Higher homotopies in commutative algebra
Project description
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Overview: This project proposes to study modules over commutative rings by applying generalized twisting cochains, which codify higher homotopies or A-infinity structures. A central tool to apply these structures will be Positselski's theory of curved objects. The first objective of this project is to generalize Positselski's theory from objects defined over a field to those defined over an arbitrary commutative ring. This will provide a robust platform to apply the homotopy-theoretical machinery of A-infinity structures to problems in commutative algebra.

A large part of the project will be focused on complete intersection (CI) rings. In this case, the machinery provides a generalization of the classic BGG correspondence, giving a duality between higher homotopies not seen before. The main objective for CI rings is to develop this BGG duality to study minimal free resolutions and the derived category of a CI ring. The outcomes of this part of the project will include new theoretical and computational tools to understand free resolutions over CI rings, a better understanding of the lattice of thick subcategories of the derived category of such a ring, and tools to study obstructions to the existence of higher homotopies on resolutions over CI rings.

Another focus of the project will be on Golod rings. The twisting cochain machinery constructs the minimal resolution of any module over such a ring in a finite number of steps. This effectiveness is a bit of a mystery, and a major objective is to understand the connection between twisting cochains for Golod rings and Massey products on the Koszul complex, whose existence characterizes Golod rings. Finding classes of rings outside of CI and Golod that have a "small" acyclic twisting cochain would shed much light on the behavior of free resolutions over these rings. The outcomes of the above will be a better overall understanding of infinite free resolutions. In the last part of the project, the PI outlines a strategy to generalize Boij-Soederberg theory from polynomial rings to hypersurfaces.

Intellectual Merit: Free resolutions of modules over commutative rings are a basic mathematical object of study. Finding such a resolution is a vast generalization of solving a system of linear equations over a field. These resolutions often exhibit beautiful and intricate structure which gives insight into the structural properties of the ring and module in question that is not easily seen elsewhere. In addition, free resolutions have strong connections to algebraic geometry, algebraic topology, combinatorics, and representation theory.

The objectives and tools in this project represent a new approach to study free resolutions over commutative rings. The existence of algebra structures on free resolutions is at the heart of some of the deepest questions in homological commutative algebra, especially the Buchsaum-Eisenbud rank conjecture. Studying algebras "up-to-homotopy" will provide some extra room to attack these questions. Success in these objectives could also have bearing on topological questions of Carlsson and Halperin.

Broader Impact: One of the main impacts of the project will be an increased exposure between commutative algebra and algebraic topology. Getting some algebraic topologists interested in classical, difficult questions in commutative algebra could have a transformative effect on the field. The proposed research will encourage this by showing how applicable a certain subset of tools of topology are to free resolutions. The PI plans to augment and stir this interest by organizing an AMS special session on higher homotopies in algebra and topology, and by writing a set of expository notes on curved objects and twisting cochains. This project will also lead to the development of mathematical software using Macaulay2 that will disseminated to the public.

The project will also impact undergraduate education through summer reading courses in subjects outside of the standard curriculum.