules over C we have the following.

[2] There was a meeting at Oberwolfach in September 2013 on "Matrix Factorizations in Algebra, Geometry, and Physics."

Theorem 3.6. *Let $R = Q/(f_1, \ldots, f_c)$ be a complete intersection. There is an equivalence*

$$[\text{gr-mf}(S, W)] \xrightarrow{\cong} \mathsf{D}^{\mathsf{b}}(R)$$

where S is the symmetric algebra $\operatorname{Hom}_Q(C, Q) \cong Q[T_1, \ldots, T_c]$ with $W = f_1T_1 + \ldots + f_cT_c \in S_{-2}$, and $[\text{gr-mf}(S, W)]$ is the homotopy category of graded matrix factorizations of W.

The two theorems above are related by the following commutative diagram.

$$\begin{tikzcd}
[\text{gr-mf}(S, W)] \arrow[r, "\cong"] \arrow[d, "\widetilde{(-)}"'] & \mathsf{D}^{\mathsf{b}}(R) \arrow[d] \\
[\text{mf}(\mathbb{P}^{\mathsf{c}-1}_Q, \mathcal{O}(1), W)] \arrow[r, "\cong"'] & \mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(R)
\end{tikzcd}$$

In [17, §7], we constructed from every object of $\text{mf}(\mathbb{P}^{\mathsf{c}-1}_Q, \mathcal{O}(1), W)$ a free resolution.

Objective 9. *Understand how the free resolution of* [17, §7] *is related to the free resolution from Theorem 2.8.*

4 Golod rings

Let Q be a local ring. For a finitely generated Q-module M, the rank of the n-th free module in a minimal free resolution of M is the *nth Betti number* of M. The *Poincare series* $\mathrm{P}^R_M(t)$ of M is the generating function of the sequence of Betti numbers. For a module finite ring homomorphism $f : (Q, \mathfrak{n}, k) \to (R, \mathfrak{m}, k)$, and an R-module M, it is a result of Serre that there is a coefficient-wise inequality

$$\mathrm{P}^R_M(t) \preccurlyeq \frac{\mathrm{P}^Q_M(t)}{1 - t\left(\mathrm{P}^Q_R(t) - 1\right)}. \tag{4.1}$$

Let $R = Q/I$, where $(Q, \mathfrak{m})$ is a regular local ring and $I \subseteq \mathfrak{m}^2$. Then an R-module M is a *Golod module* if (4.1) is an equality; R is a *Golod ring* if the residue field is a Golod module. These rings are fairly mysterious, yet they abound. For instance [29] shows that if I is any proper ideal in a regular local ring Q, then Q/I^k is a Golod ring for all large k.

The connection to twisting cochains is as follows. Let $R = Q/I$ be a Golod ring and M a finite R-module. Let $A \xrightarrow{\cong} R$ be a Q-free resolution of R, and $G \xrightarrow{\cong} M$ a Q-free resolution. Pick A_∞-structures on A and G using 2.6. Let $C = \operatorname{Bar}(A) \to A$ be the universal twisting cochain. From Theorem 2.8 we have an R-free resolution

$$A \otimes^\tau \operatorname{Bar}(A) \otimes R \xrightarrow{\cong} M,$$

and the generating series of the ranks of the free modules of this resolution is exactly the right hand side of (4.1). Thus we have a free resolution which achieves the defining bound of Golod modules. No such construction of this was previously known.

Lescot [34] proved that when R is Golod, for every R-module M, the $\dim Q - \operatorname{depth} M$th syzygy of M is a Golod module. Thus if we do this construction on the $\dim Q - \operatorname{depth} M$th syzygy of any M, we get the minimal R-free resolution of M.

Why this is happening is not clear; thus:

Objective 10. *Find a conceptual explanation for why the bar complex gives a minimal resolution of syzygies of modules over Golod rings.*

We can also study similar objectives as for complete intersections.

Objective 11. *Let $R = Q/I$ be a Golod ring and M an R-module. Use the setup of twisting cochains to relate the minimal Q-free resolution of M with higher homotopies to the minimal R-free resolution of M with higher homotopies.*

For instance, once we know the Poincare series of M over Q, we have found the Poincare series of M over R, and vice versa. This could be helpful to study the Q-free resolution of M (or R) and the higher homotopies on it.

There are several open questions on R-free resolutions that we might be able to approach. For instance, it is known by Peeva [38] that all modules over Golod rings have Poincare series of exponential growth. Avramov introduced *curvature* to measure this type of growth and asked what curvatures can arise for modules over a local ring; see [4]. If M is in the thick subcategory generated by N, then the curvature of M is at most that of N. Thus one approach to Avramov's question is the following:

Objective 12. *Use* $\mathrm{Bar}(A)$ *to set up a theory of supports that classify thick subcategories of* $\mathsf{D}^{\mathsf{f}}(R)$.

5 Non CI/Golod rings

Let $R = Q/I$, where $(Q, \mathfrak{m})$ is a regular local ring and $I \subseteq \mathfrak{m}^2$. Let A be a Q-free resolution of R and $C = \mathrm{Bar}(A)$. If the free resolution 2.8 is minimal for any module M, then as we saw in the last section, M will be a Golod module. By a result of Levin [35], this implies that R is also Golod. Thus if R is not Golod, the resolution from Theorem 2.8 with $C = \mathrm{Bar}(A)$ *will never be a minimal resolution.* On the other hand, by a result of Gulliksen [27], if the free resolution of the residue field k has polynomial growth, then R is a complete intersection. Avramov showed in [2] that if R is not a CI, then the free resolution of k has exponential growth. Thus if R is not a CI and $C \to A$ is an acyclic twisting cochain, then *the rank of C_n must grow exponentially in n.* This gives background to the following.

Objective 13. *Find classes of rings, which are neither CI nor Golod, and acyclic twisting cochains which are smaller than the bar complex and give a minimal resolution for some module.*

If $\dim Q - \operatorname{depth} R \leq 2$, then by a result of Scheja [40], R is either CI or Golod. Thus the first interesting case is $\dim Q - \operatorname{depth} R = 3$. When, in addition, R is Gorenstein, Buchsbaum and Eisenbud [15] describe explicitly the Q-free resolution of R. This will be a very good starting point for the work on this objective. An essential problem will be to take account of the symmetry present in the Q-free resolution of R.

Lurking in the background of this objective is the *homotopy Lie algebra* of R. This was defined by Avramov in [2]. It is the graded Lie algebra whose universal enveloping algebra is the Hopf algebra $\mathrm{Ext}^*_R(k, k)$. The difference between CI and Golod rings can be explained by the fact that the Lie algebra of the first is abelian and 2-dimensional over k, while the second is a free Lie algebra on a finite dimensional vector space.

Objective 14. *Understand the relation between acyclic twisting cochains and the homotopy Lie algebra of R.*

Generalized Golod rings of level n, with $n \geq 1$, were defined in [3] as those rings for which the truncation of the homotopy Lie algebra above degree $n-1$ is a free Lie algebra (thus R is Golod of level 1 if and only if R is Golod). It is shown there that if $\dim Q - \operatorname{depth} R = 3$, then R is generalized Golod of level 2. This may be helpful in constructing an acyclic twisting cochain.

6 Boij-Söderberg theory for hypersurfaces

The aim of this part of the project is to continue a generalization, started in [11], of Boij-Söderberg theory from standard graded polynomial rings to hypersurface rings.

Let S be a standard graded polynomial ring over a field. For a finitely generated graded S-module M, the *Betti table* is the matrix of positive integers which records the degrees of the minimal generators of all syzygies of M. Boij and Söderberg, motivated by the multiplicity conjecture of Herzog, Huneke, and Srinivasan [28], studied the cone spanned by the set of all Betti tables of S-modules in [13]. They conjectured that the Betti tables of pure modules spanned the cone of all Betti tables and this cone was simplicial. They showed these conjectures imply the multiplicity conjecture. Eisenbud and Schreyer proved the conjectures in [22]. See e.g. [41] for further details.

Let us take *Boij-Söderberg theory* to mean the project of describing the convex cone of Betti tables of graded modules over a ring R. There are two ways to do this: describe generators of the cone, and describe functionals that cut out the cone. One can also hope the cone is simplicial (as it has been in all cases so far). In [11], my three coauthors and I accomplished the above for rings of the form $k[x,y]/(q)$, where q is a quadric. This was the first known example of Boij-Söderberg theory for singular rings. This leads to the following:

Objective 15. *Study Boij-Söderberg theory for graded hypersurface rings $R = S/(f)$, where f is homogeneous and* $\operatorname{char}(k)$, $\deg f$ *are coprime.*

My idea to approach this is to use the following result of Orlov. In his theorem on graded Gorenstein rings [37] (on which Greg Stevenson and I have recently written an expository paper [16]), he shows that there is a functor $\mathbf{b}_i : \mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(\operatorname{gr} A) \to \mathsf{D}^{\mathsf{b}}(\operatorname{gr} A_{\geq i})$, where $\mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(\operatorname{gr} A)$ is the singularity category of A. He describes a decomposition of the minimal free resolution of every object in $\mathsf{D}^{\mathsf{b}}(\operatorname{gr} A_{\geq i})$ into a bounded complex of free modules and something in the image of $\mathbf{b}_i$. In [11, 3.5], we (implicitly) described the Betti table of all objects in the image of $\mathbf{b}_i$. Thus to describe the Betti table of all complexes, we need to describe the Betti tables of bounded complexes of free modules. But this should follow by taking a Noether normalization of R and applying the result of Erman and Eisenbud [20] on bounded complexes of free modules over a polynomial ring. The above argument should give a set of spanning elements of the cone of Betti tables of R-modules. Whether this cone is simplicial, what the equations of the facets are, etc. is still unclear and will be a large part of this objective.

7 Broader impacts

This is a multi-disciplinary project that involves techniques from algebraic topology applied to concrete problems in commutative algebra. Many of the techniques are formal homological algebra, thus may also be of interest to representation theorists considering non-commutative rings. One of the broader impacts of the project will be to foster new interaction between practitioners of these areas. If successful, this project will publicize classical, difficult problems on free resolutions in commutative algebra to topologists, whom we hope will be enticed to approach them with new techniques. Conversely, a successful project would give further evidence to commutative algebraists of the power of topological techniques. I plan to encourage specific interaction in two ways. First, by organizing an AMS Special Session on higher homotopies in algebra and topology. And second, by writing an expository set of notes covering curved objects, twisting cochains, and some of the details of the applications I've listed above. In the existing literature in the field, many details are left out and this can be a barrier to newcomers. Having an explicit reference will help encourage people to take up these techniques.

Much of this project was motivated by the desire to compute examples. The functors involved are all very concrete and potentially able to be implemented in Macaulay2. Moreover, the process of finding higher homotopies will be implemented. This could be very interesting, especially for Golod rings, as previously higher homotopies were only computed for complete intersections. All code will be released to the public, and will provide additional tools for people to explore this theory.

This project will also have an impact on undergraduate and graduate education. This summer I supervised an undergraduate student in a reading course on Lie algebras. This was a very rewarding experience, and if this project is funded it would allow me to continue and expand this activity.

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