# RESEARCH STATEMENT

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## 1. Introduction

My research is in homological commutative algebra and representation theory. Techniques from algebraic topology play a role in most of my work. I am currently in the midst of a long-term project to use $A_\infty$-structures to study a set of open problems on the shape and growth of free resolutions over commutative local rings. The first part of this project has involved generalizing the foundations of Koszul duality for $A_\infty$-structures to objects defined over an arbitrary commutative ring, as opposed to a field. I have applied this machinery successfully to Golod and complete intersection rings, and am developing ideas to apply it to larger classes of rings. I am also applying these techniques to study modular representations of restricted Lie algebras, in joint work with Eric Friedlander.

## 2. Overview of recent work

A free resolution unwraps a module into a potentially infinite sequence of matrices, giving a unique point of view on the ring and module in question. For example, the only known proof that regular rings localize uses free resolutions. A finitely generated module over a local, or graded local, ring has a minimal resolution that is unique up to isomorphism. There are a number of beautiful, easily-stated questions on minimal resolutions. These breakup broadly into two types of questions: on finite resolutions over polynomial rings, e.g. is the Koszul complex the smallest resolution of a module of finite length? And on infinite resolutions over singular rings, e.g. are the ranks in each degree eventually non-decreasing? Do they satisfy a linear recurrence relation? Avramov has asked many other questions of this kind and gives a survey of the topic in [3]. Some of these questions are known for special classes of modules or rings, but they are all wide-open in general. Much of what is known is due to a "dictionary" between local algebra and rational homotopy theory, discovered and exploited by Roos, Avramov, Halperin, and others; see [40].

Let $R$ be a singular (non-regular) local ring, so by the Auslander-Buchsbaum-Serre theorem, there exist $R$-modules with infinite minimal free resolutions. Let us assume that $R = Q/I$, for $Q$ a regular local ring (such a presentation exists for all rings of arithmetic or geometric origin, and all complete rings), so an $R$-module $M$ has a finite free $Q$-resolution. We can study the $R$-free resolution of $M$ by putting extra structure on a $Q$-free resolution and trying to deform it. Traditionally, this extra structure was a differential graded (dg) algebra/module structure. But not every $Q$-resolution has the expected dg-structure [4]. In [12] I show that $A_\infty$, or "dg up to homotopy," structures do exist where expected.

These types of higher homotopies have been not been widely used in commutative algebra, but they have the potential to become a fundamental tool. In particular, I think of them as a first step in extending the dictionary mentioned above from rational to integral homotopy theory. For instance, in [12] I show that the differentials of the Eilenberg-Moore spectral sequence for an $R$-module are given by the higher homotopies of an $A_\infty$-structure *reduced modulo the maximal ideal of* $Q$. In this, and several other instances, the classical methods study the units in the higher homotopies, but there is much more information contained there. *Trying to unpack this information, with the questions above in mind (both for commutative rings and analogues in modular representation theory), is the main theme of my recent work.* I have done this successfully for Golod rings, answering two open questions and unifying some classical results in [12]. One of the main tools to extend this to non-Golod rings will be Koszul duality for $\mathrm{A}_\infty$-structures. I have set up the foundations in the generality I need in [13], and started the application to complete intersection rings.

This project has thus far produced the two preprints cited above and three on-going projects. I describe these briefly below and give further information in Sections 3 and 4.

2.1. **Construction of infinite free resolutions.** Let $Q$ be a commutative ring, $R = Q/I$ a quotient ring, and $M$ an $R$-module. Let $A \xrightarrow{\simeq} R$ and $G \xrightarrow{\simeq} M$ be $Q$-projective resolutions. In [12], I show that $A$ is an $\mathrm{A}_\infty$-algebra (over $Q$) and $G$ an $\mathrm{A}_\infty$ $A$-module. I use a bar construction on these to construct an $R$-free resolution of $M$. This generalizes work of Iyengar [28] who used dg-structures. All of this machinery applies particularly well to Golod rings. It gives unified, conceptual proofs of several classical results on these rings. Moreover, I use it to answer a question of Eisenbud on recognizing a Golod ring, as well as construct the minimal free resolution of every finitely generated module. This is the largest class of rings for which we can write down all minimal free resolutions.

In a slightly different direction, I also generalize Avramov's obstruction theory of [4]. The context for this obstruction theory is the following. Buchsbaum and Eisenbud showed in [11] that if the minimal resolution of a module is a dg-module over a Koszul complex on a regular sequence in the annihilator of the module, then in each degree that resolution has to have at least the rank of the Koszul complex. Whether this inequality always holds is their famous rank conjecture. Avramov defined an obstruction to the existence of dg-module structures on minimal resolutions, and showed that it is non-zero in cases, but also that vanishing of these obstructions is not sufficient for a dg-module structure to exist. I generalize his obstructions slightly in [12] and show their relation to $\mathrm{A}_\infty$-structures, but this is still not sufficient for detecting dg-module structures. Pursuing this is a long term goal.

2.2. **Generalized Koszul duality.** Golod rings are the local rings with largest possible cohomology. To get sharp results for non-Golod rings, e.g. constructions of minimal resolutions, one must remove units from the bar construction without losing information on the module. This is best formulated in terms of twisting cochains, and is a general form of Koszul duality [29]. A twisting cochain is a linear map $\tau : C \to A$ from a dg-coalgebra $C$ to an $\mathrm{A}_\infty$-algebra $A$ that satisfies the Maurer-Cartan equation. This condition allows one to "twist" the usual differential on $A \otimes N$, where $N$ is a dg $C$-comodule, to get a dg $A$-module, and analogously for $\mathrm{A}_\infty$ $A$-modules. In particular, $\tau$ gives an adjoint pair of functors between dg $C$-comodules and $\mathrm{A}_\infty$ $A$-modules. A twisting cochain is *acyclic* if these functors induce an equivalence of (co)derived categories. Although there may be many such $C$, we will abuse language and say $C$ is Koszul dual to $A$. The canonical example is that the divided powers coalgebra is Koszul dual to the symmetric algebra (this is called the BGG correspondence after [8]). In the preprint [13], I carefully construct the machinery of Koszul duality for non-augmented $\mathrm{A}_\infty$-algebras defined over a commutative ring (note that a $Q$-free resolution of $Q/I$ is augmented if and only if $I$ is zero). This was previously done by Keller's student Lefévre-Hasegawa for augmented dg-algebras defined over a field in [31, §1,2], and some aspects were extended by Positselski, who in particular showed that when considering non-augmented objects, one must add a curvature term to the dual side [37]. The arguments of Lefévre-Hasegawa and Positselski make heavy use of an underlying field. To generalize their results to objects defined over commutative rings, I had to find completely new arguments and definitions for many classical results. For instance Positselski's definition of a coderived category does not behave as expected when working over a ring that is not a field. Also, many of the results of [13] have not appeared in the literature even for augmented $\mathrm{A}_\infty$-algebras defined over a field. The main result is a criterion for detecting when a twisting cochain is acyclic in terms of resolutions.

2.3. **BGG for complete intersections.** The first non-Golod rings to study with this machinery are complete intersection (CI) rings. Using the Koszul duality above, I can show that homological algebra over a CI ring is governed by a generalized BGG correspondence. Let $R = Q/I$ be a CI, so $I$ is generated by a $Q$-regular sequence, and let $A$ be the Koszul complex on that regular sequence. Thus $A$ is a $Q$-free resolution of $R$, the underlying algebra is the exterior algebra on a free $Q$-module $V$, and the differential is determined by the $Q$-linear map $V := A_1 \xrightarrow{l} A_0 = Q$. The Koszul dual of the dg-algebra $A$ (the two notions of Koszul here are coincidental) is the pair $(C, l)$, where $C$ is the divided powers coalgebra on $V[1]$ (so the usual Koszul dual of the exterior algebra plus the extra information of $l$). Dualizing this over $Q$ (to avoid coalgebras), we have a symmetric algebra $S$ and an element $l \in S_2$ (a *curved algebra* in the language of [37]). Representations of this curved algebra are "graded matrix factorizations of $l$" over the ring $S$. The machinery of Koszul duality gives an equivalence of the derived category of $R$ with these graded matrix factorizations. Greg Stevenson and I first proved this equivalence in [14], using different tools. The twisting cochain point of view gives a conceptual explanation and opens up the geometric tools of BGG [18]. It also gives an analogy with the equivariant cohomology of spaces with a torus action, as studied by Goresky, Kottwitz and MacPherson (GKM) in [21] (cohomology over $Q$ should be "singular cohomology" and that over $R$ should be "equivariant cohomology"). I can use this analogy to prove a new collapse result for a spectral sequence computing cohomology over a complete intersection, and am working on adapting further results of GKM to this context. A particular goal is to find an analogue of their computation of the equivariant cohomology *ring* of a large class of spaces, using a graph given by fixed points and 1-dimensional orbits.

Lurking in the background of all of the work on CI rings is the Buchsbaum-Eisenbud conjecture on ranks of finite free resolutions [11] mentioned above. To solve the conjecture, one has to construct an obstruction theory to being a dg-module over a Koszul complex, not an arbitrary dg-algebra. The tools above might help with this.

Part of the project on CI rings, studying higher homotopies on $R$-free resolutions and using these to construct $Q$-resolutions, will be joint work with David Eisenbud and Frank Schreyer.

2.4. **General local rings.** Complete intersection and Golod rings are two ends of a homological spectrum. For CIs one can find a very small acyclic twisting cochain inside the bar construction and for Golod rings the bar construction is the best one can do. Gulliksen showed that if the Betti numbers of the residue field $k$ are not bounded by a polynomial, then the ring is not a complete intersection [24]. Thus for a non CI/Golod ring, we look for a twisting cochain that grows exponentially, but is properly contained in the bar construction.

The story for complete intersections works so well because the Koszul complex is free as a commutative $Q$ dg-algebra. This suggests we should look for a free dg-algebra replacement of $R$, e.g. the *minimal model* of $R$ over $Q$, defined by Avramov as an analogue of Sullivan's definition in rational homotopy. A free $Q$-algebra, being a tensor product of symmetric and exterior algebras, has a well understood Koszul dual. Adding a differential, the story becomes less clear. Halperin and Stasheff calculate the cohomology of the bar construction of a free commutative dg-algebra over a field, and show that their *filtered model* does a better job at retaining higher homotopy information (they phrase it as differentials of the Eilenberg-Moore spectral sequence) [27]. Finding an optimal acyclic twisting cochain, possibly by constructing an analogue of filtered models, is work in progress. It is a tantalizing fact that the minimal $R$-resolution of $k$ is a free dg-algebra with divided powers, and its generators have the same ranks as the generators of the minimal model, shifted by one. If we could find an acyclic twisting cochain using the minimal model as above, this would be a direct result and seems to give evidence for this program. Other directions are to understand how the homotopy Lie algebra of $R$ fits in the picture, and to use the $\mathrm{A}_\infty$-structures (which are always finite when $Q$ is regular) to give finite constructions of things that have previously been out of reach. For instance calculating the algebra structure on $\mathrm{Ext}^*_R(k,k)$ for non CI/Golod rings. Any progress in these directions would likely be a huge step in answering the questions listed above.

2.5. **Modular representations of Lie algebras.** The machinery of [13] does not make any assumptions on the commutativity of objects involved, and so applies much more generally than to the cases discussed thus far. In joint work with Eric Friedlander, I am applying it to study restricted representations of a finite dimensional restricted Lie algebra. The category of such representations does not have finite global dimension, in stark constrast to e.g. representations of a complex Lie algebra. Complete intersection rings and restricted Lie algebras share many formal cohomological properties, in particular both have a well developed theory of support varieties. We have found a direct link between these two objects. The idea is that the universal enveloping algebra $U(g)$ of $g$ is a finitely generated free module over a polynomial ring $O(g)$ in its center, and the restricted enveloping algebra $V(g)$ is a quotient of $U(g)$ by a regular sequence $\mathbf{y}$ in $O(g)$ (a restricted representation is the same as a $V(g)$-module). If we let $A$ be the Koszul complex over $O(g)$ on the regular sequence $\mathbf{y}$, then $U(g) \otimes_{O(g)} A$ is a $U(g)$-free resolution of $V(g)$. Quillen showed that the Koszul dual of $U(g)$ is the exterior coalgebra $\wedge^* g[1]$ with differential induced from the bracket of $g$ [39], and the work mentioned above on CIs shows the Koszul dual of $A$ is a curved coalgebra $(C, l)$. Thus $(\wedge^* g \otimes C, 1 \otimes l)$ is Koszul dual to $U(g) \otimes_{O(g)} A \simeq V(g)$, and so one may describe the derived category of restricted representations as matrix factorizations. The theory of matrix factorizations easily recovers many formal tools for studying restricted representations, such as two spectral sequences, one converging to restricted cohomology and the other to usual cohomology, a theory of support varieties, and May's resolution of the trivial restricted representation [35]. But much more generally, the restricted enveloping algebra is a single ring in a family indexed by characters of $g$, see [20], and we can modify our approach to study the entire family, opening up the formal tools above to this family. This is entirely new and we are in the process of exploring it.

## 3. Longer Description of Recent Results

3.1. **Construction of free resolutions using $\mathbf{A}_\infty$-structures.** Let $Q$ be a regular local ring, $R = Q/I$ and $M$ an $R$-module. Let $A, G$ be a $Q$-projective resolutions of $R$ and $M$, equipped with $\mathrm{A}_\infty$-structures. In [12], I construct three tools that use the $\mathrm{A}_\infty$-structures to study resolutions. The first is the classical bar construction which puts a differential on $R \otimes_Q (\oplus_{n\geq 0} A_+^{\otimes n}) \otimes_Q G$ that makes it an $R$-projective resolution of $M$. The second is a construction of a $Q$-projective resolution of all $R$-syzygies of $M$. Once the $\mathrm{A}_\infty$-structures above are fixed, no choices are necessary to construct these $Q$-resolutions. And the third is the description of the differentials in the Eilenberg-Moore spectral sequence. Moreover, the machinery characterizes Golod modules as precisely those for which the $\mathrm{A}_\infty$-structures can be picked minimal. The classical result of Levin that if $R$ has a Golod module, it must be a Golod ring, follows almost immediately. Playing the three tools above off each other, I also give new proofs of the following results of Lescot and/or Levin: the syzygy of a Golod module is Golod, $R$ is Golod if and only if $\mathrm{Tor}_*^Q(\mathfrak{m}, k) \to \mathrm{Tor}_*^R(\mathfrak{m}, k)$ is injective where $\mathfrak{m}$ is the maximal ideal of $R$, and every module over a Golod ring has a syzygy that is a Golod [32, 33].

Golod modules are traditionally defined as those whose Poincare series (generating series of Betti numbers) achieves a known upper bound. The bar construction realizes this upper bound, so a module is Golod if and only if the bar construction is minimal. It follows from the combinatorics of the bar construction, and the fact that there are only finitely many higher homotopy maps, that if the upper bound of power series is an equality in the first $\mathrm{pd}_Q M$ degrees, then it is an equality of power series. *No bound of this form was previously known*, although whether a bound existed was a question I learned from Eisenbud. Also, by computing a fixed finite number of steps and then using a bar construction, one can construct the minimal free resolution of every finitely generated module over a Golod ring. Let us emphasize that one can construct $\mathrm{A}_\infty$-structures on a computer, e.g. using [22], and so construction of the minimal free resolution of every finitely generated module over a Golod ring can by done via machine.

The existence of $\mathrm{A}_\infty$-structures on $Q$-resolutions opens up many new questions, especially in the case $Q$ is a polynomial ring and $M$ is graded (one can talk about graded $\mathrm{A}_\infty$-structures in this context, and they exist as for non-graded objects). For instance, given a resolution with some combinatorial structure, such as a cellular resolution of a monomial ideal [6], can one use this to construct a similar combinatorial $\mathrm{A}_\infty$-structure? In another direction, the Betti table of the minimal $Q$-free resolution of $M$ has a lot of combinatorial structure due to Boij-Soederberg theory [10, 19]. The Betti table of $M$ is a rational multiple of pure diagrams and the diagrams that occur for $M$ form a chain in the poset of Betti diagrams. I think it is a very interesting question to study e.g. what restrictions a given $A_\infty$-structure puts on the Boij-Soederberg decomposition and vice versa. Both Boij-Soederberg theory and $A_\infty$-structures can be studied using Macaulay2 [22], which will help form conjectures.

3.2. **Generalized Koszul duality.** One of the key steps in generalizing the theory of [31, §1,2] from objects defined over a field to a commutative ring is to replace underlying complexes with cofibrant (aka semi-projective, K-projective, cell) complexes. (Such a resolution is the correct generalization to complexes of the notion of projective resolution of a module.) I show that $A_\infty$-algebra and module structures, and morphisms between them, transfer from a given complex to a cofibrant replacement, and under certain connectivity assumptions these transferred structures are unique up to homotopy. I also give a full proof of the fact, due to Positselski, that when considering $A_\infty$-algebras that are not augmented, the dual dg-coalgebras $C$ have a curvature map $C \to Q$, where $Q$ is the ground ring. For example, if $A$ is the resolution of a cyclic $Q$-algebra and $A_0 = Q$, then the curvature map on the bar construction will be $\operatorname{Bar} A \twoheadrightarrow A_1[1] \xrightarrow{d_1^A} A_0 = Q$.

Given an $A_\infty$-algebra $A$, a curved dg-coalgebra $C$ and $\tau : C \to A$ a twisting cochain between them, one wants to consider adjoint functors between their (co)derived categories given by a twisting cochain. Showing these functors are well-defined and are adjoint on the homotopy categories takes non-trivial work (even checking the differential on $A \otimes N$ squares to zero, for $N$ a curved $C$-comodule, is not easy). Once these functors are set up, the guiding principle is that if the twisting cochain is the universal twisting cochain $\tau_A : \operatorname{Bar} A \to A$, then the resulting functors should be an equivalence. This isn't the case for the homotopy categories so one must localize. Lefévre-Hasegawa did this by constructing a coderived category for a dg-coalgebra, and Positselski [37] generalized this by defining a coderived category for a curved dg-coalgebra, both working over a field. The curved case is difficult since curved objects don't have a differential that squares to zero, so there is no homology, and thus no quasi-isomorphisms to invert.

Positselski's definition and the results on it use the fact that everything splits additively, and this is manifestly not the case when working over a non-field. I define a coderived category for a curved dg-coalgebra over a ring by reducing everything to extended, or co-free, comodules, where quasi-isomorphisms do make sense, and then inverting these. I show that this definition of coderived category gives an equivalence for $\tau_A : \operatorname{Bar} A \to A$. A twisting cochain $\tau : C \to A$ is acyclic if the functors induced by it give an equivalence of (co)derived categories. I generalized the classical recognition criteria and showed that if the first component of the counit $A \otimes C \otimes A \to A$ is a quasi-isomorphism, then $\tau$ is acyclic. This is often computable. Also in [13] I construct finite (co)derived categories, and prove a very general change of rings of theorem for free resolutions.

## 4. Work in progress

4.1. **BGG for complete intersections.** Let us fix a complete intersection $R = Q/I$, where $I$ an ideal generated by a $Q$-regular sequence. In [15], Mark Walker and I connect homological algebra over $R$ to geometry over $\mathbb{P}_Q^{c-1}$ using a theorem of Orlov. We show that the singularity category of $R$ is equivalent to a homotopy category of matrix factorizations of locally free sheaves [16] on $\mathbb{P}_Q^{c-1}$. We were initially trying to formulate a generalization of the "Hirzebruch-Riemann-Roch" theorem for matrix factorizations of Polishchuk and Vaintrob [36] (we wanted to generalize what they did from hypersurfaces to complete intersections). We did not succeed in this (Mark showed later the quantity we aimed to compute is zero [42]), but we did generalize support varieties for complete intersections, originally introduced in [5], and gave a new construction of a free resolution of a module over a complete intersection using sheaf cohomology of the components of the matrix factorization on $\mathbb{P}_Q^{c-1}$.

To compute examples with this theory, it helps to take global sections and study things over the homogeneous coordinate ring $Q[T_1, \dots, T_c]$ of $\mathbb{P}_Q^{c-1}$. Greg Stevenson and I lifted the equivalence above from the singularity to the derived category of $R$, and on the opposite side from sheaves on $\mathbb{P}_Q^{c-1}$ to matrix factorizations of graded $S$-modules [14]. This uses a generalization of a theorem of Orlov on graded Gorenstein rings. Dave Benson gave another proof of this in the case $R$ is the group ring of an elementary abelian $p$-group [7]. As discussed above, the machinery of [13] gives a conceptual proof of this equivalence.

Let $M$ be a finitely generated $R$-module and $k$ the residue field. It is a classical fact, and follows immediately from the equivalence involving matrix factorizations above, that $\mathrm{Ext}^*_R(M,k)$ is a finitely generated module over $S = k[T_1,\dots,T_c]$, with $|T_i| = 2$. Also, $\mathrm{Ext}^*_Q(M,k)$ is a finite length $E = \mathrm{Tor}^*_Q(R,k) \cong \wedge^*(k[1]^c)$-module. At first, one hopes that these two modules determine each other through the BGG correspondence between symmetric and exterior algebras. But there isn't enough information captured. Goresky, Kottwitz and MacPherson (GKM) studied an analogous situation in equivariant cohomology [21]. If $X$ is smooth manifold on which a torus $T$ acts, then equivariant cohomology $H^*_T(X,\mathbb{R})$ is a finitely generated $\mathbb{R}[T_1,\dots,T_c] = H^*_T(\mathrm{pt})$-module and singular cohomology $H^*(X,\mathbb{R})$ is a finitely generated $H^*(T) = \wedge^*\mathbb{R}^c$-module. GKM showed that these two modules determine each other through BGG, as long as one keeps track of higher homotopies. The categorical setup they use is exactly the same for complete intersections, once we tensor everything with the residue field $k$. Higher homotopies on $\mathrm{Ext}^*_Q(M,k)$ have been used to study $M$ since they were introduced in [17], but higher homotopies on $\mathrm{Ext}^*_R(M,k)$ have not been studied before. I show their existence can be deduced from operators on an $R$-free resolution of $M$. This uses results in [23]. David Eisenbud and Frank Schreyer have independently discovered these operators, and we are in the process of pooling our results into a paper.

GKM show that the higher homotopies on the two sides are in some sense dual. Indeed, they show that there is a commutative diagram

$$\begin{tikzcd}
\mathsf{D}^{\mathsf{b}}(S) \arrow[rr, shift left] & & \mathsf{D}^{\mathsf{b}}(E) \arrow[ll, shift left, "\cong"] \\
& \mathsf{D}^{\mathsf{b}}_T(\mathrm{pt}) \arrow[ul, "\cong"] \arrow[ur, "\cong"'] & \\
& \mathsf{D}^{\mathsf{b}}_T(X) \arrow[u, "p_*"'] &
\end{tikzcd}$$

where $\mathsf{D}^{\mathsf{b}}_T(X)$ is the equivariant derived category of sheaves on $X$ [9], the top functors are the BGG equivalence, $p : X \to \mathrm{pt}$ is projection to a point, and the diagonal functors are defined in *loc. cit.* GKM show that if one takes the constant sheaf on $X$, the corresponding object in $\mathsf{D}^{\mathsf{b}}(S)$ is the singular chains of $X$ with higher homotopy data, and the corresponding object in $\mathsf{D}^{\mathsf{b}}(E)$ is equivariant chains with higher homotopy data. In particular, the higher homotopies of one determine those of the other.

Using this picture, I propose to think of an $R$-module $M$ as the pushforward of a sheaf on some non-existent space with $T$-action, and use the same tools to study the homological algebra of $M$. In particular, the higher homotopies on a $Q$-resolution should be related to the higher homotopies on an $R$-resolution. This is still work in progress, but I can adapt a result of GKM that shows that the spectral sequence resulting from $R$-higher homotopies collapses if and only if the spectral sequence from $Q$-higher homotopies collapses. This points to a bigger picture that I am working to uncover. GKM call spaces for which these spectral sequences collapse *equivariantly formal* (these include all smooth complex projective varieties) and compute the ring structure on both sides from the orbit graph of $X$. This has vertices fixed points and edges one-dimensional orbits. This is a fantastic result that has been used widely, e.g. in [30]. I am very excited about working to adapt it to cohomology of modules over CI rings. One preliminary idea is that the analogue of fixed points for a module over a CI should be (related to) free summands in the $S$-module $\mathrm{Ext}^*_R(M,k)$.

BGG is a very useful as a computational tool. For instance, it is shown in [18] how to compute sheaf cohomology as the homology of a complex of finite dimensional modules over the exterior algebra. Properties of the graded $S$-module $\mathrm{Ext}^*_R(M,k)$ (and its sheafification), e.g. its Castelnuovo-Mumford regularity, are very helpful to know, but they are very difficult to compute since they possibly depend on the entire infinite resolution of $M$. I hope to use the generalized BGG of [13] to compute some of these quantities. The tools of [18] don't apply directly. They are formulated for complexes of graded $S$-modules and we are working with dg $S$-modules, but there is a connection between the two that I am in the process of working out. It should give a spectral sequence that converges to the classical BGG and this should help get a handle on some of these geometric quantities of $\operatorname{Ext}_R^*(M,k)$, and the spectral sequence should degenerate for high $R$-syzygies of a module.

4.2. **General local rings.** Consider a local ring $R = Q/I$, where $(Q, \mathfrak{n}, k)$ is a regular local ring and $I \subseteq \mathfrak{n}^2$ (such a presentation of $R$ is unique in some precise sense). We can consider the *acyclic closure*, or Tate resolution, of the residue field. This is a free divided power dg $R$-algebra $R\langle Y\rangle \xrightarrow{\simeq} k$ that is an $R$-free resolution of $k$. It was introduced in [41] and shown to be minimal in [25]. We can also consider the *minimal model* of $R$ defined in [2]. This is the smallest free dg $Q$-algebra $Q[X]$ that resolves $R$ over $Q$. These two constructions are connected by the *homotopy Lie algebra* of $R$, denoted $\pi^*(R)$. This can be defined as the homology of the Lie algebra of derivations of the acyclic closure, or as $X^*[1] \otimes_Q k$ with bracket determined by the "quadratic part" of the differential of $Q[X]$. In particular, it follows that $Y_{n+1} \cong X_n$. We also have that the universal enveloping algebra of $\pi^*(R)$ is $\operatorname{Ext}_R^*(k,k)$, and when $\operatorname{char} k = 0$, $\pi^*(R)$ is isomorphic to the Andre-Quillen, or cotangent, homology of $R$. In general, there are spectral sequences connecting the two.

These tools are at the center of much of the work that has been done on infinite free resolutions over local rings. For example, Halperin used a comparison of the acyclic closure and minimal model to prove his rigidity theorem [26]. If we set $e_n := \dim_Q X_n$, Halperin showed that if $e_n = 0$ for some $n \geq 3$, then $e_n = 0$ for all $n \geq 3$ and $R$ is CI. This is one of the deepest results in the cohomology of local rings. As another example, Avramov used the homotopy Lie algebra to show that the growth of the minimal free resolution of the residue field is either polynomial or exponential [2].

These structures are intimately connected to $\mathrm{A}_\infty$-structures and Koszul duality, and in fact there is evidence that $\mathrm{A}_\infty$-structures contain more information, but the entire picture isn't clear yet. One ingredient is that if $D \xrightarrow{\simeq} k$ is an $R$-free resolution of $k$ with a diagonal map $D \to D \otimes D$ that commutes with differentials, e.g. if $D$ is a dg-coalgebra, then the dual of this diagonal gives the algebra structure on $\operatorname{Ext}_R^*(k,k)$ (I proved this, but it is probably well known). Let $A \xrightarrow{\simeq} R$ be the minimal $Q$-free resolution equipped with an $\mathrm{A}_\infty$-algebra structure and let $K \xrightarrow{\simeq} k$ be the Koszul complex of $Q$, equipped with an $\mathrm{A}_\infty$ $A$-module structure (which exists since $k$ is an $R$-module). Let $C \to A$ be an acyclic twisting cochain, e.g. $C = \operatorname{Bar} A$. Then $(R \otimes_Q C) \otimes_R (R \otimes_Q K) \xrightarrow{\simeq} k$ is an $R$-free resolution of $k$, with differential given by higher homotopies. I'm working on putting a diagonal on $(R \otimes_Q C) \otimes_R (R \otimes_Q K)$ to compute the Ext-algebra structure. This will use the fact that $R \otimes_Q C$ is a dg-coalgebra, and should be a semi-tensor product as in [34]. For instance, when $R$ is Golod, the homotopy Lie algebra is a semi-direct product of a free Lie algebra and a finite dimensional Lie algebra. The higher homotopies on $A$ and $K$ will be minimal in this case, so $k \otimes_R (R \otimes_Q \operatorname{Bar} A) \otimes_R (R \otimes_Q K) \cong \operatorname{Ext}_R^*(k,k)$ and we can see how the free part of the homotopy Lie algebra is coming from $\operatorname{Bar} A$ and the finite dimensional part is from the Koszul complex $K$. In small cases, we can also try removing units from $\operatorname{Bar} A$ by hand, thus giving a finite computation of $\pi^*(R)$. For instance when $R$ is a codimension 3 Gorenstein ring, the higher homotopies modulo the residue field have rank 1. I'm working on finding a basis of $\operatorname{Bar} A$ to be able to describe concretely what happens when we factor out this rank 1 non-minimal part. This should give a closed description of the minimal free resolution of $k$ over a codimension 3 Gorenstein ring and a description of the algebra $\operatorname{Ext}_R^*(k,k)$. Both of these are unknown, and are the tip of the iceberg of this machinery. Another point of view, that I learned from Alexander Berglund, is that the minimal model makes $X^*[1]$ an $\mathrm{L}_\infty$-algebra over $Q$, whose homology is $\pi^*(R)$. This is the same as putting a dg-coalgebra structure on $\wedge^* X^*[1]$, and this should surely fit into the machinery of Koszul duality, but exactly how is not clear. The ideas of Anick [1] on the relation between Quillen's Lie algebra model of a space and Sullivan's minimal model should be relevant to this work.

Any success on the above topics would shed a lot of light on the homotopy theory of local rings. In particular, it could have strong ramifications for the many open questions on infinite free resolutions. And no matter what, it opens up large tracts of unexplored mathematical territory.

4.3. **Modular representations of Lie algebras.** A Lie algebra defined over a field of positive characteristic $p$ is restricted if it has a "$p$-th power operation." If we pick a faithful representation, then being restricted means that the Lie algebra is closed under $p$th powers of matrices. The Lie algebra of an algebraic group in positive characteristic is restricted.

Zassenhaus showed that a restricted Lie algebra $g$ has a symmetric algebra $O(g)$ in the center of $U(g)$ and $U(g)$ is free of finite rank over $O(g)$. The algebra $O(g)$ is generated by the Frobenius twist of $g$. For any character $\chi : g \to k$, one can consider the corresponding algebra map $\chi : O(g) \to k$, and the ring $A_\chi = U(g) \otimes_{O(g)} k_\chi$. This gives a family of rings indexed by characters of $g$. When $\chi = 0$, $A_\chi$ is the restricted enveloping algebra of $g$. This family of rings was studied by Friedlander and Parshall in e.g. [20] and was an essential ingredient in Premet's proof of the Kac-Weisfeller conjecture [38]. An important feature of this family is that every irreducible $U(g)$-module is naturally a module over $A_\chi$ for a unique $\chi$. Using the Koszul duality setup described above, Eric Friedlander and I can study the family $A_\chi$. Specifically, we can describe a category of "generalized matrix factorizations" from which one can easily recover the derived category of each $A_\chi$. In particular this provides a category to study not necessarily finite dimensional $g$-modules that correspond to more than a single $A_\chi$. This is (vaguely and optimistically) some modular version of category $\mathcal{O}$. The machinery of matrix factorizations and Koszul duality gives several new tools to study this family, and we are excited about exploring their applications.

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