KOSZUL DUALITY FOR REPRESENTATIONS OF AN A-INFINITY ALGEBRA DEFINED OVER A COMMUTATIVE RING (PRELIMINARY VERSION)

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ABSTRACT. We generalize the Koszul duality of Lefévre-Hasegawa and Positselski to A_∞-algebras and modules defined over an arbitrary commutative ring. This is formulated in terms of generalized twisting cochains. Much of the theory we present has not previously appeared, even for objects defined over a field. We show that a projective resolution of the complex underlying an A_∞-algebra has an A_∞-algebra structure that makes the augmentation map a strict morphism. This gives both motivating examples and an important tool in generalizing results from a field to an arbitrary ring. One of the main results is a characterization of acyclic twisting cochains, which we show can considerably reduce the complexity of representations of an A_∞-algebra.

As applications of the theory, we give a change of rings theorem for projective resolutions, and generalize the classical BGG correspondence between the exterior and symmetric algebras to a correspondence between a Koszul complex and a curved symmetric algebra. This recovers much of the classical theory of homological algebra over a complete intersection ring and provides a path to a generalization to noncommutative versions, in particular to modular representations of finite dimensional p-restricted Lie algebras.

CONTENTS

1. Introduction

1.1. **Notation.** We fix a commutative ring k. All modules, complexes, homomorphisms, tensor products, and (co)algebras are defined over k unless stated otherwise. For graded modules M, N, $\operatorname{Hom}(M, N)$ is the graded module which in degree n is $\prod_{i\in\mathbb{Z}} \operatorname{Hom}(M_i, N_{i+n})$. For complexes, all degrees are homological and so differentials lower degree. If M and N are complexes, then $\operatorname{Hom}(M, N)$ is a complex with differential $d_{\operatorname{Hom}}(f) = fd_M - (-1)^{|f|}d_N f$. A morphism of complexes is a cycle of degree zero in $\operatorname{Hom}(M, N)$. If M is a complex, $M[1]$ is the complex with $M[1]_n = M_{n-1}$ and $d_{M[1]} = -d_M$. We set

$$s : M \to M[1]$$

to be the degree 1 map with $s(m) = m$.

For graded modules M, N, $M \otimes N$ is the graded module that in degree n is $\bigoplus_{i\in\mathbb{Z}} M_i \otimes N_{n-i}$. If M, N are complexes, $M \otimes N$ is a complex with differential $d_\otimes = d_M \otimes 1 + 1 \otimes d_N$. When applying tensor products of homogeneous maps we use the sign convention $(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$. All elements of graded objects are assumed to be homogeneous. For such an x we set $\underline{x} = (-1)^{|x|+1}x$. All (co)modules are assumed to be left (co)modules.

For a pair of functors F, G between categories $\mathcal{A}, \mathcal{B}$, we write

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "F"] & \mathcal{B} \arrow[l, shift left, "G"] \end{tikzcd}$$

to indicate they form an adjoint pair with F *left adjoint to* G.

2. Non-augmented A_∞-algebras and curvature

In this section we recall the definition of strictly unital A_∞-algebras, formulated in terms of the tensor coalgebra. We show how the failure of an A_∞-algebra to be augmented can be compensated for with a curvature map on the bar construction. This idea is due to Positselski. To construct A_∞-structures, we modify Lefévre-Hasegawa's obstruction theory to the non-augmented case. We use this to show that if A is an A_∞-algebra, then a semiprojective, or cofibrant, resolution of the complex underlying A has an A_∞-algebra structure. We give conditions for such a structure to be unique up to homotopy. From this it follows that a k-projective resolution of a k-algebra has a unique up-to-homotopy A_∞-algebra structure.

2.1. **Background on coalgebras.** In this subsection we collect some basic material on graded coalgebras. For proofs of unsubstantiated claims, see e.g. (REF Montgomery).

Fix a graded counital coalgebra (C, Δ, ϵ), where $\Delta : C \to C \otimes C$ is comultiplication and $\epsilon : C \to k$ the counit. We will use Sweedler notation, so if $\Delta(x) = \sum_i x_{1i} \otimes x_{2i}$, we write $\Delta(x) = x_{(1)} \otimes x_{(2)}$. More generally, set $\Delta^{(2)} = \Delta$ and $\Delta^{(n)} = (1 \otimes \Delta^{(n-1)})\Delta$ for $n \geq 3$, and write $\Delta^{(n)}(x)$ as $x_{(1)} \otimes \ldots \otimes x_{(n)}$. If N is a graded left C-comodule, we write $\Delta_N(y) = y_{(-1)} \otimes y_{(0)} \in C \otimes N$ for comultiplication on N.

The coalgebra C is *coaugmented* if there is a morphism of graded coalgebras $\eta : k \to C$ that is a splitting of the counit map ϵ. When C is coaugmented, there is an isomorphism of graded modules $C \cong k \oplus \overline{C}$, where $\overline{C} = \ker \epsilon$. We always assume that C has coaugmention η, and throughout, we write $p : C \to \overline{C}$ for the projection induced by η.

We will work exclusively with the following class of coaugmented coalgebras.

Definition 2.1.1. Let C be a coaugmented coalgebra. Set $\overline{\Delta}^{(n)} := p^{\otimes n}\Delta^{(n)} : C \to \overline{C}^{\otimes n}$ and define the k-submodule of nth primitives as

$$C_{[n]} := \ker(\overline{\Delta}^{(n)}) \subseteq C.$$

The coalgebra C is *cocomplete* if $C = \bigcup_{n \geq 1} C_{[n]}$.

Example 2.1.2. If G is a k-linear algebraic group with coordinate coalgebra $C = k[G]$, then C is cocomplete if and only if G is unipotent. is this right?

Example 2.1.3. If C is graded, and satisifes....then C is connected.

Example 2.1.4. If C is cocomplete, then the counit is the only group-like element. Thus for G a non-trivial finite group and a field k, the coalgebra structure making the group ring a Hopf algebra is never cocomplete.

The following will be for us the most important example of a cocomplete coalgebra.

Definition 2.1.5. Let B be a graded module. The *tensor coalgebra of* B, denoted $T^c(B)$, has underlying graded module $\bigoplus_{n \geq 0} B^{\otimes n}$ and comultiplication

$$\Delta(x_1 \otimes \ldots \otimes x_n) = \sum_{i=0}^{n} (x_1 \otimes \ldots \otimes x_i) \otimes (x_{i+1} \otimes \ldots \otimes x_n).$$

The counit is projection $\epsilon : T^c(B) \to B^{\otimes 0} = k$ and the inclusion $k \hookrightarrow T^{\mathsf{co}}(B)$ is a coaugmentation. We have $\overline{T^{\mathsf{co}}(B)} = \ker \epsilon = \bigoplus_{n \geq 1} B^{\otimes n}$ and

$$T^c(B)_{[n]} = \bigoplus_{i=0}^{n} B^{\otimes i}.$$

In particular, the tensor coalgebra is cocomplete.

Definition 2.1.6. A *coderivation* of a graded coalgebra (C, Δ) is a homogeneous k-linear map $d : C \to C$ such that $(d \otimes 1 + 1 \otimes d)\Delta = \Delta d$.

Definition 2.1.7. Let C be a cocomplete graded coalgebra, B a graded k-module, $\varphi : C \to T^{\mathsf{co}}(B)$ a homogeneous k-linear map. For any $m \geq 1$, define

$$\varphi^m = p_m \varphi : C \to B^{\otimes m},$$

where $p_m : T^{\mathsf{co}}(B) \twoheadrightarrow B^{\otimes m}$ is the canonical projection. If φ is a graded, coaugmented, coalgebra map, then $\sum_{m \geq 0} \varphi^m(x)$ is finite for every $x \in C$, since C is cocomplete, and $\varphi = \sum_m \varphi^m$. If $C = T^{\mathsf{co}}(A)$, for some graded module A, set

$$\varphi^m_n = p_m \varphi i_n : B^{\otimes n} \to B^{\otimes m}.$$

The following universal properties of the tensor coalgebra will be essential in the sequel. For proofs see e.g. [18, §2.5] or [8, §2.1, 2.2].

Lemma 2.1.8. *Let C be a cocomplete graded coalgebra and B a graded module.*

(1) *Let $\varphi : C \to T^{\mathsf{co}}(B)$ be a graded, coaugmented, morphism of coalgebras and let $\iota : \overline{C} = \ker \epsilon \to C$ be inclusion. For every $m \geq 1$, there is an equality:*

$$\varphi^m = (\varphi^1)^{\otimes m} \iota^{\otimes m} \overline{\Delta}^{\otimes m}.$$

In particular, the map

$$\mathrm{Hom}_{\mathrm{coalg}}(C, T^c(B)) \to \mathrm{Hom}(\overline{C}, B)$$

$$\varphi \mapsto \varphi^1 \circ i,$$

is an isomorphism, natural in C and B.

(2) *Let $\partial : T^{\mathsf{co}}(B) \to T^{\mathsf{co}}(B)$ be a graded coderivation. For every $n \geq 1$ and $1 \leq i \leq n$, there is an equality*

$$\partial^{n-i+1}_n = \sum_{j=0}^{n-i} 1^{\otimes j} \otimes \partial^1_i \otimes 1^{\otimes n-j-i} : B^{\otimes n} \to B^{\otimes n-i+1}.$$

In particular, the map

$$\mathrm{Coder}(T^c(B)) \to \mathrm{Hom}(\overline{T}^{\mathsf{co}}(B), B)$$

$$\partial \mapsto \partial^1 \circ \iota$$

is an isomorphism, natural in B, where $\iota : \overline{T}^{\mathsf{co}}(B) \to T^{\mathsf{co}}(B)$ is inclusion.

2.2. A_∞**-algebras and curved coalgebras.** In this subsection we recall the definitions of A_∞-algebra and morphism, and give details on the idea of Positselski that a curvature term can compensate for a lack of augmentation.

Some context for this technical section is the following. It is important that our constuctions are strictly unital. If we want to e.g. construct a strictly unital morphism between *augmented* A_∞-algebras A, B, we can construct a non-unital morphism $\overline{A} = A/k \cdot 1_A \to \overline{B}$ and formally extend it to a morphism $A \to B$. This works since we have lost no information by passing from A to $\overline{A}$ (by the definition of augmentation). When we work with non-augmented algebras, we also pass to $\overline{A}$, but we have to keep track of the curvature maps. This is notationally and conceptually a bit burdensome, as we are carrying around a possibly infinite number of ideals of k.

Definition 2.2.1. An A_∞*-algebra* (A, m) is a graded module A and a degree -1 map $m : \overline{T}^{\mathsf{co}}(A[1]) \to A[1]$ such that the induced coderivation $\partial : T^c(A[1]) \to T^c(A[1])$ satisfies $d^2 = 0$. In this case $(T^{\mathsf{co}}(A[1]), \partial)$ is a coaugmented dg-coalgebra (a dg-coalgebra is coaugmented if the underlying graded coalgebra is coaugmented and $\partial(1) = 0$). An A_∞*-morphism* $A \to B$ is a morphism of coaugmented dg-coalgebras $(T^c(A[1]), \partial) \to (T^c(B[1]), \partial)$.

Remark 2.2.2. We can unpack these definitions using the notation of 2.1.7. For $m : \overline{T}^{\mathsf{co}}(A[1]) \to A[1]$ a degree -1 linear map, let $\partial : T^{\mathsf{co}}(A[1]) \to T^{\mathsf{co}}(A[1])$ be the corresponding coderivation. We set $m_n = p_1 m i_n : A[1]^{\otimes n} \to A[1]$. The restriction of ∂ to $A[1]^{\otimes n}$ is $\sum_{i=1}^n \partial_n^{n-i+1}$, with

$$\partial_n^{n-i+1} = \sum_{j=0}^{n-i} 1^{\otimes j} \otimes m_i \otimes 1^{\otimes n-i-j} : A[1]^{\otimes n} \to A[1]^{\otimes n-i+1}.$$

(Note that $m_n = \partial_n^1$). By definition, m is an A_∞-structure when $\partial^2 = 0$, and this is equivalent to $m\partial = 0$, i.e. for all $n \geq 1$,

$$\sum_{i=1}^{n} \sum_{j=0}^{n-i} m_{n-i+1}(1^{\otimes j} \otimes m_i \otimes 1^{\otimes n-i-j}) = 0. \tag{2.2.3}$$

Similarly, a map of graded coalgebras $T^{\mathsf{co}}(A[1]) \to T^{\mathsf{co}}(B[1])$ is determined by its post-composition with projection $p : T^{\mathsf{co}}(B[1]) \to B[1]$, by 2.1.8.(1). Given a map of that form, $\alpha : T^{\mathsf{co}}(A[1]) \to B[1]$, write the components $\alpha_n : A[1]^{\otimes n} \to B[1]$ for $n \geq 1$ ($\alpha_0 = 0$ since α is coaugmented). The map of graded coalgebras determined by α is a map of dg-coalgebras if and only if

$$\sum_{i=1}^{n} \alpha_{n-i+1} \left(\sum_{j=0}^{n-i} 1^{\otimes j} \otimes m_i^A \otimes 1^{\otimes n-i-j} \right) = \sum_{i=1}^{n} m_i^B \left(\sum_{\substack{j_1+\ldots+j_i=n \\ j_k \geq 1}} \alpha_{j_1} \otimes \ldots \otimes \alpha_{j_i} \right)$$

for all $n \geq 1$. In this case, we abuse language and say (α_n) is a a morphism of A_∞-algebras. The morphism is *strict* if $\alpha_n = 0$ for $n \geq 2$.

Example 2.2.4. There is a fully faithful functor from the category of dg-algebras over k to the category of A_∞-algebras over k whose image is the subcategory with objects (A, m^A) such that $m_n^A = 0$ for $n \geq 3$ and morphisms strict morphisms of A_∞-algebras.

In the following, and throughout, we will use the standard notation $[a_1|\dots|a_n]$ for $s(a_1)\otimes\dots\otimes s(a_n)\in A[1]^{\otimes n}$ (where $s: A\to A[1]$ is the degree -1 suspension map).

Definition 2.2.5. Let $m^A : \overline{T^{\mathsf{co}}(A[1])} \to A[1]$ be an A_∞-algebra. An element 1_A of A_0 is a *strict unit* for A if

(1) for every $a\in A$
$$m_2^A([a|1_A]) = (-1)^{|a|+1}[a] \quad \text{and} \quad m_2^A([1_A|a]) = -[a],$$
(2) $m_1^A(1_A)=0$ and for every sequence of elements $a_1,\dots,a_n$ of A, with $n\ge 2$,
$$m_{n+1}^A([a_1|\dots|a_{i-1}|1_A|a_{i+1}|\dots|a_n]) = 0$$
for all $1\le i\le n$.

A strict unit is clearly unique if it exists, in which case we say A is strictly unital. A morphism between strictly unital A_∞-algebras $(\alpha_n): A\to B$ is *strictly unital* if $\alpha_1(1_A)=1_B$ and
$$\alpha_n([a_1|\dots|1_A|\dots|a_n]) = 0 \text{ for all } n\ge 2.$$

Remark 2.2.6. There is a choice being made in the signs for strict unit. Indeed, we are implicitly assuming that there is a multiplication $\widetilde{m}_2 : A\otimes A\to A$, related to m_2 by the following commutative diagram
$$\begin{tikzcd} A[1]\otimes A[1] \arrow[r, "m_2"] \arrow[d, "s^{-1}\otimes s^{-1}"'] & A[1] \\ A\otimes A \arrow[r, "\widetilde{m}_2"'] & A, \arrow[u, "s"'] \end{tikzcd}$$
such that $\widetilde{m}_2(a\otimes 1_A) = a = \widetilde{m}_2(1_A\otimes a)$. With this convention, the diagram
$$\begin{tikzcd} A[1]\otimes A[1] \arrow[r, "m_2"] & A[1] \arrow[d, "s^{-1}"] \\ A\otimes A \arrow[r, "\widetilde{m}_2"'] \arrow[u, "s\otimes s"] & A, \end{tikzcd}$$
does *not* commute. One could also work with this convention.

Definition 2.2.7. Let A be an A_∞-algebra with strict unit 1_A.

(1) A linear map $v : A[1]\to k[1]$ is a *split unit* of A if it splits the inclusion $\eta : k\cdot 1_A[1]\hookrightarrow A[1]$.
(2) A split unit v is an *augmentation* of A if it is a strict, strictly unital A_∞-morphism, i.e. $vm_n^A = 0$ for all $n\ne 2$, and $vm_2^A - m_2^k(v\otimes v) = 0$, where $m_2^k : k[1]\otimes k[1]\to k[1]$ sends $[a|b]$ to $-[ab]$ (the sign is due to the convention explained in the previous remark).

If A has a strict unit, then a split unit is a mild extra condition. For instance, if k is a field or A_0 is a rank one free module, then A has a split unit. More generally, if A is any graded k-module, a *marked point* is an element $1_A\in A_0$ and a linear map $v: A[1]\to k[1]$ splitting the inclusion $k\cdot 1_A[1]\hookrightarrow A[1]$. We write $\overline{A} = \ker v$ and let

$$0 \longrightarrow k[1] \underset{\eta}{\overset{v}{\rightleftarrows}} A[1] \underset{p}{\overset{b}{\rightleftarrows}} \overline{A}[1] \longrightarrow 0 \tag{2.2.8}$$

be the corresponding split short exact sequence of graded k-modules.

An augmentation is a much more restrictive assumption – it allows one to pass from A to the non-unital object $\overline{A}$ without losing information. Let us expand upon this point. If A has a split unit, then each multiplication map m_n is determined by the map $\overline{A}[1]^{\otimes n} \hookrightarrow A[1]^{\otimes n} \to A[1]$, using that 1_A is a strict unit. If the split unit v is an augmentation, then the maps

$$\overline{A}[1]^{\otimes n} \hookrightarrow A[1]^{\otimes n} \xrightarrow{m_n} A[1] \xrightarrow{v} k[1] \tag{2.2.9}$$

are zero, and thus m_n is completely determined by

$$\overline{A}[1]^{\otimes n} \hookrightarrow A[1]^{\otimes n} \xrightarrow{m_n} A[1] \twoheadrightarrow \overline{A}[1]. \tag{2.2.10}$$

Moreover, the coderivation on $T^{\mathsf{co}}(\overline{A}[1])$ induced by the maps (2.2.10) is a differential (this is contained in 2.2.17 below), thus when v is an augmentation, we can replace the dg-coalgebra $T^{\mathsf{co}}(A[1])$ by $T^{\mathsf{co}}(\overline{A}[1])$.

When v is not an augmentation, then by definition we lose information by passing to $T^{\mathsf{co}}(\overline{A}[1])$, and even more, the induced coderivation may not square to zero, as the following example shows.

Example 2.2.11. Let A be a dg-algebra with $A_i = 0$ for $i < 0$, $A_0 = k$, and strict unit $1_A = 1_k \in A_0$. (For instance, let A be the Koszul complex of a map $l : V \to k$.) Let $v : A[1] \to k[1]$ be projection onto A_0. Then A is augmented if and only if v is an augmentation if and only if $(m_1^A)_1 : A_1 \to A_0$ is the zero map.

go over this example closely

Set $\overline{m}_1 = (m_1^A)_{\geq 2} : \overline{A}[1] \to \overline{A}[1]$ and $\overline{m}_2 = m_2^A|_{\overline{A}[1]\otimes\overline{A}[1]} : \overline{A}[1] \otimes \overline{A}[1] \to \overline{A}[1]$. Note that for degree reasons the image of $\overline{m}_2$ is contained in $\overline{A}[1]$. Let $\overline{\partial}$ be the coderivation of $T^c(\overline{A}[1])$ defined by these maps, and let $x \in A_1$ and $y \in \overline{A}$. Then

$$\begin{aligned}\overline{\partial}[x|y] &= [\overline{m}_1(x)|y] + [x|\overline{m}_1(y)] + \overline{m}_2[x|y] \\ &= [x|m_1(y)] + m_2[x|y].\end{aligned}$$

Applying $\overline{\partial}$ again gives

$$(\overline{\partial})^2[x|y] = m_2[x|m_1(y)] + m_1m_2[x|y].$$

Using (2.2.3), this is $-m_2[m_1(x)|y] = -m_2[m_1(x) \cdot 1_A|y] = m_1(x)y$. In particular, if $(m_1)_1 \neq 0$, then $\overline{\partial}^2 \neq 0$.

Positselski connects the non-triviality of the maps (2.2.9) to the non-triviality of the square of the coderivation on $T^{\mathsf{co}}(\overline{A}[1])$ resulting from the maps (2.2.10). Moreover, he sets up a framework to work with such objects and their representations. To recall this, we first need the following definitions.

Definition 2.2.12. Let C be a graded coalgebra. The graded dual, $C^* = \operatorname{Hom}_k(C, k)$, is a graded algebra and C is a graded C^*-bimodule via the action

$$\gamma \cdot x = \gamma(x_{(1)})x_{(2)} \qquad x \cdot \gamma = (-1)^{|\gamma||x_{(1)}|}\gamma(x_{(2)})x_{(1)}$$

for $\gamma \in C^*$ and $x \in C$, with $\Delta(x) = x_{(1)} \otimes x_{(2)} \in C \otimes C$. For such a pair, set should be sign here?

$$[\gamma, x] := \gamma \cdot x - x \cdot \gamma. \tag{2.2.13}$$

Definition 2.2.14. A *curved differential graded coalgebra* (cdgc) is a triple (C, ∂, h) with C a graded coalgebra, ∂ a degree –1 coderivation, and $h : C \to k$ a homogeneous k-linear morphism of degree –2 such that $h\partial = 0$ and

$$\partial^2 = [h, -].$$

A morphism of cdgcs $C \to D$ is a pair (β, a) with $\beta : C \to D$ a degree zero morphism of graded coalgebras and $a : C \to k$ a degree -1 map, called the change of curvature, such that:

$$\partial\beta = \beta(\partial + [a, -]),$$
$$h^D\beta = h^C + a\partial - a^2.$$

The composition of morphisms $(\beta, a) : C \to D$ and $(\gamma, b) : D \to E$ is $(\gamma\beta, b\beta + a)$.

Remark 2.2.15. Can decompose any cdgc morphism as isomorphism followed by a morphism with zero change of curvature...

The name curvature is due to the following example.

Example 2.2.16. Let X be a smooth manifold...

Let us now make explicit how curved coalgebras are related to failure of v to be an augmentation.

Theorem 2.2.17. *Let A be a graded module with a marked point $1_A \in A_0$. Consider the splitting of graded k-modules,*

$$\begin{tikzcd} \overline{T}^{\mathsf{co}}(A[1]) \arrow[r, shift right, "\tilde{p}"'] & \overline{T}^{\mathsf{co}}(\overline{A}[1]), \arrow[l, shift right, "\tilde{b}"'] \end{tikzcd}$$

where $\tilde{p} = \overline{T}^{\mathsf{co}}(p), \tilde{b} = \overline{T}^{\mathsf{co}}(b)$, and p, b are the splitting maps of (2.2.8).

(1) *Let $m : \overline{T}^{\mathsf{co}}(A[1]) \to A[1]$ be an A_∞-algebra structure on A with v a split unit. Define maps $\overline{m}, h$ so that the following commute:*

$$\begin{tikzcd} & \overline{T}^{\mathsf{co}}(\overline{A}[1]) \arrow[dl, dashed, "h"'] \arrow[d, "m\tilde{b}"] \arrow[dr, dashed, "\overline{m}"] & \\ k[1] & A[1] \arrow[l, "v"] \arrow[r, "p"'] & \overline{A}[1]. \end{tikzcd}$$

and let $\overline{\partial}$ be the coderivation of $T^{\mathsf{co}}(\overline{A}[1])$ induced by $\overline{m}$. Then

$$(T^{\mathsf{co}}(\overline{A}[1]), \overline{\partial}, s^{-1}h) \textit{ is a curved dg-coalgebra,}$$

and v is an augmentation if and only if $h = 0$.

Conversely, given a cdg coalgebra $(T^{\mathsf{co}}(\overline{A}[1]), \overline{\partial}, s^{-1}h)$, there exists a unique A_∞-algebra structure, with v a split unit, on A such that the diagram above commutes.

(2) *Let A, B be A_∞-algebras with split units and let $\alpha : \overline{T}^{\mathsf{co}}(A[1]) \to B[1]$ be a strictly unital A_∞-morphism. Define k-linear maps $\overline{\alpha}, a$:*

$$\begin{tikzcd} & \overline{T}^{\mathsf{co}}(\overline{A}[1]) \arrow[dl, dashed, "a"'] \arrow[d, "\alpha\tilde{b}"] \arrow[dr, dashed, "\overline{\alpha}"] & \\ k[1] & B[1] \arrow[l, "v"] \arrow[r, "p"'] & \overline{B}[1], \end{tikzcd}$$

and let $\overline{\beta} : T^{\mathsf{co}}(\overline{A}[1]) \to T^{\mathsf{co}}(\overline{B}[1])$ be the graded coalgebra map induced by $\overline{\alpha}$. Then:

$$(\overline{\beta}, s^{-1}a) \textit{ is a morphism of curved dg-coalgebras.}$$

Conversely, given a morphism of cdg coalgebras $(\bar{\beta}, s^{-1}a) : T^{\mathsf{co}}(\overline{A}[1]) \to T^{\mathsf{co}}(\overline{B}[1])$*, there exists a unique strictly unital* A_∞*-morphism* $\alpha : T^{\mathsf{co}}(A[1]) \to B[1]$ *such that the above diagram is commutative.*

This result is due to Positselski. See [17, pp. 81-82] for the statement and a sketch of a proof. We give a detailed proof in 9.1.

Remark. By (REF LH, Positselski), when k is a field there is a Quillen equivalence between the category of augmented dg k-algebras and the category of coaugmented dg k-coalgebras. While we don't attempt to construct model structures here, the above result roughly shows we can extend this equivalence between dg k-algebras with split unit and curved dg k-coalgebras.

Definition 2.2.18. If A is an A_∞-algebra with split unit, define the following cdgc

$$\operatorname{Bar} A = (T^c(\overline{A}[1]), \bar{d}, h).$$

Example 2.2.19. Let A be the Koszul complex on a single element f of k, so

$$A = 0 \to ke \xrightarrow{f} k1_A \to 0.$$

This is a dga by setting $e^2 = 0$. We have

$$\operatorname{Bar} A = (\ldots \to 0 \to k\{se \otimes se\} \to 0 \to k\{se\} \to 0 \to k \to 0, 0, \tilde{f})$$

with $\tilde{f}((se)^{\otimes n})$ equal to f when $n = 1$ and zero otherwise.

Remark 2.2.20. Conversely, given maps $\overline{m}$ and h, set

$$m_n^A b^{\otimes n} = b\overline{m}_n^A + \eta_A s h_n.$$

(If we hope to make a strictly unital A_∞-algebra, this will determine m_n^A since the strict unit determines m_n^A on the kernel of $p^{\otimes n} : A[1]^{\otimes n} \to \overline{A}[1]^{\otimes n}$.)

$$\overline{m}_n^A = p m_n^A b^{\otimes n} : \overline{A}[1]^{\otimes n} \to \overline{A}[1]$$
$$h_n = s^{-1} v m_n^A b^{\otimes n} : \overline{A}[1]^{\otimes n} \to k,$$

We record here that $(T^c(\overline{A}[1]), \bar{d}, h)$ is a cdgc if and only if the following hold for all $n \geq 1$:

$$\sum_{i=1}^{n} \sum_{j=0}^{n-i} \overline{m}_{n-i+1}^A (1^{\otimes j} \otimes \overline{m}_i^A \otimes 1^{\otimes n-i-j}) = h_{n-1} \otimes 1 - 1 \otimes h_{n-1},$$

$$\sum_{i=1}^{n} \sum_{j=0}^{n-i} \bar{h}_{n-i+1} (1^{\otimes j} \otimes \overline{m}_i^A \otimes 1^{\otimes n-i-j}) = 0.$$

2.3. **Obstruction theory.** In this subsection we set up some technical tools we will need to inductively construct A_∞-algebras and morphisms. These tools were first used by Lefévre-Hasegawa [13, Appendix B]. We adapt them here to handle strict units of non-augmented A_∞-algebras.

For a graded module B, set $T_n^c(B) = \bigoplus_{i=0}^n B^{\otimes i}$. Analogously to $T^c(B)$, this is a graded coalgebra and coderivations of $T_n^c(B)$ correspond to homogeneous maps $\overline{T_n^c}(B) = \bigoplus_{i=1}^n B^{\otimes i} \to B$.

Definition 2.3.1. Let A be a graded module. An *A_n-algebra structure on A* is a degree -1 map $m|_n : \overline{T_n^c}(A[1]) \to A[1]$ such that the induced coderivation $d|_n$ of $T_n^c(A[1])$ satisfies $(d|_n)^2 = 0$; A has a split unit if the analogous conditions for a split unit in an A_∞-algebra hold.

Let $n \geq 1$ and assume that A is an A_n-algebra. Consider the Hom-complex $\mathrm{Hom}(A[1]^{\otimes n+1}, A[1])$ between the complexes $(A[1], m_1)$ and $(A[1]^{\otimes n+1}, m_1^{(n+1)})$, where $m_1^{(n+1)} := m_1 \otimes 1^{\otimes n} + 1 \otimes m_1 \otimes 1^{\otimes n-1} + \ldots + 1^{\otimes n} \otimes m_1$. Define

$$c(m|_n) = \sum_{i=2}^{n} \sum_{j=0}^{n-i+1} m_{n-i+2}(1^{\otimes j} \otimes m_i \otimes 1^{\otimes n-i-j+1}). \tag{2.3.2}$$

By [13, B.1.2], $c(m|_n)$ is a cycle in $\mathrm{Hom}(A[1]^{\otimes n+1}, A[1])$ and a degree -1 map $m_{n+1} \in \mathrm{Hom}(A[1]^{\otimes n+1}, A[1])$ extends $m|_n$ to an A_{n+1}-structure on A if and only if

$$d(m_{n+1}^A) + c(m^A|_n) = 0 \in \mathrm{Hom}(A[1]^{\otimes n+1}, A[1]).$$

If A is an A_n-algebra with a split unit, we need to be able to determine when an extension m_{n+1} preserves the split unit. If A were augmented, we could put an A_{n+1} structure on $\overline{A}$ and then formally extend it to an A_{n+1} structure on A with a strict unit. In general, we need to include the map h in the definition of c as follows:

Proposition 2.3.3. *Let $m|_n : \overline{T_n^c}(A[1]) \to A[1]$ be an A_n-structure, and assume A has a split unit v. Set $\overline{A} = A/k \cdot 1_A$ and let $b : \overline{A} \to A$ be the splitting induced by v. Let $\overline{m}_i$ and h_i be the maps constructed from m_i in 2.2.17.*

The map

$$\bar{c}(m|_n) := \sum_{i=2}^{n} \sum_{j=0}^{n-i+1} m_{n-i+2}^A b^{\otimes n-i+2}(1^{\otimes j} \otimes \overline{m}_i^A \otimes 1^{\otimes n-i-j+1}) - h_n \otimes b + b \otimes h_n$$

is a degree –2 cycle in $\mathrm{Hom}(\overline{A}[1]^{\otimes n+1}, A[1])$, where the differential is the Hom-complex differential and we view $\overline{A}[1]^{\otimes n+1}$ as a complex with differential $\overline{m}_1^{(n+1)}$.

A degree –1 map $\widetilde{m}_{n+1} \in \mathrm{Hom}(\overline{A}[1]^{\otimes n+1}, A[1])$ extends $m|_n$ to an A_{n+1}-structure in which v is a split unit ($\widetilde{m}_{n+1}$ will equal $m_{n+1}b^{\otimes n+1}$, which determines m_{n+1}) if and only if

$$d(\widetilde{m}_{n+1}) + \bar{c}(m|_n) = 0.$$

The proof is given in 9.3.

Definition 2.3.4. Let A and B be A_n-algebras. An *A_n-morphism* is a morphism of coaugmented dg-coalgebras $T_n^c(A[1]) \to T_n^c(B[1])$.

As for A_∞-algebras, this is equivalent to linear maps

$$\alpha_i : A[1]^{\otimes i} \to B[1]$$

for $i = 1, \ldots, n$, such that

$$\sum_{i=1}^{l} \alpha_{l-i+1} \left(\sum_{j=0}^{l-i} 1^{\otimes j} \otimes m_i^A \otimes 1^{\otimes l-i-j} \right) = \sum_{i=1}^{l} m_i^B \left(\sum_{\substack{j_1+\ldots+j_i=l \\ j_i \geq 1}} \alpha_{j_1} \otimes \ldots \otimes \alpha_{j_i} \right)$$

for $l = 1, \dots, n$. If A, B are strictly unital, then α is strictly unital if the analogous conditions as for an A_∞-morphism hold.

Assume that A, B are A_{n+1}-algebras and $\alpha|_n : T^c_n(A[1]) \to B[1]$ is an A_n morphism. Define

$$c(\alpha|_n) = \sum_{i=2}^{n+1} \alpha_{n-i+2}\left(\sum_{j=0}^{n-i+1} 1^{\otimes j} \otimes m^A_i \otimes 1^{\otimes n-i-j+1}\right) - \sum_{i=2}^{n+1} m^B_i \left(\sum_{\substack{j_1+\ldots+j_i=n+1\\ j_k\geq 1}} \alpha_{j_1}\otimes \ldots \otimes \alpha_{j_i}\right).$$

By [13, B.1.5], $c(\alpha|_n)$ is a cycle in $\mathrm{Hom}(A[1]^{\otimes n+1}, B[1])$ and $\alpha_{n+1} : A[1]^{\otimes n+1} \to B[1]$ extends $\alpha|_n$ to an A_{n+1}-morphism if and only if $d(\alpha_{n+1}) + r(\alpha|_n) = 0$.

As above, we need to modify this to take split units into account.

Proposition 2.3.5. *Let A, B be strictly unital A_{n+1}-algebras, and assume that A has a split unit v. Set $\overline{A} = A/k \cdot 1_A$.*

Let $\alpha|_n : T^c_n(\overline{A}[1]) \to B[1]$ be a strictly unital A_n morphism with components $\alpha_i : \overline{A}[1]^{\otimes i} \to B[1]$. Set $\widetilde{\alpha}_i = \alpha_i b^{\otimes i} : \overline{A}[1]^{\otimes i} \to B[1]$. Then

$$\bar{c}(\widetilde{\alpha}|_n) := \sum_{i=2}^{n+1} \widetilde{\alpha}_{n-i+2}\left(\sum_{j=0}^{n-i+1} 1^{\otimes j} \otimes \overline{m}^A_i \otimes 1^{\otimes n-i-j+1}\right)$$
$$- \sum_{i=2}^{n+1} m^B_i \left(\sum_{\substack{j_1+\ldots+j_i=n+1\\ j_k\geq 1}} \widetilde{\alpha}_{j_1}\otimes \ldots \otimes \widetilde{\alpha}_{j_i}\right) + \eta_B s h^A_{n+1}$$

is a degree –1 cycle in $\mathrm{Hom}(\overline{A}[1]^{\otimes n+1}, B[1])$. A degree zero map

$$\widetilde{\alpha}_{n+1} \in \mathrm{Hom}(\overline{A}[1]^{\otimes n+1}, B[1])$$

extends $\alpha|_n$ to a strictly unital A_{n+1}-morphism if and only

$$d(\widetilde{\alpha}_{n+1}) + \bar{c}(\widetilde{\alpha}|_n) = 0.$$

The proof is similar to the previous one.

Definition 2.3.6.

(1) Let $\alpha, \beta : C \to D$ be degree zero morphisms of graded coalgebras. A degree –1 linear map $r : C \to D$ is a (α, β)-coderivation if
$$\Delta_D r = (\alpha \otimes r + r \otimes \beta)\Delta_C.$$
(2) If C, D are dg-coalgebras, the morphisms $\alpha, \beta : C \to D$ are *(coderivation) homotopic* if there exists an (α, β)-coderivation r such that $d_{\mathrm{Hom}}(r) = \alpha - \beta$.
(3) Let A, B be A_n-algebras. Two morphisms of A_n-algebras $\alpha, \beta : T^c_n(A[1]) \to T^c_n(B[1])$ are *homotopic* if they are homotopic as morphisms of dg-coalgebras.

An (α, β)-coderivation $T^c_n(A[1]) \to T^c_n(B[1])$ is determined by the induced map $\overline{T}^c_n(A[1]) \to B[1]$. Given any linear map $r : \overline{T}^c_n(A[1]) \to B[1]$, the corresponding (check this) (α, β)-coderivation restricted to $\overline{A}[1]^{\otimes m}$ is

$$\sum_{\substack{j_1+\ldots+j_i=m\\ j_l\geq 1}} \left(\sum_{k=1}^{i} \alpha_{j_1} \otimes \ldots \otimes \alpha_{j_{k-1}} \otimes r_{j_k} \otimes \beta_{j_{k+1}} \otimes \ldots \otimes \beta_{j_i}\right).$$

The equation $d(r) = \alpha - \beta$ holds if and only if it holds after composing with $p_1 : T^c_n(B[1]) \to B[1]$. The restriction of $p_1 d(r)$ to $A[1]^{\otimes m}$ is

$$\sum_{i=1}^{m}\sum_{j=0}^{i-1} r_i(1^{\otimes j}\otimes m^A_{m-i+1}\otimes 1^{i-j-1}) + \sum_{\substack{j_1+\ldots+j_i=m\\ j_l\geq 1}} m^B_i\left(\sum_{k=1}^{i}\alpha_{j_1}\otimes\ldots\otimes\alpha_{j_{k-1}}\otimes r_{j_k}\otimes\beta_{j_{k+1}}\otimes\ldots\otimes\beta_{j_i}\right).$$

Thus r is a homotopy between α and β if and only if

$$\alpha_m - \beta_m = \sum_{i=1}^{m}\sum_{j=0}^{i-1} r_i(1^{\otimes j}\otimes m^A_{m-i+1}\otimes 1^{i-j-1})$$

$$+\sum_{i=1}^{m} m^B_i\left(\sum_{\substack{j_1+\ldots+j_i=m\\ j_l\geq 1}}\left(\sum_{k=1}^{i}\alpha_{j_1}\otimes\ldots\otimes\alpha_{j_{k-1}}\otimes r_{j_k}\otimes\beta_{j_{k+1}}\otimes\ldots\otimes\beta_{j_i}\right)\right)$$

for $m = 1, \ldots, n$.

The following is similar to the other results in [13, Appendix B], but is not proved there. For completeness we give a proof following the outline of *loc. cit.*

Proposition 2.3.7. *Let A, B be A_n-algebras and let $\alpha|_n, \beta|_n : T^c_n(A[1]) \to T^c_n(B[1])$ be A_n-morphisms. Let $r|_{n-1} : T^c_{n-1}(A[1]) \to B[1]$ be a homotopy between $\alpha|_{n-1}$ and $\beta|_{n-1}$. The map*

$$c(r|_{n-1}) := \alpha_n - \beta_n - \sum_{i=1}^{n-1}\sum_{j=0}^{i-1} r_i(1^{\otimes j}\otimes m^A_{n-i+1}\otimes 1^{i-j-1})$$

$$-\sum_{i=2}^{n} m^B_i\left(\sum_{\substack{j_1+\ldots+j_i=n\\ j_l\geq 1}}\left(\sum_{k=1}^{i}\alpha_{j_1}\otimes\ldots\otimes\alpha_{j_{k-1}}\otimes r_{j_k}\otimes\beta_{j_{k+1}}\otimes\ldots\otimes\beta_{j_i}\right)\right)$$

is a cycle in $\mathrm{Hom}(A[1]^{\otimes n}, B[1])$. *A map r_n extends $r|_{n-1}$ to a homotopy between $\alpha|_n$ and $\beta|_n$ if and only if $d(r_n) + c(r|_{n-1}) = 0$.*

Proof. Let r_n be any map of degree –1 in $\mathrm{Hom}(A[1]^{\otimes n}, B[1])$. Let $r : T^c_n(A[1]) \to T^c_n(B[1])$ be the (α, β)-coderivation induced by $r|_{n-1}$ and r_n. Set

$$\gamma = \alpha|_n - \beta|_n - d(r) : T^c_n(A[1]) \to T^c_n(B[1]).$$

By assumption, we have that γ is zero when restricted $T^c_{n-1}(A[1])$, and thus factors through the projection $p_n : T^c_n(A[1]) \twoheadrightarrow A[1]^{\otimes n}$. Also by construction and assumption, the image of this map is contained in $B[1]$. Thus we have the following diagram:

$$\begin{tikzcd} T^c_n(A[1]) \arrow[r, "\gamma"] \arrow[d, "p_n"'] & T^c_n(B[1]) \\ A[1]^{\otimes n} \arrow[r, "\gamma_n"] & B[1]. \arrow[u, "i_1"'] \end{tikzcd}$$

By definition of γ, we have

$$\gamma_n = r_n m^{(n)}_1 + c(r|_{n-1}) + m^B_1 r_n.$$

Let d^B be the coderivation of $T^c_n(B[1])$ giving the A_n-structure on B. We have

$$i_1 m_1^B \gamma_n p_n = d^B i_1 \gamma_n p_n = d^B \gamma.$$

We also have

$$d^B \gamma = d^B(\alpha - \beta - d(r)) = (\alpha - \beta - d(r))d^A = \gamma d^A.$$

Here we are using that $d^2(r) = 0$, and so $d^B d(r) = d(r)d^A$. Since $\ker \gamma$ contains $T^c_{n-1}(A[1])$, it follows that $\gamma d^A = i_1 \gamma_n m_1^{(n)} p_n$. We have shown that

$$i_1 \overline{m}_1^B \gamma_n p_n = i_1 \gamma_n m_1^{(n)} p_n$$

and thus that

$$m_1^B \gamma_n - \gamma_n m_1^{(n)} = 0,$$

which shows that γ_n is a cycle in $\mathrm{Hom}(A[1]^{\otimes n}, B[1])$. Since $\gamma_n - c(r|_{n-1}) = r_n m_1^{(n)} + m_1^B r_n$ is a boundary, hence a cycle, $c(r|_{n-1})$ must also be a cycle.

Finally, we have that r is a homotopy between α and β if and only if $\gamma = 0$, and this happens if and only if $\gamma_n = 0$ if and only if $d_{\mathrm{Hom}}(r_n) + c(r|_{n-1}) = 0$. □

A homotopy r between strictly unital morphisms is strictly unital if r_i is zero on any element containing 1_A for $i \geq 1$.

Proposition 2.3.8. *Let A, B be strictly unital A_n-algebras, A with split unit, α, β strictly unital A_n-morphisms and $r|_{n-1} : T^c_{n-1}(A[1]) \to B[1]$ a strictly unital homotopy between $\alpha|_{n-1}$ and $\beta|_{n-1}$. Let $\widetilde{r} = rb$, and define $\widetilde{\alpha}, \widetilde{\beta}$ analogously. The map*

$$\bar{c}(\widetilde{r}|_{n-1}) := \widetilde{\alpha}_n - \widetilde{\beta}_n - \sum_{i=1}^{n-1} \sum_{j=0}^{i-1} \widetilde{r}_i (1^{\otimes j} \otimes \overline{m}^A_{n-i+1} \otimes 1^{i-j-1})$$
$$- \sum_{i=2}^{n} m_i^B \left(\sum_{\substack{j_1 + \ldots + j_i = n \\ j_l \geq 1}} \left(\sum_{k=1}^{i} \widetilde{\alpha}_{j_1} \otimes \ldots \otimes \widetilde{\alpha}_{j_{k-1}} \otimes \widetilde{r}_{j_k} \otimes \widetilde{\beta}_{j_{k+1}} \otimes \ldots \otimes \widetilde{\beta}_{j_i} \right) \right)$$

is a cycle in $\mathrm{Hom}(\overline{A}[1]^{\otimes n}, B[1])$. An element $\widetilde{r}_n$ extends $\widetilde{r}|_{n-1}$ to a strictly unital homotopy between α and β if and only if $d(\widetilde{r}_n) + \bar{c}(\widetilde{r}|_{n-1}) = 0$.

This is a similar (but easier) adjustment as in the proof 9.1: one checks that $c(r|_{n-1})$ is zero on any element containing 1_A, and that $\bar{c}$ is the formula for cb.

2.4. **Transfer of A_∞-structures to resolutions.** Let B be a strictly unital A_∞-algebra. We show here that if $A \xrightarrow{\pi} B$ is a semiprojective resolution, a.k.a. cofibrant replacement, of the complex (B, m_1), then A has the structure of a strictly unital A_∞-algebra such that π is a strict morphism of A_∞-algebras. Classicaly, transfer results of this type were done between homotopy equivalences using the homotopy perturbation lemma, see e.g. [9, 11]. However this is not appicable in this situation: there may not even be a k-linear map from B to A. should be more transition

We first recall some homological algebra of complexes. If P is a complex of k-modules, we view $\mathrm{Hom}(P, -)$ as an endo-functor on the category of k-complexes.

Definition 2.4.1. A complex of k-modules P is *semiprojective* if $\mathrm{Hom}(P, -)$ preserves surjective quasi-isomorphisms. A *semiprojective resolution* of a complex M is a quasi-isomorphism $pM \to M$ with pM semiprojective.

Semiprojective complexes are the cofibrant objects in the projective model structure on the category of k-complexes [10, 2.3]. A large source of examples is the following: if P is a complex with P_n projective for all n and zero for $n \ll 0$, then P is semiprojective. In particular a k-free resolution of a k-module is semiprojective. Semifree resolutions, of which semiprojective are summands, were first defined in [3]. Semiprojective complexes are called K-projective complexes of projectives in [20] and cell k-modules in [12].

Note that if k is a field, then $\operatorname{Hom}(P, -)$ is exact for every complex, so every complex is semiprojective.

We need the following properties. For proofs see [1] or [10].

2.4.2. Let P, Q be semiprojective complexes.

(1) P_n is projective for all n;
(2) $\operatorname{Hom}(P, -)$ preserves quasi-isomorphisms;
(3) if $P \to Q$ is a quasi-isomorphism, then it is a homotopy equivalence;
(4) every complex has a surjective semiprojective resolution.

Definition 2.4.3. If A, B are A_∞-algebras with split units, two morphisms from A to B are *homotopic* if the corresponding morphisms of dg-coalgebras $T^c(A[1]) \to T^c(B[1])$ are homotopic as defined in 2.3.6.(1). A homotopy is strictly unital if it is zero on any term involving 1_A.

Definition 2.4.4. A *split unit* for a complex (A, d^A) is an element $1_A \in A_0$ such that $k \cdot 1_A \to A$ is split over k and such that $d^A(1_A) = 0$.

Theorem 2.4.5.

(1) *Let B be an A_∞-algebra with strict unit 1_B. Let*

$$\pi : A \to B$$

be a surjective semiprojective resolution of the complex (B, m_1). Assume that the complex A has a split unit 1_A such that $\pi(1_A) = 1_B$. Let $v : A[1] \to k[1]$ be a k-linear splitting of the inclusion $k \cdot [1_A] \to A[1]$. Then A has the structure of an A_∞-algebra with split unit v such that π is a strict morphism of A_∞-algebras.

(2) *Let A' be an A_∞-algebra with split unit such that (A', m_1) is semiprojective. Let A, B be arbitrary strictly unital A_∞-algebras and $\pi : A \to B$ a strict morphism such that $\pi = \pi_1$ is a surjective quasi-isomorphism. Then for any strictly unital $\alpha : A' \to B$, there exists a strictly unital $\beta : A' \to A$ such that $\pi\alpha = \beta$:*

$$\begin{tikzcd} & A \arrow[d, "\pi"] \\ A' \arrow[ur, dashed, "\beta"] \arrow[r, "\alpha"'] & B. \end{tikzcd}$$

If, for all $n \geq 2$,

$$H_1 \operatorname{Hom}(\overline{A}'[1]^{\otimes n}, A[1]) \cong H_n \operatorname{Hom}((\overline{A}')^{\otimes n}, A) = 0,$$

then any two such liftings are homotopic by a strictly unital homotopy.

Proof. We first prove part 1. Assume by induction that there is an A_n-structure on A in which 1_A is a split unit and such that the following diagram is commutative

$$\begin{tikzcd} A[1]^{\otimes i} \arrow[r, "m_i^A"] \arrow[d, "\pi^{\otimes i}"'] & A[1] \arrow[d, "\pi"] \\ B[1]^{\otimes i} \arrow[r, "m_i^B"'] & B[1] \end{tikzcd} \tag{2.4.6}$$

for $i = 1, \ldots, n$. This holds for $n = 1$ by definition.

We now use 2.3.3 to construct m_{n+1}^A such that the above holds for $n+1$. Since A and $k \cdot 1_A$ are semiprojective, $\overline{A} = A/k \cdot 1_A$ is semiprojective, and thus $\overline{A}[1]^{\otimes n+1}$ is also semiprojective. Since π is a surjective quasi-isomorphism,

$$\varphi := \mathrm{Hom}(\overline{A}[1]^{\otimes n+1}, \pi) : \mathrm{Hom}(\overline{A}[1]^{\otimes n+1}, A[1]) \xrightarrow{\simeq} \mathrm{Hom}(\overline{A}[1]^{\otimes n+1}, B[1])$$

is also a surjective quasi-isomorphism. Set

$$\phi := \mathrm{Hom}(\pi^{\otimes n+1} b^{\otimes n+1}, B[1]) : \mathrm{Hom}(B[1]^{\otimes n+1}, B[1]) \to \mathrm{Hom}(\overline{A}[1]^{\otimes n+1}, B[1]),$$

and

$$c_B := \sum_{i=2}^{n} \sum_{j=0}^{n-i+1} m_{n-i+2}^B (1^{\otimes j} \otimes m_i^B \otimes 1^{\otimes n-i-j+1}) \in \mathrm{Hom}(B[1]^{\otimes n+1}, B[1]).$$

By [13, B.1.2], we have $d(m_{n+1}^B) + c_B = 0$. Let

$$\bar{c}_A = \bar{c}(m|_n) \in \mathrm{Hom}(\overline{A}[1]^{\otimes n+1}, A[1])$$

be the map defined in 2.3.3.

We claim that

$$\varphi(\bar{c}_A) = \phi(c_B).$$

We have

$$\begin{aligned} \phi(c_B) &= \sum_{i=2}^{n} \sum_{j=0}^{n-i+1} m_{n-i+2}^B (1^{\otimes j} \otimes m_i^B \otimes 1^{\otimes n-i-j+1}) \pi^{\otimes n+1} b^{\otimes n+1} \\ &= \sum_{i=2}^{n} \sum_{j=0}^{n-i+1} m_{n-i+2}^B (\pi b)^{\otimes n-i+2} (1^{\otimes j} \otimes \overline{m}_i \otimes 1^{\otimes n-i-j+1}) \\ &\quad + \sum_{i=2}^{n} \sum_{j=0}^{n-i+1} m_{n-i+2}^B \pi^{\otimes n-i+2} (b^{\otimes j} \otimes \eta s h_i \otimes b^{\otimes n-i-j+1}) \\ &= \sum_{i=2}^{n} \sum_{j=0}^{n-i+1} \pi m_{n-i+2}^A b^{\otimes n-i+2} (1^{\otimes j} \otimes \overline{m}_i \otimes 1^{\otimes n-i-j_1}) - h_n \otimes \pi b + \pi b \otimes h_n, \end{aligned}$$

where we have used (2.4.6) and the strict unit of A,

$$= \pi \bar{c}_A = \varphi(\bar{c}_A).$$

Using the surjectivity of φ, pick $m'_{n+1} \in \mathrm{Hom}(\overline{A}[1]^{\otimes n+1}, A[1])$ with $\varphi(m'_{n+1}) = \phi(m_{n+1}^B)$. Then we have

$$\varphi(d(m'_{n+1}) + \bar{c}_A) = d(\varphi(m'_{n+1})) + \varphi(\bar{c}_A) = \phi(d(m_{n+1}^B) + c_B) = \phi(0) = 0.$$

Thus $d(m'_{n+1}) + \bar{c}_A \in \ker\varphi$. By 2.3.3, $\bar{c}_A$ is a cycle and so $d(m'_{n+1}) + \bar{c}_A$ is a cycle. Since φ is a quasi-isomorphism, $\ker\varphi$ is acyclic. So there exists m''_{n+1} in $\ker\varphi$ with

$$d(m''_{n+1}) = d(m'_{n+1}) + \bar{c}_A.$$

Set $\widetilde{m}_{n+1} = m'_{n+1} - m''_{n+1}$ for any such m''_{n+1}. Then we have

$$d(\widetilde{m}_{n+1}) + \bar{c}_A = 0,$$

which by 2.3.3 shows that $\widetilde{m}_{n+1}$ extends $m|_n$ to an A_{n+1}-structure in which 1_A is a strict unit. We also have that

$$\varphi(\widetilde{m}_{n+1}) = \varphi(m'_{n+1} - m''_{n+1}) = \varphi(m'_{n+1}) = \phi(m^B_{n+1}).$$

Coupled with the fact that $\pi(1_A) = 1_B$, this shows that (2.4.6) holds for $i = n+1$.

We now prove part 2. Assume $\widetilde{\alpha} : T^c(\overline{A}'[1]) \to B[1]$ determines a strictly unital morphism. We inductively construct $\widetilde{\beta}_n : \overline{A}'[1]^{\otimes n} \to A[1]$. Since $\overline{A}'[1]$ is semiprojective, and π is a surjective quasi-isomorphism, there exists a morphism of complexes (i.e. an A_1-morphism) $\widetilde{\beta}_1 : \overline{A}'[1] \to A[1]$ such that $\pi\widetilde{\alpha}_1 = \widetilde{\beta}_1$. Assume that we have constructed an A_n-morphism $\widetilde{\beta}|_n : T^c_n(\overline{A}'[1]) \to A[1]$ such that $\pi\widetilde{\beta}|_n = \widetilde{\alpha}|_n$. Set

$$\psi = \operatorname{Hom}(\overline{A}'[1]^{\otimes n+1}, \pi) : \operatorname{Hom}(\overline{A}'[1]^{\otimes n+1}, A[1]) \to \operatorname{Hom}(\overline{A}'[1]^{\otimes n+1}, B[1]).$$

Since $\overline{A}'[1]^{\otimes n+1}$ is semiprojective and π is a surjective quasi-isomorphism, ψ is a surjective quasi-isomorphism. It follows from the induction hypothesis that $\psi(\bar{c}(\widetilde{\beta}|_n)) = \bar{c}(\widetilde{\alpha}|_n)$. We also have that $d(\widetilde{\alpha}_{n+1}) + \bar{c}(\widetilde{\alpha}|_n) = 0$. Using surjectivity of ψ, pick an element $\widetilde{\beta}'_{n+1}$ such that $\psi(\widetilde{\beta}_{n+1}) = \widetilde{\alpha}_{n+1}$. Then we have $\psi(d(\widetilde{\beta}'_{n+1}) + \bar{c}(\widetilde{\beta}|_n)) = 0$. Using 2.3.5, $d(\widetilde{\beta}'_{n+1}) + \bar{c}(\widetilde{\beta}|_n)$ is a cycle. Since ψ is a quasi-isomorphism, $\ker\psi$ is acyclic. Thus, pick $\widetilde{\beta}''_{n+1}$ such that $d(\widetilde{\beta}''_{n+1}) = d(\widetilde{\beta}'_{n+1}) + \bar{c}(\widetilde{\beta}|_n)$. Setting $\widetilde{\beta}_{n+1} = \widetilde{\beta}'_{n+1} - \widetilde{\beta}''_{n+1}$, we see that this extends $\widetilde{\beta}|_n$ to an A_{n+1}-morphism, and $\phi(\widetilde{\beta}_{n+1}) = \widetilde{\alpha}_{n+1}$.

Finally, assume that $\widetilde{\beta}, \widetilde{\gamma} : T^c(\overline{A}'[1]) \to A[1]$ determine strictly unital morphisms and $\pi\widetilde{\beta} = \pi\widetilde{\gamma}$. We construct a strictly unital homotopy $r : T^c(\overline{A}'[1]) \to A[1]$ inductively. In the case $n = 1$, we have $\pi\widetilde{\beta}_1 = \pi\widetilde{\gamma}_1$, and thus $\widetilde{\beta}_1 - \widetilde{\gamma}_1$ is in $\ker\psi$. Since β_1 and γ_1 are morphisms of complexes, $d(\widetilde{\beta}_1) = 0 = d(\widetilde{\gamma}_1)$. Thus $\widetilde{\beta}_1 - \widetilde{\gamma}_1$ is a cycle in $\ker\psi$, and hence a boundary. Let r_1 be some map with $d(r_1) = \widetilde{\beta}_1 - \widetilde{\gamma}_1$. Now, assume by induction that for some $n \geq 2$ we have a homotopy $r|_{n-1} : T^c_{n-1}(\overline{A}'[1]) \to A[1]$ between $\widetilde{\beta}|_{n-1}$ and $\widetilde{\gamma}|_{n-1}$. By 2.3.8, $\bar{c}(r|_{n-1})$ is a degree zero cycle in $\operatorname{Hom}(\overline{A}'[1]^{\otimes n}, A[1]) \cong \operatorname{Hom}((\overline{A}')^{\otimes n}, A)[-n+1]$. By the assumption that $H_{n-1}\operatorname{Hom}((\bar{A}')^{\otimes n}, A) = 0$, we may choose r_n such that $d(r_n) + \bar{c}(r|_{n-1}) = 0$, and hence this extends $r|_{n-1}$ to a homtopy between $\widetilde{\alpha}|_n$ and $\widetilde{\beta}|_n$. □

Note that we do not assume that B has a split unit in the above theorem. If we did, it would shorten the proof considerably. However, a case of interest is the following, where B does not have a split unit. This is studied further in [6].

Example 2.4.7. Let $B = k/I$ be a cyclic k-algebra and let $A \xrightarrow{\cong} B$ be a k-projective resolution with $A_0 = k$. Set $1_A = 1_k \in A_0$. The complex A is semiprojective with split unit $A \twoheadrightarrow A_0$; we have $\overline{A} = A_{\geq 1}$, $m_1 = -d^A$, $h_1 = \overline{A}[1] \twoheadrightarrow A_1[1] \xrightarrow{m_1} k$ and

$h_n = 0$ for $n \neq 1$. To find $\overline{m}_2$, we find a homotopy for the map $h_1 \otimes 1 - 1 \otimes h_1$.

$$\begin{tikzcd}
0 & A_1 \otimes A_1 \arrow[l] \arrow[d, "m_1\otimes 1 - 1 \otimes m_1"'] & {\substack{A_1\otimes A_2 \\ A_2 \otimes A_1}} \arrow[l] \arrow[d, "\begin{pmatrix} m_1 \otimes 1 \\ 1 \otimes m_1 \end{pmatrix}"'] & \dots. \arrow[l] \\
0 & A_1 \arrow[l] & A_2 \arrow[l] & \dots \arrow[l]
\end{tikzcd}$$

Such a homotopy exists by the classical lifting lemma. There is a quasi-isomorphism

$$\operatorname{Hom}(\overline{A}[1]^{\otimes n+1}, A[1]) \xrightarrow{\simeq} \operatorname{Hom}(\overline{A}^{\otimes n+1}, B)[-n].$$

In particular, $H_{-2} \operatorname{Hom}(\overline{A}[1]^{\otimes n+1}, A[1]) \cong H_{n-2} \operatorname{Hom}(\overline{A}^{\otimes n+1}, B) = 0$ for $n \geq 2$ since $\operatorname{Hom}(\overline{A}^{\otimes n+1}, B)$ is concentrated in negative homological degrees. Thus by 2.3.3, given $m|_n$ for $n \geq 2$, we can extend to m_{n+1}.

Finally, since $H_1 \operatorname{Hom}(\overline{A}[1]^{\otimes n+1}, A[1]) = 0$ for all $n \geq 0$, the A_∞-structure on A is unique up to homotopy.

2.5. **Transfer of A_∞-algebra structures by homotopy equivalences.** The following is classical, and one of the motivations for A_∞-algebras. It will let us remove the surjective assumption of 2.4.5. One can use the modified obstruction theory above to modify Lefévre-Hasegawa 's proof.

Theorem 2.5.1. *Let (B, d^B) be a complex with a split unit $1_B \in B_0$. Assume that A is an A_∞-algebra with split unit and $\varphi : B \to A$ is a homotopy equivalence of complexes with $\varphi(1_B) = 1_A$. Then there is an A_∞-structure m^B on B with split unit, such that $m_1^B = d^B$, and there is a morphism of A_∞-algebras $\psi : B \to A$ with $\psi_1 = \varphi$. The morphism ψ is a homotopy equivalence of A_∞-algebras.*

We can remove the surjective assumption from 2.4.5.(1).

Corollary 2.5.2. *Let B be an A_∞-algebra and let $\pi : A \to B$ be a semiprojective resolution of the complex (B, m_1^B). Assume that A has a split unit and $\pi(1_A) = 1_B$. Then there is an A_∞-algebra structure on A and a morphism of A_∞-algebras $\psi : A \to B$ such that $\psi_1 = \pi$.*

Proof. Let $A' \to B$ be a surjective semiprojective resolution of B. Then there is a strictly unital A_∞-algebra structure on A' and a strict morphism $A' \to B$. But by the defining property of semiprojective complexes, A and A' are homotopic by maps that commute with the augmentations to B. Therefore by Theorem 2.5.1, A has a strictly unital A_∞-algebra structure and there is a morphism of A_∞-algebras $A \to A'$ that commutes with the augmentations to B. Therefore we get an A_∞-morphism $A \to B$. □

We emphasize that if $A \to B$ is not surjective, then the morphism $A \to B$ may not be strict.

3. A_∞-MODULES AND CURVED COMODULES

In this section we recall the definition of A_∞-modules, formulated in terms of extended comodule structures over the tensor coalgebra. We state a version of the obstruction theory for modules and use it to prove the analogue of Theorem 2.4.5 for A_∞-modules.

3.1. A_∞**-modules.** Let C be a graded coalgebra with graded coderivation d and N a graded C-comodule. A homogeneous map $d_N : N \to N$ with $|d_N| = |d|$ is a *coderivation* of N (with respect to d) if the following diagram commutes:

$$\begin{tikzcd} N \arrow[r, "d_N"] \arrow[d, "\Delta_N"'] & N \arrow[d, "\Delta_N"'] \\ C \otimes N \arrow[r, "d\otimes 1 + 1 \otimes d_N"'] & C \otimes N. \end{tikzcd}$$

Recall that if N is a graded comodule over a graded coalgebra C, it is a graded left C^*-module via the action

$$h \cdot x = h(x_{-1})x_{(0)}$$

for $h \in C^*$ and $x \in N$, where $\Delta_N(x) = x_{(-1)} \otimes x_{(0)} \in C \otimes N$.

Definition 3.1.1. Let (C, d, h) be a cdgc. A *curved differential graded C-comodule* (cdg C-comodule) is a pair (N, d_N), with N a graded C-comodule and $d_N : N \to N$ a coderivation, such that

$$d_N^2(x) = h \cdot x \quad \text{for all } x \in N.$$

If N, P are cdg C-comodules and $f \in \operatorname{Hom}(N, P)$, then $d_{\operatorname{Hom}}(f) = fd^N - (-1)^{|f|}d^P f$ satisfies $(d_{\operatorname{Hom}})^2 = 0$. Thus $\operatorname{Hom}(N, P)$ a complex. A *morphism of cdg comodules* is a degree zero C-colinear map $\alpha : N \to P$ such that $d_{\operatorname{Hom}}(\alpha) = 0$.

The following definition and properties are essential in what follows. They are the linear analogue of the tensor coalgebra and its properties given in 2.1.8.

Definition 3.1.2. Let C be a graded coalgebra and M a graded k-module. The *extended comodule* determined by M is the graded comodule $C \otimes M$ with comultiplication $\Delta_C \otimes 1$.

Lemma 3.1.3. *Let C be a graded coalgebra, N a graded C-comodule, and $C \otimes M$ an extended C-comodule.*

(1) *The map*

$$\varphi : \operatorname{Hom}(N, M) \xrightarrow{\cong} \operatorname{Hom}_C(N, C \otimes M),$$
$$\alpha \mapsto (1 \otimes \alpha)\Delta_N$$

is an isomorphism. It is natural in N and M. The inverse sends β to $N \xrightarrow{\beta} C \otimes M \xrightarrow{\epsilon\otimes 1} k \otimes M \cong M$.

(2) *Let d be a coderivation of C of degree n. The map*

$$\operatorname{Hom}(C \otimes M, M)_n \xrightarrow{\cong} \operatorname{Coder}(C \otimes M, C \otimes M)$$
$$m \mapsto d_M = d \otimes 1 + (1 \otimes m)(\Delta_C \otimes 1)$$

is an isomorphism. The inverse sends a coderivation d_M to $(\epsilon_C \otimes 1)d_M$.

(3) *Let C be a cdgc, and $(N, d^N), (C \otimes M, d^{C\otimes M})$ cdg C-comodules. Define*

$$m^M = (\epsilon_C \otimes 1)d^{C\otimes M} : C \otimes M \to M.$$

Define an endomorphism d' of $\operatorname{Hom}(M, N)$ by

$$d'(\alpha) = \alpha d^N - (-1)^{|\alpha|}m^M(1 \otimes \alpha)\Delta_N.$$

Then $(d')^2 = 0$ *and the isomorphism of modules given in part 1 is an isomorphism of complexes*

$$\varphi : (\mathrm{Hom}(N, M), d') \to (\mathrm{Hom}_C(N, C \otimes M), d_{\mathrm{Hom}}).$$

The proof is straightforward. The first two are dual to well-known results for free modules over an algebra, and the third follows from part 1.

Definition 3.1.4. Let A be an A_∞-algebra with corresponding dg-algebra $(T^c(A[1]), d)$. Let M be a graded module. An A_∞ *A-module structure* on M is a degree –1 map

$$m^M : T^c(A[1]) \otimes M \to M$$

such that the induced coderivation $d^M : T^c(A[1]) \otimes M \to T^c(A[1]) \otimes M$ (using 3.1.3.(2) applied to $(T^c(A[1]), d)$) satisfies $(d^M)^2 = 0$. A *morphism of* A_∞ *A-modules* $M \to N$ is a morphism of dg-comodules

$$\alpha : T^c(A[1]) \otimes M \to T^c(A[1]) \otimes N.$$

If we label the components of m^M as

$$m^M_n : A[1]^{\otimes n-1} \otimes M \to M,$$

then by Lemma 3.1.3.(2), m^M is an A_∞ A-module structure on M if and only if this is not immediate; need a diagram chase

(3.1.5) $$\sum_{i=1}^{n} \sum_{j=0}^{n-i} m^M_{n-i+1}(1^{\otimes j} \otimes m_i \otimes 1^{\otimes n-i-j}) = 0$$

for all $n \geq 1$, where m_i is m^M if $j = i - 1$ and m^A otherwise. For a map

$$\alpha : T^c(A[1]) \otimes M \to T^c(A[1]) \otimes N,$$

label the components of $p_1\alpha : T^c(A[1]) \otimes M \to N$ as $\alpha_n : A[1]^{\otimes n-1} \otimes M \to N$. By 3.1.3.(1), α is a morphism of A_∞-modules if and only if

(3.1.6) $$\sum_{i=1}^{n} \alpha_i \left(\sum_{j=0}^{i-1} 1^{\otimes j} \otimes m_{n-i+1} \otimes 1^{\otimes i-j-1} \right) = \sum_{i=1}^{n} m^N_i (1^{\otimes i-1} \otimes \alpha_{n-i+1})$$

for all $n \geq 1$. A morphism α is *strict* if $\alpha_n = 0$ for $n \geq 2$.

Definition 3.1.7. Let A be an A_∞-algebra with strict unit 1_A. An A_∞ A-module M is *strictly unital* if $m^M_2([1_A] \otimes m) = m$ and $m^M_n([a_1|\ldots|1_A|\ldots|a_{n-1}] \otimes m) = 0$ for all $n \geq 3$. A morphism α is *strictly unital* if $\alpha_n([a_1|\ldots|1_A|\ldots|a_{n-1}] \otimes m) = 0$ for all $n \geq 2$.

When A has a split unit, we can characterize strictly unital A_∞ A-modules using the cdgc $\mathrm{Bar}\, A$.

Lemma 3.1.8. *Let A be an A_∞-algebra with split unit v and let $b : \mathrm{Bar}\, A \hookrightarrow T^c(A[1])$ be the splitting of $p : T^c(A[1]) \to \mathrm{Bar}\, A$ that it induces. Let M and N be graded modules.*

(1) *A degree –1 map $m^M : T^c(A[1]) \otimes M \to M$ is a strictly unital A_∞ A-module structure on M if and only if the* $\mathrm{Bar}\, A$ *coderivation induced by*

$$\overline{m}^M = m^M(b \otimes 1) : \mathrm{Bar}\, A \otimes M \to M$$

makes $\mathrm{Bar}\, A \otimes M$ *a cdg* $\mathrm{Bar}\, A$*-comodule.*

(2) *If M, N are strictly unital A_∞ A-modules, a map*
$$g : T^c(A[1]) \otimes M \to T^c(A[1]) \otimes N$$
is a strictly unital morphism of A_∞ A-modules if and only if $f = pgb : \operatorname{Bar} A \otimes M \to \operatorname{Bar} A \otimes N$ is a morphism of cdg $\operatorname{Bar} A$-comodules.

The proof is given in 9.4.

Let us record that a map $\overline{m}^M : \operatorname{Bar} A \otimes M \to M$ with components $\overline{m}^M_i : \overline{A}[1]^{\otimes i-1} \otimes M \to M$ is a strictly unital A_∞ A-module structure if and only if
$$\sum_{i=1}^{k} \sum_{j=0}^{k-i} \overline{m}^M_{k-i+1}(1^{\otimes j} \otimes \overline{m}_i \otimes 1^{\otimes k-i-j}) = h_{k-1} \otimes 1$$
for all $k \geq 1$, where $\overline{m}_i$ is $\overline{m}^M$ if $j = i-1$ and $\overline{m}^A$ otherwise.

3.2. **Obstruction theory for modules.**

Definition 3.2.1. Let A be an A_n-algebra. An A_n A-*module* structure on a module M is a degree –1 linear map $m|^M_n : T^c_{n-1}(A[1]) \otimes M \to M$ such that the induced coderivation of $T^c_{n-1}(A[1]) \otimes M$ squares to zero. If A is strictly unital, then M is strictly unital if the analagous conditions as in the A_∞case hold.

An A_n A-module structure on M is equivalent to a set of degree –1 maps $m^M_i : A[1]^{\otimes i-1} \otimes M \to M$ that satisfy (3.1.5) for $k = 1, \ldots, n$. Note that whether M is an A_n-module only depends on the A_{n-1}-structure of A, and so it makes sense to extend M to a A_{n+1}-module over A. If M is an A_n-module, set
$$c(m|^M_n) = \sum_{i=2}^{n} \sum_{j=0}^{n-i} m^M_{n-i+2}(1^{\otimes j} \otimes m_i \otimes 1^{n-i-j}).$$
By [13, B.2.1], $c(m|^M_n)$ is a degree –2 cycle in $\operatorname{Hom}(A[1]^{\otimes n} \otimes M, M)$ (with Hom-differential between the complexes with differentials $m_1^{(n)} \otimes 1 + 1 \otimes m^M_1$ and m^M_1) and a map $m^M_{n+1} : A[1]^{\otimes n} \otimes M \to M$ extends $m|^M_n$ to an A_{n+1}-structure on M if and only if
$$d(m^M_{n+1}) + c(m|^M_n) = 0.$$
Analogously for algebras, we adjust this for strict units.

Proposition 3.2.2. *Let A be an A_n-algebra with split unit and M a strictly unital A_n-module with multiplication maps $\overline{m}^M_i : \overline{A}[1]^{\otimes i-1} \otimes M \to M$. The map*
$$c(\overline{m}|^M_n) := \sum_{i=2}^{n} \sum_{j=0}^{n-i} \overline{m}^M_i(1^{\otimes j} \otimes \overline{m}_{n-i+2} \otimes 1^{i-j-1}) + h_n \otimes 1$$
is a cycle in $\operatorname{Hom}(\overline{A}[1]^{\otimes n} \otimes M, M)$. A morphism $\overline{m}^M_{n+1}$ extends M to a strictly unital A_{n+1}-module if and only if
$$d(\overline{m}^M_{n+1}) + c(\overline{m}|^M_n) = 0.$$

The proof is similar to the proof of 2.3.3.

Definition 3.2.3. Let M, N be A_n-modules. A A_n-morphism from M to N is a map of dg $(T^c_{n-1}(A[1]), d)$-comodules
$$\alpha : T^c_{n-1}(A[1]) \otimes M \to T^c_{n-1}(A[1]) \otimes N.$$

Two A_n-morphisms

$$\alpha, \beta : T^c_{n-1}(A[1]) \otimes M \to T^c_{n-1}(A[1]) \otimes N$$

are *homotopic* if there is a degree 1 $T^c_{n-1}(A[1])$-colinear map $r : T^c_{n-1}(A[1]) \otimes M \to T^c_{n-1}(A[1]) \otimes N$ such that $d(r) = \alpha - \beta$.

A homotopy r is determined by $p_1 r : T^c_{n-1}(A[1]) \otimes M \to N$. If we label the components as $r_i : A[1]^{\otimes i-1} \otimes M \to N$ and the components of α, β as α_i, β_i, then r is a homotopy if and only if for $1 \leq l \leq n$, we have:

$$\sum_{i=1}^{l} \sum_{j=0}^{i-1} r_i(1^{\otimes j} \otimes m_{l-i+1} \otimes 1^{\otimes i-j-1}) + \sum_{i=1}^{l} m_i^N (1^{\otimes i-1} \otimes r_{l-i+1}) = \alpha_l - \beta_l.$$

As above, if A has a strict unit, we can define strictly unital morphisms between A-modules and strictly unital homotopies. When A has a split unit, strictly unital A_n-morphisms between M and N correspond to maps $T^c_{n-1}(\overline{A}[1]) \otimes M \to T^c_{n-1}(\overline{A}[1]) \otimes N$ of cdg $(T^c_{n-1}(\overline{A}[1]), d)$ morphisms and similarly for homotopies.

Proposition 3.2.4. *Let A be an A_n-algebra with split unit, and M, N strictly unital A_{n+1}-modules.*

(1) *Let $\alpha|_n : T^c_{n-1}(\overline{A}[1]) \otimes M \to T^c_{n-1}(\overline{A}[1]) \otimes N$ be a strictly unital A_n-morphism. The map*

$$c(\alpha|_n) = \sum_{i=1}^{n} \alpha_i \left(\sum_{j=0}^{i-1} 1^{\otimes j} \otimes \overline{m}_{n-i+2} \otimes 1^{i-j-1} \right) - \sum_{i=2}^{n+1} \overline{m}_i^N (1 \otimes \alpha_{n-i+2})$$

is a cycle. A morphism $\alpha_{n+1} : A[1]^{\otimes n} \otimes M \to N$ extends $\alpha|_n$ to an A_{n+1}-morphism if and only if

$$d(\alpha_{n+1}) + c(\alpha|_n) = 0.$$

(2) *Let α, β be strictly unital A_{n+1}-morphisms from M to N. Let $r|_n : T^c_{n-1}(\overline{A}[1]) \otimes M \to T^c_{n-1}(\overline{A}[1]) \otimes N$ be a homotopy between $\alpha|_n$ and $\beta|_n$. The map*

$$c(r|_n) = \alpha_{n+1} - \beta_{n+1}$$
$$- \sum_{i=2}^{n+1} \sum_{j=0}^{i-1} r_i(1^{\otimes j} \otimes m_{n-i+2} \otimes 1^{\otimes i-j-1}) - \sum_{i=1}^{n} m_i^N (1^{\otimes i-1} \otimes r_{n-i+2})$$

is a cycle in $\operatorname{Hom}(\overline{A}[1]^{\otimes n} \otimes M, N)$. A morphism r_{n+1} extends $r|_n$ to a homotopy between α and β if and only if

$$d(r_{n+1}) + c(r|_n) = 0.$$

3.3. **Semiprojective resolutions of $\mathbf{A}_\infty$ A-modules.**

Definition 3.3.1.

(1) Let C be a cdgc and M, N cdg C-comodules. Morphisms $\alpha, \beta : M \to N$ are *homotopic* if there exists a C-colinear map $r : M \to N$ of degree 1 such that $d_{\mathrm{Hom}}(r) = \alpha - \beta$.
(2) If A is an A_∞-algebra with split unit, and M, N are strictly unital A_∞ A-modules, two morphisms $\alpha, \beta : \operatorname{Bar} A \otimes M \to \operatorname{Bar} A \otimes N$ are *homotopic* if they are homotopic as cdg $\operatorname{Bar} A$-comodules.

Theorem 3.3.2. *Let A be an A_∞-algebra with split unit such that (A, m_1) is semiprojective.*

(1) *Let M be a strictly unital A_∞ A-module and $\epsilon : P \xrightarrow{\simeq} M$ a surjective semiprojective resolution of the complex (M, m_1^M). There is a strictly unital A_∞ A-module structure on P such that ϵ is a strict morphism of A_∞ A-modules.*

(2) *Let N be a strictly unital A_∞ A-module with (N, m_1^N) semiprojective. Let M, P be arbitrary strictly unital A_∞ A-modules and $\epsilon : M \to P$ a strict A_∞ A-module morphism with $\epsilon = \epsilon_1$ a surjective quasi-isomorphism. Then for any strictly unital A_∞ A-module morphism $\alpha : T^c(\overline{A}[1]) \otimes N \to M$, there exists a strictly unital A_∞ A-module morphism $\beta : T^c(\overline{A}[1]) \otimes N \to P$ such that $\epsilon\beta = \alpha$.*

$$\begin{tikzcd} & P \arrow[d, "\epsilon"] \\ N \arrow[ur, dashed, "\beta"] \arrow[r, "\alpha"'] & M. \end{tikzcd}$$

If for all $n \geq 2$,

$$H^{n-1}(\operatorname{Hom}(\overline{A}^{\otimes n-1} \otimes N, P)) = 0,$$

then any two such liftings are homotopic by a strictly unital homotopy.

Proof. We assume by induction that for some $n \geq 1$, P is a strictly unital A_n-module and the diagram

$$\begin{tikzcd} \overline{A}[1]^{\otimes i-1} \otimes P \arrow[r, "\overline{m}_i^P"] \arrow[d, "1\otimes\epsilon"'] & P \arrow[d, "\epsilon"', "\simeq"] \\ \overline{A}[1]^{\otimes i-1} \otimes M \arrow[r, "\overline{m}_i^M"'] & M \end{tikzcd}$$

is commutative for $i = 2, \ldots, n$. This holds for $n = 1$ by assumption.

Since A is semiprojective, so is $\overline{A}$, and hence so is $\overline{A}[1]^{\otimes n-1} \otimes P$. Since ϵ is a surjective quasi-isomorphism,

$$\varphi := \operatorname{Hom}(\overline{A}[1]^{\otimes n} \otimes P, \epsilon) : \operatorname{Hom}(\overline{A}[1]^{\otimes n} \otimes P, P) \xrightarrow{\simeq} \operatorname{Hom}(\overline{A}[1]^{\otimes n} \otimes P, M)$$

is a surjective quasi-isomorphism. Set

$$\phi := \operatorname{Hom}(1 \otimes \epsilon, M) : \operatorname{Hom}(\overline{A}[1]^{\otimes n} \otimes M, M) \to \operatorname{Hom}(\overline{A}[1]^{\otimes n} \otimes P, M).$$

By the induction hypothesis, we have

$$\varphi(c(\overline{m}|_n^P)) = \phi(c(\overline{m}|_n^M)),$$

where $c(-)$ is as defined in 3.2.2. Now one follows the same steps as in the proof of 2.4.5 to find $\overline{m}_{n+1}^P : \overline{A}[1]^{\otimes n-1} \otimes P \to P$ that extends P to an A_{n+1}-module and such that the induction hypothesis holds.

Let $\alpha : T^c(\overline{A}[1]) \otimes N \to M$ be a strictly unital A_∞-morphism. We assume by induction that there exists an A_n-morphism $\beta|_n : T^c_{n-1}(\overline{A}[1]) \otimes N \to P$ such that $\epsilon\beta|_n = \alpha|_n$. For $n = 1$, we can pick a lift of α_1 since N is semiprojective. Now, one follows the same steps as in the proof of 2.4.5, using 3.2.2. The homotopy uniqueness of β is also analogous to 2.4.5 and uses 3.2.2. $\square$

Example 3.3.3. Let $B = k/I$ be a cyclic k-algebra, M a B-module, and P a k-projective resolution of M. To construct an A_∞ A-module structure on P, we find a homotopy for the map

$$\begin{tikzcd} 0 & A_1 \otimes P_0 \arrow[l] \arrow[d, "h_1\otimes 1"'] & {\substack{A_1\otimes P_1 \\ A_2\otimes P_0}} \arrow[l] \arrow[d, "\left(\substack{h_1\otimes 1 \\ 0}\right)"'] & \dots \arrow[l] \\ 0 & P_0 \arrow[l] & P_1 \arrow[l] & \dots \arrow[l] \end{tikzcd} \tag{3.3.4}$$

and then degree considerations as in 2.4.7 show that we can find boundaries $\overline{m}_3^P, \overline{m}_4^P, \dots$ and that this A_∞ A-module structure is unique up to homotopy.

3.4. **Homotopy equivalences.** We record here the module version of Theorem 2.5.1.

Theorem 3.4.1. *Let A be an A_∞-algebra with split unit, M, N complexes, $f : M \to N$ a homotopy equivalence of complexes, and assume that N is an A_∞ A-module. Then there is a structure of A_∞ A-module on M and a morphism $\phi : M \to N$ of A_∞ A-modules with $\phi_1 = f$. Moreover ϕ is a homotopy equivalence of A_∞ A-modules.*

Corollary 3.4.2. *Let A be a semiprojective A_∞-algebra and M an A_∞ A-module. If $P \to M$ is a semiprojective resolution of (M, m_1^M), then there is an A_∞ A-module structure on P and a morphism of A_∞ A-modules $P \to M$.*

4. Twisting cochains

A twisting cochain is a linear map from a cdgc to an A_∞-algebra that allows one to define functors between their (co)module categories. In this section we give the definition and show that the functors given by a twisting cochain are an adjoint pair on homotopy categories. This is well known in the case of dg-objects, but for A_∞-algebras it does not seem to appear in the literature.

From this point on, all algebras, modules and morphisms are strictly unital.

4.1. **Primitive filtration and cocomplete comodules.** We collect here some technical facts on graded coalgebras that we will need throughout this section.

Let C be a graded coalgebra with coaugmentation η and let $p : C \to \overline{C} = C/\operatorname{im}\eta$ be projection. Recall that $\overline{\Delta}^{(n)} = p^{\otimes n}\Delta^{(n)} : C \to \overline{C}^{\otimes n}$ for $n \geq 2$. Set $\overline{\Delta}^{(0)} = \epsilon_C$ and $\overline{\Delta}^{(1)} = p$. For N a graded C-comodule, define

$$\overline{\Delta}_N^{(n)} : N \to \overline{C}^{\otimes n-1} \otimes N$$

by $\overline{\Delta}_N^{(1)} = 1_N$, $\overline{\Delta}_N = \overline{\Delta}_N^{(2)} = (p \otimes 1)\Delta_N$ and

$$\overline{\Delta}_N^{(n)} = (1^{\otimes n-2} \otimes \overline{\Delta}_N)\overline{\Delta}_N^{(n-1)} \text{ for } n \geq 3.$$

Note that $(1^{\otimes i-1} \otimes \overline{\Delta}_N^{(j)})\overline{\Delta}_N^{(i)} = \ \overline{\Delta}_N^{(i+j-1)}$ (as opposed to $\overline{\Delta}_N^{(i+j)}$).

Definition 4.1.1. Define

$$N_{[n]} = \ker \overline{\Delta}_N^{(n)} \subseteq N.$$

This is a k-submodule of N and there is a chain of inclusions

$$0 \subseteq N_{[1]} \subseteq N_{[2]} \subseteq \dots \subseteq N_{[n]} \subseteq \dots$$

The graded comodule N is *cocomplete* if $N = \bigcup_{n\geq 1} N_{[n]}$.

Lemma 4.1.2. *There are equalities*

$$\begin{aligned}\overline{\Delta}_N^{(n)} &= (1 \otimes \overline{\Delta}_N^{(n-1)})\overline{\Delta}_N \\ &= (p^{\otimes n-1} \otimes 1)\Delta_N^{(n)} \\ &= (\overline{\Delta}^{(n-1)} \otimes 1)\Delta_N.\end{aligned}$$

These are easily checked using induction. From the last equality, we have:

Corollary 4.1.3. *If C is cocomplete, then every C-comodule is cocomplete.*

From the first equality, we have $\overline{\Delta}_N(N_{[n]}) \subseteq \operatorname{im}(C \otimes N_{[n-1]} \to C \otimes N)$, so

$$\Delta_N(N_{[n]}) \subseteq \operatorname{im}(C \otimes N_{[n]} \to C \otimes N),$$

and so $N_{[n]}$ is a subcomodule of N.

If C has a coderivation d_C and N has a coderivation d_N, then

$$\Delta_N^{(n)} d_N = (d_C^{(n-1)} \otimes 1_N + 1^{\otimes n-1} \otimes d_N)\Delta_N^{(n)}, \tag{4.1.4}$$

where

$$\begin{aligned}d_C^{(n-1)} &= d_C \otimes 1_C^{\otimes n-2} + 1_C \otimes d_C \otimes 1_C^{\otimes n-3} \\ &\quad + \ldots + 1_C^{\otimes n-2} \otimes d_C.\end{aligned}$$

Let $\bar{d}_C : \overline{C} \to \overline{C}$ be the map induced by d_C. By definition, we have $pd_C = \bar{d}_C p$. Applying $p^{\otimes n-1} \otimes 1$ to both sides of (4.1.4) and using 4.1.2, we have

$$\overline{\Delta}_N^{(n)} d_N = (\bar{d}_C^{(n-1)} \otimes 1_N + 1^{\otimes n-1} \otimes d_N)\overline{\Delta}_N^{(n)}. \tag{4.1.5}$$

In particular, this implies that

$$d_N(N_{[n]}) \subseteq N_{[n]}.$$

Finally, we assume that (C, d, h) is a cdgc. A map $\eta : k \to C$ is a cdgc coaugmentation of C if η is a coaugmentation of graded coalgebras, $d\eta = 0$ and $h\eta = 0$. The cdgc C is cocomplete if it has a cdgc coaugmentation and is cocomplete as a graded coalgebra with respect to this coaugmentation. A cdg C-comodule N is cocomplete if it is cocomplete as a graded C-comodule.

Similarly as for $\overline{\Delta}_N^{(n)}$, one checks by induction that

$$\overline{\Delta}^{(n)} = (p \otimes \overline{\Delta}^{(n-1)})\Delta = (\overline{\Delta}^{(n-1)} \otimes p)\Delta.$$

Thus if C is flat over k, $\Delta(C_{[n]}) \subseteq (\overline{C} \otimes C_{[n-1]}) \cap (C_{[n-1]} \otimes \overline{C})$. In particular, since $C_{[n-1]}$ is closed under d_C by (4.1.5) applied to $N = C$, we have that:

Lemma 4.1.6. *If C is a cdgc, with C a graded flat module over k, then $C_{[n]}/C_{[n-1]}$ is a complex under the map induced by d_C.*

4.2. **Some dg-categories and functors between them.** We define here categories that we will use for the rest of the paper.

Let C be a cdgc. If N, P are cdg C-comodules, the map $d_{\mathrm{Hom}}(f) = fd^N - (-1)^{|f|}d^P f$ satisfies $(d_{\mathrm{Hom}})^2 = 0$. Thus $\operatorname{Hom}(N, P)$ a complex. We set $\operatorname{Hom}_C(N, P)$ to be the subcomplex of C-colinear maps.

Definition 4.2.1. Let C be a cdgc. We consider the following categories with objects cocomplete cdg comodules.

(1) dg-comod(C) is the dg-category with morphism complexes $\operatorname{Hom}_C(N, P)$;

(2) $\mathrm{comod}(C) = Z^0(\text{dg-comod}(C))$ is the category with morphisms

$$\mathrm{Hom}_{\mathrm{comod}(C)}(N, P) := Z^0 \,\mathrm{Hom}_C(N, P),$$

where $Z^0(-)$ denotes the degree zero cycles of a complex;

(3) $\underline{\mathrm{comod}}(C) = H^0(\text{dg-comod}(C))$ is the category with morphisms

$$\mathrm{Hom}_{\underline{\mathrm{comod}}(C)}(N, P) := H^0 \,\mathrm{Hom}_C(N, P),$$

where $H^0(-)$ denotes the degree zero cohomology of a complex.

Remark. A morphism in $\mathrm{comod}(C)$ is exactly a morphism between cdg-comodules as defined in 3.1.1, and $\underline{\mathrm{comod}}(C)$ is the quotient of $\mathrm{comod}(C)$ by homotopy equivalence as defined in 3.3.1.

The following will be one of the central concepts in the rest of the paper.

Definition 4.2.2. Let C be a cdgc.

(1) An *extended cdg C-comodule* is a cdg C-comodule whose underlying graded comodule is extended, see 3.1.2. By 3.1.3.(2), an extended cdg C-comodule $(C \otimes N, d)$ is determined by the linear map

$$m := (\epsilon_C \otimes 1)d : C \otimes N \to N.$$

We will write the extended comodule as the pair $(C \otimes M, m)$, and refer to m as the structure map of the extended comodule. The coderivation corresponding to an arbitrary degree zero map $m : C \otimes N \to N$ need not make $C \otimes N$ a cdg C-comodule. If it does, we say m gives $C \otimes N$ a cdg C-comodule structure.

(2) $\text{dg-comod}^{\mathrm{ext}}(C)$ is the full dg-subcategory of $\text{dg-comod}(C)$ with objects extended cdg C-comodules and $\underline{\mathrm{comod}}^{\mathrm{ext}}(C) \to \underline{\mathrm{comod}}(C)$ the induced functor on homotopy categories.

Definition 4.2.3. Let $(\varphi, a) : C \to D$ be a morphism of cdgcs.

(1) The pushforward of a cdg C-comodule (N, d^N), denoted $\varphi_*(N, d_N)$, is the cdg D-comodule with comultiplication

$$N \xrightarrow{\Delta_N} C \otimes N \xrightarrow{\varphi \otimes 1} D \otimes N$$

and differential $d^N(x) + a * x$, where $a * (-)$ is defined in (**??**). There is an obvious map of complexes $\mathrm{Hom}_C(N, M) \to \mathrm{Hom}_D(\varphi_* N, \varphi_* M)$, so there is a dg-functor

need to fix reference to §2

$$\varphi_* : \text{dg-comod}(C) \to \text{dg-comod}(D).$$

(2) To define the pullback $\varphi^*(M, d^M)$ of a cdg D-comodule, write $(\varphi, a) = (\phi, 0) \circ (\psi, a)$ with (ψ, a) an isomorphism of cdg coalgebras, using (**??**). For the isomorphism ψ, define $\psi^*(M, d^M) = (\psi^{-1})_*(M, d^M)$. Thus to define φ^*, we may assume $a = 0$. In such a case, $\varphi^*(M, d^M) = (D \boxtimes_C M, 1 \boxtimes d^M)$.

need reference to §2 and to match up notation

Proposition 4.2.4. *Let $(\varphi, a) : C \to D$ be a morphism of cdgcs. There is a strict*[1] *dg-adjoint pair*

$$\text{dg-comod}(C) \underset{\varphi^*}{\overset{\varphi_*}{\rightleftarrows}} \text{dg-comod}(D).$$

[1]the unit and counit are isomorphisms of complexes, not just quasi-isomorphisms

The definition of φ^* is a bit convoluted. For the class of extended comodules, the only comodules we will need to evaluate it on, there is a straightforward description of the pullback functor and the adjunction isomorphism.

Lemma 4.2.5. *Let $(\varphi, a) : C \to D$ be a morphism of cdg coalgebras and $(D \otimes N, m)$ and $(D \otimes P, n)$ extended cdg D-comodules.*

(1) *There is an equality*
$$\varphi^*(D \otimes N, m) = (C \otimes N, m(\varphi \otimes 1) + a \otimes 1).$$
(2) *The morphism of complexes*
$$\mathrm{Hom}_D(D \otimes N, D \otimes M) \to \mathrm{Hom}_C(\phi^*(D \otimes N), \phi^*(D \otimes M)).$$
corresponds to the morphism of complexes
$$\mathrm{Hom}(D \otimes N, M) \to \mathrm{Hom}(C \otimes N, M)$$
$$\alpha \mapsto \alpha(\phi \otimes 1),$$
using 3.1.3.(3).

(3) *The map*
$$\mathrm{Hom}_C(N, \phi^*(D \otimes M)) \to \mathrm{Hom}_D(\phi_* N, D \otimes M)$$
$$\alpha \mapsto (\phi \otimes 1)\alpha.$$
is an isomorphism of complexes that is natural in M and N. The inverse sends β to $(1 \otimes (\epsilon_D \otimes 1)\beta)\Delta_N$.

Proof. By 3.1.3.(3), we have isomorphisms of complexes
$$\mathrm{Hom}(N, M) \to \mathrm{Hom}_C(N, C \otimes^\tau M) \quad \mathrm{Hom}(\phi_* N, M) \to \mathrm{Hom}_D(\phi_* N, D \otimes M).$$
The graded k-modules $\mathrm{Hom}(N, M)$ and $\mathrm{Hom}(\phi_* N, M)$ are the same; we show that differentials on $\mathrm{Hom}(N, M)$ and $\mathrm{Hom}(\phi_* N, M)$ given in 3.1.3.(3) are the same. Let $m^M : C \otimes^\tau M \to M$ and $\widetilde{m}^M : D \otimes M \to M$ be the structure maps of these extended cdg comodules. We have
$$m^M = \widetilde{m}^M(\phi \otimes 1) + a \otimes 1$$
by 4.2.5. Let $d' = d^{\phi_* N} = d^N + a * (-)$ be the differential of $\phi_*(N)$. By 3.1.2.(3), the differential on $\mathrm{Hom}(M, N)$ sends α to
$$\alpha d^N + m^M(1 \otimes \underline{\alpha})\Delta_N$$
$$= \alpha d^N + \widetilde{m}^M(\phi \otimes 1)(1 \otimes \underline{\alpha})\Delta_N + (a \otimes 1)(1 \otimes \underline{\alpha})\Delta_N.$$
By the same result, the differential of $\mathrm{Hom}(\phi_* N, M)$ sends α to
$$\alpha d' + \widetilde{m}^M(1 \otimes \underline{\alpha})\Delta_{\phi_* N}$$
$$= \alpha d^N + \widetilde{m}^M(1 \otimes \underline{\alpha})(\phi \otimes 1)\Delta_N + \alpha(a * (-)),$$
and we see the differentials are the same.

Under the composition
$$\mathrm{Hom}_C(N, C \otimes^\tau M) \xrightarrow{\cong} \mathrm{Hom}(N, M) \xrightarrow{\cong} \mathrm{Hom}_D(\phi_* N, D \otimes M),$$
α gets sent to $(1 \otimes \overline{\alpha})\Delta_{\phi_* N} = (\phi \otimes 1)(1 \otimes \overline{\alpha})\Delta_N = (\phi \otimes 1)\alpha$, where $\overline{\alpha} = (\epsilon_C \otimes 1)\alpha$, and one checks the similar formula for the inverse. Finally, note that as 3.1.3.(3) is natural in N and M, so is the isomorphism here. $\square$

We need analogous dg-categories for an A_∞-algebra.

Definition 4.2.6. Let A be a A_∞-algebra with split unit. We consider the following categories with objects *strictly unital* A_∞ A-modules.

(1) dg-mod$^\infty(A)$ is the dg-category with morphism complexes
$$\mathrm{Hom}_A^\infty(M,N) := \mathrm{Hom}_{\mathrm{Bar}\,A}(\mathrm{Bar}\,A\otimes M, \mathrm{Bar}\,A\otimes N);$$
(2) $\mathrm{mod}^\infty(A) = Z^0(\text{dg-mod}(A))$ is the category with morphisms
$$\mathrm{Hom}_{\mathrm{mod}^\infty(A)}(M,N) := Z^0\,\mathrm{Hom}_{\mathrm{Bar}\,A}(\mathrm{Bar}\,A\otimes M, \mathrm{Bar}\,A\otimes N);$$
(3) $\underline{\mathrm{mod}}^\infty(A) = H^0(\text{dg-mod}^\infty(A))$ is the category with morphisms
$$\mathrm{Hom}_{\underline{\mathrm{mod}}^\infty(A)}(M,N) := H^0\,\mathrm{Hom}_{\mathrm{Bar}\,A}(\mathrm{Bar}\,A\otimes M, \mathrm{Bar}\,A\otimes N).$$

Remark. By 3.1.8, a morphism in $\mathrm{mod}^\infty(A)$ is a morphism of A_∞ A-modules and $\underline{\mathrm{mod}}^\infty(A)$ is the quotient of $\mathrm{mod}^\infty(A)$ by homotopy equivalence as defined in 3.3.1.

Make connection between A_∞-modules and extended Bar A-comodules...say that one of the functors for a morphism doesn't exist.

We need one last formality. Define the shift of a cdg C-comodule and the cone of morphism between such modules exactly as for complexes. These constructions make dg-comod(C) into a *pre-triangulated dg-category* and so by [4, §3, Prop. 2] the homotopy category $\underline{\mathrm{comod}}(C)$ is triangulated with triangles those isomorphic to $X \xrightarrow{f} Y \to \mathrm{cone}(f) \to X[1]$ (see [19] for a detailed exposition on pre-triangulated categories). The functors φ_*, φ^* preserve cones and shifts, thus induce triangulated functors between homotopy categories. Even more is true.

Proposition 4.2.7. *Let $(\varphi, a) : C \to D$ be a morphism of cdgcs.*

(1) *Arbitrary coproducts exist in the category $\underline{\mathrm{comod}}(C)$.*
(2) *The functors*
$$\underline{\mathrm{comod}}(C) \underset{\varphi^*}{\overset{\varphi_*}{\rightleftarrows}} \underline{\mathrm{comod}}(D).$$
preserve coproducts.

Should give careful proof/references for this...very important.

4.3. **Twisting cochains.**

Recall our stated goal of finding a Morita invariant of A that is smaller than Bar A. This will come from a morphism of cdgcs $C \to \mathrm{Bar}\,A$. As a morphism of graded coalgebras, such a morphism is determined by...

Here we recall the definition of twisting cochains, introduce the universal twisting cochain, and define two functors determined by a twisting cochain.

Definition 4.3.1. Let (C, d, h_C) be a cdgc, cocomplete with respect to a coaugmentation $\eta : k \to C$, and let A be an A_∞-algebra with structure map
$$m^A : T^c(A[1]) \to A[1].$$
Set $\overline{C} = C/\operatorname{im}\eta$. A *twisting cochain from C to A* is a degree zero map of graded modules $\tau : C \to A[1]$ such that $\tau\eta = 0$ and
$$\bar\tau\,\bar d - \sum_{n\geq 1} m_n^A\,\bar\tau^{\otimes n}\,\overline{\Delta}^{(n)} + \bar h = 0, \tag{4.3.2}$$
where $\bar\tau : \overline{C} \to A[1]$ is induced by τ, $\bar d : \overline{C} \to \overline{C}$ is induced by d, and $\bar h$ is the map $\overline{C} \xrightarrow{\bar h_C} k \xrightarrow{s} k\cdot[1_A] \xrightarrow{\eta_A} A[1]$. We set $\mathrm{Tw}(C,A)$ to be the set of twisting cochains between C and A.

Example 4.3.3. Start with divided powers coalgebra to symmetric algebra? Mention it generalizes in two ways? i.e. to Lie algebras and to the curved case...

Example 4.3.4. Let A be the Koszul complex of a linear map $l : V \to k$, see e.g. [5, §1.6], with V a finitely generated free module concentrated in degree 1. Since A is a dg-algebra, it is an A_∞-algebra.

Let $C \subseteq T^c(V[1])$ be the sub-coalgebra of symmetric tensors, or the divided powers coalgebra (the dual is the symmetric algebra on $V^*[-1]$), and consider the cdgc $(C, 0, h)$, where

$$h : C \twoheadrightarrow V[1] \xrightarrow{s^{-1}} V \xrightarrow{-l} k.$$

We claim

$$\tau : C \twoheadrightarrow V[1] \xrightarrow{s^{-1}} V \hookrightarrow A$$

is a twisting cochain. The equation (4.3.2) simplifies to

$$-m_1^A\bar{\tau} - m_2^A(\bar{\tau} \otimes \bar{\tau})\overline{\Delta}^2 + \bar{h} = 0.$$

This is zero on all n-tensors for $n \geq 3$. For $[x] \in V[1]$,

$$(-m_1^A\bar{\tau} - m_2^A(\bar{\tau} \otimes \bar{\tau})\overline{\Delta}^2 + \bar{h})([x]) = -m_1^A(x) + h([x]) = l(x) - l(x) = 0,$$

using that $m_1^A = -d_A$. We also have

$$(-m_1^A\bar{\tau} - m_2^A(\bar{\tau} \otimes \bar{\tau})\overline{\Delta}^2 + \bar{h})([x \otimes x]) = m_2^A(x \otimes x) = 0.$$

Since symmetric 2-tensors are k-linear combinations of $x \otimes x$, for $x \in V$, this shows that τ is a twisting cochain.

We will show in 6.4.4 that this twisting cochain is *acyclic*, or gives an equivalence between appropriate (co)derived categories.

Definition 4.3.5. Let $(\phi, a) : C \to D$ be a morphism of cdg coalgebras, and

$$\widetilde{\tau} : D \to A[1]$$

a twisting cochain. The *composition of* $\widetilde{\tau}$ *and* ϕ is

$$\tau = \widetilde{\tau}\phi + \eta_A s a : C \to A[1].$$

One checks this is a twisting cochain. In this situation, we say the following diagram is commutative.

$$\begin{tikzcd} D \arrow[dr, "\widetilde{\tau}"] & \\ C \arrow[u, "(\phi,a)"] \arrow[r, "\tau"'] & A \end{tikzcd}$$

Lemma 4.3.6. *Let A be an A_∞-algebra with split unit v and*

$$b : \overline{A}[1] = A[1]/k \cdot [1_A] \to A[1]$$

the induced splitting of $p : A[1] \to \overline{A}[1]$. The map

$$\tau_A : \operatorname{Bar} A \xrightarrow{p_1} \overline{A}[1] \xrightarrow{b} A[1]$$

is a twisting cochain. If $\tau : C \to A$ is any twisting cochain, then the morphism of cdg coalgebras

$$(\phi, a) : C \to \operatorname{Bar} A,$$

with ϕ induced by $p_1\tau$, using 2.1.8, and $a = s^{-1}v\tau$, is the unique morphism of coaugmented cdg coalgebras such that

$$\begin{tikzcd} \operatorname{Bar} A \arrow[dr, "\tau_A"] & \\ C \arrow[u, "(\phi,a)"] \arrow[r, "\tau"'] & A, \end{tikzcd}$$

is commutative (in the sense of 4.3.5).

Thus there is a bijection of sets, natural in both arguments,

$$\operatorname{Tw}(C, A) \cong \operatorname{Hom}_{\mathrm{cdgc}}(C, \operatorname{Bar} A).$$

In particular, if $C = \operatorname{Bar} B$ for some A_∞-algebra with split unit, then we have

$$\operatorname{Tw}(\operatorname{Bar} B, A) \cong \operatorname{Hom}_{\mathrm{cdgc}}(\operatorname{Bar} B, \operatorname{Bar} A) \cong \operatorname{Hom}_{A_\infty}(B, A).$$

Proof. To show τ_A is a twisting cochain, we need to show that

$$\bar\tau\,\bar d - \sum_{n\geq 1} m_n^A\,\bar\tau^{\otimes n}\,\overline{\Delta}^{(n)} + \bar h = 0.$$

Since $\overline{\Delta}^{(l)}$ vanishes on $\overline{A}[1]^{\otimes n}$ if $l > n$ and τ_A is only nonzero on $\overline{A}[1]$, we have that $\sum_{l\geq 1} m_l\tau^{\otimes l}\overline{\Delta}^{(l)}([a_1|\dots|a_n]) = m_n[a_1|\dots|a_n]$. Thus

$$\begin{aligned}(\bar\tau\,\bar d - \sum_{n\geq 1} m_n^A\,\bar\tau^{\otimes n}\,\overline{\Delta}^{(n)} + \bar h)[a_1|\dots|a_n]\\ = b\overline{m}_n[a_1|\dots|a_n] - m_n[ba_1|\dots|ba_n] + h_n([a_1|\dots|a_n])1_A = 0,\end{aligned}$$

where the last equality is by Theorem 2.2.17.

By 2.1.8, there is a correspondence between pairs (ϕ, a), with $\phi : C \to T^c(\overline{A}[1])$ a graded coalgebra morphism and $a : C \to k$ a linear map, and morphisms of graded modules $\tau : C \to A[1]$. The proof 9.2, replacing $\widetilde{\alpha}$ there by τ, shows that the pair $(\phi, a) : C \to \operatorname{Bar} A$ is a morphism of cdg coalgebras if and only if τ is a twisting cochain. □

Definition 4.3.7. Let A be an A_∞-algebra with split unit. The twisting cochain

$$\tau_A : \operatorname{Bar} A \to A[1]$$

defined in the lemma above is the *universal twisting cochain* of A.

The goal of this subsection is to define a pair of functors given by a twisting cochain. The following is the main step in defining the functors on objects.

Proposition 4.3.8. *Let C be a cocomplete cdgc, A an A_∞-algebra and $\tau : C \to A[1]$ a twisting cochain.*

(1) *For N a cdg C-comodule, $A[1] \otimes N$ has a structure of A_∞ A-module given by the maps*

$$m_1^{A[1]\otimes N} = 1 \otimes d^N + \sum_{j\geq 1}(m_j^A \otimes 1_N)(1 \otimes \bar\tau^{\otimes j-1} \otimes 1_N)(1 \otimes \overline{\Delta}_N^{(j)})$$

$$m_n^{A[1]\otimes N} = \sum_{j\geq 1}(m_{n+j-1}^A \otimes 1_N)(1^{\otimes n} \otimes \bar\tau^{\otimes j-1} \otimes 1_N)(1^{\otimes n} \otimes \overline{\Delta}_N^{(j)}) \text{ for } n \geq 2.$$

Note that if A is a dg-algebra, then $A[1]\otimes N$ is a dg A-module, i.e. $m_n^{A[1]\otimes N} = 0$ for $n \geq 3$, and the underlying module is free.

(2) *If A has a split unit v, and M is an A_∞ A-module with structure map*
$$\overline{m}^M : \operatorname{Bar} A \otimes M \to M,$$
then $C \otimes M$ has the structure of a cdg C-comodule given by
$$\sum_{n\geq 1} \overline{m}_n^M(\overline{\tau}^{\otimes n-1} \otimes 1)(\overline{\Delta}_C^{(n-1)} \otimes 1) + s^{-1}v\tau \otimes 1 : C \otimes M \to M.$$

Part 1 is proved in 9.5 by a direct, but involved, computation showing that $m_n^{A[1]\otimes N}$ satisfy the equations (3.1.5). Part 2 is straightforward, using the following definition and lemma.

Let $\tau : C \to A[1]$ be a twisting cochain. Part 2 of 4.3.8 follows by applying 4.2.5.(1) to the map of cdg coalgebras $C \to \operatorname{Bar} A$ induced by τ. (We will need 4.2.5.(2) in the sequel.)

Definition 4.3.9. Let M be an A_∞ A-module with structure map $m^M : \operatorname{Bar} A \otimes M \to M$. The *shift of* M is the A_∞ A-module structure on $M[1]$ given by the map
$$m^{M[1]} = -s^1 m^M(1 \otimes s^{-1}) : \operatorname{Bar} A \otimes M[1] \to M[1].$$
Define an A_∞ A-module structure on $M[-1]$ by switching s and s^{-1} in the above.

Definition 4.3.10. Let $\tau : C \to A[1]$ be a twisting cochain.

(1) For N a cdg C-comodule, the *twisted tensor product of A and N* is the A_∞ A-module
$$A \otimes^\tau N = (A[1] \otimes N)[-1]$$
where $A[1]\otimes N$ has the A_∞ A-module structure of 4.3.8.(1). The underlying module of $A \otimes^\tau N$ is $A \otimes N$ and the structure maps are
$$m_n^{A\otimes^\tau N} = -(s^{-1} \otimes 1)\, m_n^{A[1]\otimes N}\,(1^{\otimes n-1} \otimes s \otimes 1).$$

(2) If A has a split unit and M is an A_∞ A-module, the *twisted tensor product of C and M*, denoted,
$$C \otimes^\tau M,$$
is the cdg C-comodule of 4.3.8.(2).

Example 4.3.11. Let $\tau_A : \operatorname{Bar} A \to A[1]$ be the universal twisting cochain. For an A_∞ A-module M, $\operatorname{Bar} A \otimes^{\tau_A} M$ is the comodule defined in 3.1.8.

Example 4.3.12. Let $\tau : C \to A[1]$ be the generalized BGG twisting cochain of Example 4.3.4 and let N be a cdg C-comodule. Then $A \otimes^\tau N$ is a dg A-module. The multiplication is determined by that on A. Let $\zeta_1, \dots, \zeta_n$ be a basis of A_1 and $\xi_1 = s(\zeta_i) \in A_1[1] = C_2$. For $n \in N$, we have
$$d(1_A \otimes n) = \sum_{i=1}^n \zeta_i \otimes n_i + 1_A \otimes d_N(n),$$
where $\sum_{i=1}^n \xi_i \otimes n_i = \overline{\Delta}_N^{(1)}(n) - \overline{\Delta}_N^{(2)}(n)$, and this determines d.

We have that $C \otimes^\tau A$ is a cdg C-comodule, with comultiplication $\Delta_C \otimes 1$ and the differential on e.g. $\xi_i \otimes x$ is
$$d(\xi_i \otimes x) = 1 \otimes \zeta_i \cdot x + \xi_i \otimes d_A(x)$$

Example 4.3.13. Let A be an A_∞-algebra with split unit, and let $\tau_A : \mathrm{Bar}\, A \to A[1]$ be the universal twisting cochain. Let N be a cdg $\mathrm{Bar}\, A$-comodule. We describe the maps $m_n^{A\otimes^{\tau_A} N} : A[1]^{\otimes n-1} \otimes A \otimes N \to A \otimes N$.

We fix an element x of N and write, for some $k \geq 1$,

$$\Delta_N(x) = \sum_{j=1}^{k} c_j \otimes x_j$$

with $c_j = \sum_l [a_1^{jl}| \dots |a_{j-1}^{jl}] \in \overline{A}[1]^{\otimes j-1}$ and $x_j \in N$. Note that this later sum is finite since $\mathrm{Bar}\, A$ is cocomplete, and hence N is a cocomplete comodule. We have that

$$(\overline{\tau}_A^{j-1} \otimes 1)\overline{\Delta}_N^{(j)}(x) = (\overline{\tau}_A^{j-1} \otimes 1)(\overline{\Delta}^{(j-1)} \otimes 1)(x) = c_j \otimes x_j,$$

where the first equality uses $\overline{\Delta}_N^{(j)} = (\overline{\Delta}^{(j-1)} \otimes 1)$, see 4.1.2, and the second uses that τ_A is zero on $\overline{A}[1]^{\otimes i}$ for $i \geq 2$. Thus, we have

$$m_1^{A\otimes^\tau N}(a \otimes x) = \underline{a} \otimes d_N(x)$$

$$-(s^{-1} \otimes 1) \sum_{j \geq 1} (m_j^A \otimes 1_N)(1 \otimes \overline{\tau}^{\otimes j-1} \otimes 1_N)(1 \otimes \overline{\Delta}_N^{(j)})([a] \otimes x)$$

$$= \underline{a} \otimes d_N(x) - \sum_{j=1}^{k} \sum_l s^{-1} m_j^A([a|ba_1^{jl}| \dots |ba_{j-1}^{jl}]) \otimes x_j.$$

For $y = [y_1| \dots |y_{n-1}] \in A[1]^{\otimes n-1}$, we have

$$m_n^{A\otimes^\tau N}(y \otimes a \otimes x) = -\sum_{j=1}^{k} \sum_l s^{-1} m_{n+j-1}^A [\underline{y}_1| \dots |\underline{y}_{n-1}|a|ba_1^{jl}| \dots |ba_{j-1}^{jl}] \otimes x_j.$$

We can now define the pair of functors given by a twisting cochain.

Definition 4.3.14. Let C be a cocomplete cdgc, A an A_∞-algebra with split unit and $\tau : C \to A[1]$ a twisting cochain.

Define dg-functors

$$L_\tau := A \otimes^\tau - : \text{dg-comod}(C) \to \text{dg-mod}^\infty(A)$$

$$R_\tau := C \otimes^\tau - : \text{dg-mod}^\infty(A) \to \text{dg-comod}(C)$$

with $L_\tau(N) = A \otimes^\tau N$, $R_\tau(M) = C \otimes^\tau M$, and maps of complexes

$$\mathrm{Hom}_C(N, P) \to \mathrm{Hom}_A^\infty(A \otimes^\tau N, A \otimes^\tau P)$$

$$f \mapsto 1 \otimes 1 \otimes f$$

and

$$\mathrm{Hom}_A^\infty(L, M) = \mathrm{Hom}_{\mathrm{Bar}\, A}(\mathrm{Bar}\, A \otimes^{\tau_A} L, \mathrm{Bar}\, A \otimes^{\tau_A} M)$$

$$\to \mathrm{Hom}_C(C \otimes^\tau L, C \otimes^\tau M)$$

the map 4.2.5.(2) applied to the morphism $\phi : C \to \mathrm{Bar}\, A$ given by τ.

We record the following for later use. It follows almost immediately from the definitions.

Lemma 4.3.15. *Let C be a cocomplete cdgc, A an A_∞-algebra with split unit and $\tau : C \to A[1]$ a twisting cochain. Let τ_A be the universal twisting cochain and $(\phi, a) : C \to \operatorname{Bar} A$ the morphism of cdg coalgebras corresponding to τ. The following diagram of functors is commutative,*

$$\begin{tikzcd} \text{dg-mod}^{\infty}(A) \arrow[r, "R_{\tau_A}"] \arrow[dr, "R_\tau"'] & \text{dg-comod}^{\text{ext}}(\operatorname{Bar} A) \arrow[d, "\phi^*"] \\ & \text{dg-comod}^{\text{ext}}(C), \end{tikzcd} \tag{4.3.16}$$

noting that the image of R_τ is contained in $\text{dg-comod}^{\text{ext}}(C)$*, and where ϕ^* is defined in* **??**.

4.4. **L_τ and R_τ form an adjoint pair.** Let $\tau : C \to A[1]$ be a twisting cochain. Our goal is to show the dg-functors 4.3.14 induce an adjoint pair of functors on the homotopy categories of cdg C-comodules and A$_\infty$ A-modules. We first show this is the case for the universal twisting cochain τ^A. The following detailed description of the differential on $\operatorname{Bar} A \otimes^{\tau_A} A \otimes^{\tau_A} N$, for a cdg $\operatorname{Bar} A$-comodule N, will be essential.

Lemma 4.4.1. *Let A be an A_∞-algebra with split unit v, $b : \overline{A} \to A$ the induced splitting, $\tau_A : \operatorname{Bar} A \to A[1]$ the universal twisting cochain, and N a cdg* $\operatorname{Bar} A$*-comodule. Let $\widetilde{d}$ be the* $\operatorname{Bar} A$*-coderivation of* $\operatorname{Bar} A \otimes^{\tau_A} A \otimes^{\tau_A} N$.

For an element x of N, write

$$\Delta_N(x) = \sum_{j=1}^{k} c_j \otimes x_j,$$

for some $k \geq 0$ depending on x, with $c_j = \sum_l [a_1^{jl}|\dots|a_{j-1}^{jl}] \in \overline{A}[1]^{\otimes j-1}$ and $x_j \in N$. For $a \in A$ and $y = [y_1|\dots|y_{n-1}] \in \overline{A}[1]^{\otimes n-1} \subseteq \operatorname{Bar} A$, we have

$$\widetilde{d}(y \otimes a \otimes x) = d_{\operatorname{Bar} A}(y) \otimes a \otimes x$$

$$- \sum_{i=2}^{n} \sum_{j=1}^{k} \sum_l [\underline{y}_1|\dots|\underline{y}_{n-i}] \otimes s^{-1} m^A_{i+j-1}[b\underline{y}_{n-i+1}|\dots|b\underline{y}_{n-1}|a|ba_1^{jl}|\dots|ba_{j-1}^{jl}] \otimes x_j$$

$$+ \underline{y} \otimes \underline{a} \otimes d_N(x) - \underline{y} \otimes \sum_{j=1}^{k} \sum_l s^{-1} m^A_j([a|a_1^{jl}|\dots|a_{j-1}^{jl}]) \otimes x_j.$$

Proof. We described the maps $m_n^{A\otimes^{\tau_A} N} : A[1]^{\otimes n-1} \otimes A \otimes N \to A \otimes N$ in 4.3.13. Since the A$_\infty$ A-module $A \otimes^\tau N$ is strictly unital, it is a cdg $\operatorname{Bar} A$-comodule with structure map $\overline{m}^{A\otimes^\tau N} = m^{A\otimes^\tau N}(b \otimes 1) : \operatorname{Bar} A \otimes (A \otimes^\tau N) \to A \otimes^\tau N$ by 3.1.8.(1). By 3.1.3.(2), the corresponding $\operatorname{Bar} A$-coderivation of $\operatorname{Bar} A \otimes^{\tau_A} A \otimes^{\tau_A} N$ is

$$\widetilde{d} = d_{\operatorname{Bar} A} \otimes 1 \otimes 1 + (1 \otimes \overline{m}^{A\otimes^\tau N})(\Delta_{\operatorname{Bar} A} \otimes 1 \otimes 1)$$

and this is the formula above applied to $y \otimes a \otimes x$. □

Definition 4.4.2. Let A be an A_∞-algebra with split unit and N a cdg $\operatorname{Bar} A$-comodule. Define

$$\eta_N : N \to \operatorname{Bar} A \otimes^{\tau_A} A \otimes^{\tau_A} N$$
$$x \mapsto x_{(-1)} \otimes 1_A \otimes x_{(0)}.$$

Lemma 4.4.3. *The map η_N is a morphism of cdg* $\operatorname{Bar} A$*-comodules and is natural with respect to N.*

Proof. Define a k-linear map $N \to A \otimes^{\tau} N$ by $x \mapsto 1_A \otimes x$. Then η_N is the C-colinear map corresponding to this map via 3.1.3.(1), and so in particular η_N is C-colinear. Since the k-linear map is clearly natural, η is as well. To finish the proof, we need to show that $\eta_N d_N = \widetilde{d}\eta_N$, where $\widetilde{d}$ is the coderivation of $\operatorname{Bar} A \otimes^{\tau_A} A \otimes^{\tau_A} N$. Applying 4.4.1 to $x_{(-1)} \otimes 1_A \otimes x_{(0)}$, the first and third summands are equal to $d_N \eta_N$, using the definition of coderivation. So we have to show the second and fourth summands are zero. This follows since $(\Delta_{\operatorname{Bar} A} \otimes 1)\Delta_N = (1 \otimes \Delta_N)\Delta_N$; for the signs, recall 2.2.5. □

Definition 4.4.4. Let A be an A_∞-algebra with split unit and M an A_∞ A-module. Define a degree zero map

$$\epsilon_M : \operatorname{Bar} A \otimes^{\tau_A} A \otimes^{\tau_A} \operatorname{Bar} A \otimes^{\tau_A} M \to \operatorname{Bar} A \otimes^{\tau_A} M$$

on the component $\overline{A}[1]^{\otimes n-1} \otimes A \otimes \overline{A}[1]^{\otimes k-1} \otimes M$ by

$$\sum_{j=0}^{n-1} 1^{\otimes n-j-1} \otimes m^M_{k+j+1}(b^{\otimes j} \otimes s \otimes b^{k-1} \otimes 1).$$

Proposition 4.4.5. *Let A be an A_∞-algebra with split unit and M an A_∞ A-module. Let $\eta = \eta_{\operatorname{Bar} A \otimes^{\tau_A} M}$ be the map defined in 4.4.2.* should spend more time on this, see if can give better proof

(1) *The map $\epsilon_M : \operatorname{Bar} A \otimes^{\tau_A} A \otimes^{\tau_A} \operatorname{Bar} A \otimes^{\tau_A} M \to \operatorname{Bar} A \otimes^{\tau_A} M$ is a morphism of cdg* $\operatorname{Bar} A$*-comodules, i.e. an A_∞ A-module morphism.*

(2) *There is an equality*

$$\epsilon_M \eta = 1_{\operatorname{Bar} \otimes^{\tau_A} M}.$$

(3) *The map*

$$r : \operatorname{Bar} A \otimes^{\tau_A} A \otimes^{\tau_A} \operatorname{Bar} A \otimes^{\tau_A} M \to \operatorname{Bar} A \otimes^{\tau_A} A \otimes^{\tau_A} \operatorname{Bar} A \otimes^{\tau_A} M$$

$$[x_1|\dots|x_{n-1}] \otimes a \otimes y \otimes z \mapsto$$

$$\sum_{i=0}^{n-1} [x_1|\dots|x_i] \otimes 1_A \otimes [\underline{x}_{i+1}|\dots|\underline{x}_{n-1}|p(a)|y] \otimes z.$$

is $\operatorname{Bar} A$*-colinear and satisfies*

$$d_{\mathrm{Hom}}(r) = \eta\epsilon_M - 1.$$

Thus ϵ_M and η are inverse isomorphisms in $\underline{\mathrm{mod}}^\infty(A)$.

(4) *ϵ_M is natural up to homotopy: for $\beta : \operatorname{Bar} A \otimes^{\tau_A} M \to \operatorname{Bar} A \otimes^{\tau_A} N$ an A_∞-morphism, the diagram*

$$\begin{tikzcd}
\operatorname{Bar} A \otimes^{\tau_A} A \otimes^{\tau_A} \operatorname{Bar} A \otimes^{\tau_A} M \arrow[r, "\epsilon_M"] \arrow[d, "1\otimes 1\otimes\beta"'] & \operatorname{Bar} \otimes^{\tau_A} M \arrow[d, "\beta"] \\
\operatorname{Bar} A \otimes^{\tau_A} A \otimes^{\tau_A} \operatorname{Bar} A \otimes^{\tau_A} N \arrow[r, "\epsilon_N"] & \operatorname{Bar} \otimes^{\tau_A} N
\end{tikzcd}$$

is commutative in $\underline{\mathrm{mod}}^\infty(A)$.

This may be checked by a fairly involved computation, applying Lemma 4.4.1 to the cdg $\operatorname{Bar} A$-comodule $\operatorname{Bar} A \otimes^{\tau_A} M$. should say something about the proof

The following is now almost a formality. should change this sentence

Proposition 4.4.6. *Let A be an A_∞-algebra with split unit and universal twisting cochain $\tau_A : \operatorname{Bar} A \to A[1]$. The dg-functors 4.3.14 induce an adjoint pair of functors between homotopy categories,*

$$\begin{tikzcd} \underline{\text{comod}}(\operatorname{Bar} A) \arrow[r, shift left, "L_{\tau_A}"] & \underline{\text{mod}}^\infty(A) \arrow[l, shift left, "R_{\tau_A}"] \end{tikzcd},$$

with unit

$$\eta_N : N \to \operatorname{Bar} A \otimes^{\tau_A} A \otimes^{\tau_A} N$$

and counit

$$\epsilon_M : \operatorname{Bar} A \otimes^{\tau_A} A \otimes^{\tau_A} \operatorname{Bar} A \otimes^{\tau_A} M \to \operatorname{Bar} A \otimes^{\tau_A} M$$

defined above.

Note that R_{τ_A} is the inclusion of the full subcategory of extended $\operatorname{Bar} A$-comodules; accordingly the counit ϵ is an isomorphism by 4.4.5.(3).

Proof. Since 4.3.14 give dg-functors, taking homology gives functors between the homotopy categories. By 4.4.3 and 4.4.5.(4), there are natural transformations

$$\eta : 1 \to R_{\tau_A} L_{\tau_A} \qquad \epsilon : R_{\tau_A} L_{\tau_A} R_{\tau_A} \to R_{\tau_A}.$$

Since R_{τ_A} is fully faithful, we will also consider ϵ as a natural transformation $L_{\tau_A} R_{\tau_A} \to 1$. By [14, Theorem 4.1.2], to show that these functors are an adjoint pair, it is enough to show that $\eta_N : N \to R_{\tau_A} L_{\tau_A} N$ is a universal arrow from N to R_{τ_A}. So given $g : N \to R_{\tau_A} M$, let $\widetilde{g} = \epsilon_M R_{\tau_A} L_{\tau_A}(g) : R_{\tau_A} L_{\tau_A} N \to R_{\tau_A} M$. Then by naturality of η, we have

$$\eta g = R_{\tau_A} L_{\tau_A}(g)\eta.$$

Applying ϵ_M to the above, and using that $\epsilon_M \eta_{R_{\tau_A} M} = 1_{R_{\tau_A} M}$, 4.4.5.(3), we have

$$g = \epsilon_M R_{\tau_A} L_{\tau_A}(g)\eta = \widetilde{g}\eta.$$

Since R_{τ_A} is fully faithful, $\widetilde{g} = R_{\tau_A}(f)$ for some map $f : L_{\tau_A} N \to M$, and thus η is a universal arrow from N to R_{τ_A}. $\square$

Lemma 4.4.7. *Let $(\phi, a) : C \to D$ be a morphism of cdg coalgebras. Let A be an A_∞-algebra with split unit and assume we have a commutative diagram*

$$\begin{tikzcd} D \arrow[dr, "\tau'"] & \\ C \arrow[u, "{(\phi,a)}"] \arrow[r, "\tau"'] & A \end{tikzcd}$$

in the sense of 4.3.5, with τ and τ' twisting cochains.

(1) *Let N be a cocomplete cdg C-comodule. There is an equality of A_∞ A-modules*

$$A \otimes^\tau N = A \otimes^{\tau'} \phi_* N.$$

(2) *Let M be an A_∞ A-module. There is an isomorphism*

$$C \otimes^\tau M \cong \phi^*(D \otimes^{\tau'} M).$$

Proof. For part 1, we note that both have the same underlying module and the formulas 4.3.8.(1) are the same, using the commutativity of the diagram. For part 2, we use the fact that τ and τ' correspond to morphisms of cdg coalgebras to $\psi : C \to \operatorname{Bar} A$ and $\psi' : D \to \operatorname{Bar} A$, respectively, and $\psi = \phi\psi'$. The claim now follows from the fact that $(\psi')^*\phi^* = (\phi\psi')^*$. $\square$

Definition 4.4.8. Let $\tau : C \to A[1]$ be a twisting cochain and N a cdg C-comodule. Define the degree zero map

$$\eta_N^\tau : N \to C \otimes^\tau A \otimes^\tau N$$

by

$$\eta_N^\tau(x) = x_{(-1)} \otimes 1_A \otimes x_{(0)}.$$

Putting the above pieces together, we have:

Theorem 4.4.9. *Let C be a cocomplete cdgc, A an A_∞-algebra with split unit and $\tau : C \to A$ a twisting cochain. The dg-functors 4.3.14 induce an adjoint pair*

$$\underline{\mathrm{comod}}(C) \underset{R_\tau}{\overset{L_\tau}{\rightleftarrows}} \underline{\mathrm{mod}}^\infty(A)$$

with unit

$$\eta_N^\tau : N \to C \otimes^\tau A \otimes^\tau N$$

and counit

$$\epsilon_M^\tau : \operatorname{Bar} A \otimes^{\tau_A} (A \otimes^\tau C \otimes^\tau M) \xrightarrow{\epsilon_M(1\otimes 1\otimes\phi\otimes 1)} \operatorname{Bar} A \otimes^{\tau_A} M,$$

where $\phi : C \to \operatorname{Bar} A$ is the map of coalgebras induced by τ, η^τ is defined in 4.4.8 and ϵ_M is defined in 4.4.4.

5. (Co)derived categories

The adjoint 4.4.9 will only be an equivalence in trivial situations, e.g. when $A = k$ and $C = \operatorname{Bar} A \cong k$. We define here quotient categories of $\underline{\mathrm{comod}}(C)$ and $\underline{\mathrm{mod}}^\infty(C)$ such that when $C = \operatorname{Bar} A$, the functor $\underline{\mathrm{comod}}(\operatorname{Bar} A) \to \underline{\mathrm{mod}}^\infty(A)$ induces an equivalence of quotient categories. We then determine conditions on $\tau : C \to A$ when this induced functor is an equivalence.

is this the only case?

functor is not induced anymore

To define the coderived category of a cdgc C, we first use 5.1.8 to find the largest quotient of $\underline{\mathrm{comod}}(C)$ where every comodule is isomorphic to an extended comodule. Quasi-isomorphisms make sense between extended comodules (but not for arbitrary cdg comodules). Inverting quasi-isomorphisms in this quotient, we arrive at the definition of the coderived category.

match this up with intro

5.1. **Cobar construction and extended comodules.** We describe the cobar construction $\Omega(C)$ of a cdgc C and use this to prove a key techincal result about the category of extended C-comodules. Throughout, C is a cdgc, cocomplete with respect to a coaugmentation η and $p : C \to \overline{C} = C/\operatorname{im}\eta$ is projection.

The underlying algebra of $\Omega(C)$ is the tensor algebra $T(\overline{C}[-1]) := \oplus_{n\geq 0}\overline{C}[-1]^{\otimes n}$. Dual to the tensor coalgebra, $\Omega(C)$ is a graded algebra via the multiplication

$$(x_1 \otimes \ldots \otimes x_i)(x_{i+1} \otimes \ldots \otimes x_n) = x_1 \otimes \ldots \otimes x_n$$

and $|x_1 \otimes \ldots \otimes x_i| = \sum_{j=1}^{i} |x_j|$. We want to define a differential on $\Omega(C)$ such that it becomes a dga and

$$\tau^C : C \xrightarrow{p} \overline{C} \xrightarrow{s^{-1}} \overline{C}[-1] \xrightarrow{j} T(\overline{C}[-1]) \xrightarrow{s} T(\overline{C}[-1])[1] = \Omega(C)[1]$$

is a twisting cochain. We have $\tau^C \eta = 0$ and the induced map $\overline{\tau}^C$ is given by $\overline{C} \xrightarrow{s^{-1}} \overline{C}[-1] \xrightarrow{j} T(\overline{C}[-1]) \xrightarrow{s} T(\overline{C}[-1])[1]$. The definition of a twisting cochain (4.3.2) reduces to

$$\overline{\tau}_C \overline{d}_C - m_1^{\Omega(C)} \overline{\tau}_C - m_2^{\Omega(C)} (\overline{\tau}_C \otimes \overline{\tau}_C) \overline{\Delta}_C - \eta_{\Omega(C)} s \overline{h}_C = 0. \tag{5.1.1}$$

Applying $s^{-1}(-)s$ to the above and setting $d = -s^{-1} m_1^{\Omega(C)} s$ we have

$$-js^{-1}\overline{d}_C s + dj + (js^{-1} \otimes js^{-1})\overline{\Delta}_C s + \eta_{\Omega(C)} \overline{h}_C s = 0$$

or

$$dj = js^{-1}\overline{d}_C s - (js^{-1} \otimes js^{-1})\overline{\Delta}_C s - \eta_{\Omega(C)} \overline{h}_C s.$$

Dual to the tensor coalgebra, derivations of the tensor algebra $T(\overline{C}[-1])$ are determined by linear maps $\overline{C}[-1] \to T(\overline{C}[-1])$. Thus the above equation determines d. Using [18, 2.16], for an element of $x = \langle x_1 | \ldots | x_n \rangle \in \overline{C}[-1]^{\otimes n} \subseteq \Omega(C)$, we have

$$\begin{aligned} d(x) = & \sum_{k=0}^{n-1} \overline{h}(x_{k+1}) \langle \underline{x}_1 | \ldots | \underline{x}_k | x_{k+2} | \ldots | x_n \rangle \\ & - \sum_{k=0}^{n-1} \langle \underline{x}_1 | \ldots | \underline{x}_k | \overline{d}(x_{k+1}) | x_{k+2} | \ldots | x_n \rangle \\ & + \sum_{k=0}^{n-1} \langle \underline{x}_1 | \ldots | \underline{x}_k | \underline{x}_{k+1(1)} | x_{k+1(2)} | x_{k+2} | \ldots | x_n \rangle \end{aligned} \tag{5.1.2}$$

where $\overline{\Delta}(x_k) = x_{k(1)} \otimes x_{k(2)}$. One checks that $d^2 = 0$, and thus $\Omega(C)$ is a dg-algebra.

Definition 5.1.3. The *cobar construction* of C is the dg-algebra

$$\Omega(C) := (T(\overline{C}[-1]), d)$$

with d the derivation above.

It is classical that $\tau^C : C \to \Omega(C)$ is the universal twisting cochain from C to dg-algebras. The proof of e.g [18, 2.11] is easily adapted from the case of a dgc to the case a cdgc, so we have the following.

Lemma 5.1.4. *Let A be a dg-algebra and C a cocomplete cdgc. If $\tau : C \to A[1]$ is a twisting cochain, there is a unique map of dg-algebras $\varphi : \Omega(C) \to A$ such that $\tau = \varphi\tau^C$.*

If A is an A_∞-algebra, as opposed to a dg-algebra, it is not clear that 5.1.4 holds. It surely must, up to some sort of homotopy, possibly for simply connected C, but there seems to be nothing in the literature treating this case. Instead of trying to formulate this extension, we will make do with Proposition 5.1.8 below.

We will need the following in the sequel.

Definition 5.1.5. For C be a cocomplete cdgc, set $(\phi_C, 0) : C \to \operatorname{Bar}\Omega(C)$ to be the unique map of cdg coalgebras such that the diagram

$$\begin{tikzcd} \operatorname{Bar}\Omega(C) \arrow[dr, "\tau_{\Omega(C)}"] & \\ C \arrow[u, "(\phi_C, 0)"] \arrow[r, "\tau^C"'] & \Omega(C), \end{tikzcd}$$

commutes in the sense of 4.3.5. Such a ϕ_C exists by applying 4.3.6 to the twisting cochain $\tau_{\Omega(C)} : \operatorname{Bar}\Omega(C) \to \Omega(C)$. (This also shows that $a = 0$.)

Definition 5.1.6. Let C be a cocomplete cdgc with coaugmentation η, and let $C \otimes X$ be an extended comodule. If $C = k \oplus \overline{C}$ is the linear decomposition of C, where $\overline{C} = C/\operatorname{im}\eta$, and $m : C \otimes X \to X$ is the structure map of $C \otimes X$, see 4.2.2, then define maps d^X and $\overline{m}$ by

$$m : C \otimes X \cong (k \otimes X) \oplus (\overline{C} \otimes X) \xrightarrow{(\ d^X \quad \overline{m}\ )} X.$$

It follows that $(d^X)^2 = 0 : X \to X$, so (X, d^X) is a complex. Conversely, if (X, d^X) is a complex, an *extended comodule structure on* X is a linear map $\overline{m} : \overline{C} \otimes X \to X$ such that $m := (d^X \ \ \overline{m}) : C \otimes X \to X$ makes X a cdg C-comodule.

The following shows that an extended comodule structure is determined by a twisting cochain. The proof is an unwinding of definitions.

Lemma 5.1.7. *Let C be a cocomplete cdgc and X a complex. A degree –1 map*

$$\overline{m} : \overline{C} \otimes X \to X$$

is a cdg comodule structure on X if and only if the corresponding map

$$\tau : C \to \operatorname{Hom}(X, X),$$

extended from $\overline{C}$ by setting $\tau(1) = 0$, is a twisting cochain, where $\operatorname{Hom}(X, X)$ *is the endomorphism dga of X.*

The following is key to the rest of the section.

Proposition 5.1.8. *Let C be a cocomplete cdgc and*

$$\tau^C : C \to \Omega(C)[1] \quad \tau_{\Omega(C)} : \operatorname{Bar}\Omega(C) \to \Omega(C)[1]$$

the universal twisting cochains.

(1) *The functor*

$$R_{\tau^C} : \underline{\mathrm{mod}}^{\infty}(\Omega(C)) \to \underline{\mathrm{comod}}(C)$$

is fully faithful with image $\underline{\mathrm{comod}}^{\mathrm{ext}}(C)$.

(2) *Let $\phi_C : C \to \operatorname{Bar}\Omega(C)$ be the morphism of cdg coalgebras resulting from 5.1.5. The homology of the functor* **??***, denoted*

$$\phi_C^* : \underline{\mathrm{comod}}^{\mathrm{ext}}(\operatorname{Bar}\Omega(C)) \to \underline{\mathrm{comod}}^{\mathrm{ext}}(C),$$

is an equivalence.

Proof. Let X be an A_∞ $\Omega(C)$-module and let

$$\epsilon_X^{\tau^C} : \operatorname{Bar}\Omega(C) \otimes \Omega(C) \otimes C \otimes X \to \operatorname{Bar}\Omega(C) \otimes X$$

be the counit. To see the that R_{τ^C} is fully faithful, we show that $\epsilon_X^{\tau^C}$ is a homotopy equivalence. Write

$$\operatorname{Bar}\Omega(C) \otimes \Omega(C) \otimes C \otimes X \;\cong\; \begin{array}{c} \big(\operatorname{Bar}\Omega(C) \otimes \Omega(C) \otimes \overline{C} \otimes X\big) \oplus \\ \Big(\operatorname{Bar}\Omega(C) \otimes \overline{\Omega(C)} \otimes k \otimes X\Big) \oplus \\ \big(\operatorname{Bar}\Omega(C) \otimes k \otimes k \otimes X\big). \end{array} \tag{5.1.9}$$

The differential of $\operatorname{Bar}\Omega(C) \otimes \Omega(C) \otimes C \otimes X$ can be written

$$\begin{bmatrix} d & 0 & 0 \\ \varphi & d' & 0 \\ * & * & d'' \end{bmatrix},$$

with

$$\varphi : \operatorname{Bar}\Omega(C) \otimes \Omega(C) \otimes \overline{C} \otimes X \to \operatorname{Bar}\Omega(C) \otimes \overline{\Omega(C)} \otimes k \otimes X$$

a degree -1 invertible $\operatorname{Bar} A$-colinear map such that $\varphi d + d'\varphi = 0$ and $d'' = d^{\operatorname{Bar}\Omega(C)\otimes X}$. There is a morphism of $\operatorname{Bar}\Omega(C)$-comodules

$$\eta : \operatorname{Bar}\Omega(C) \otimes X \to \operatorname{Bar}\Omega(C) \otimes \Omega(C) \otimes C \otimes X$$

identifying $\operatorname{Bar}\Omega(C) \otimes X$ with the third summand in (5.1.9). Define a $\operatorname{Bar}\Omega(C)$-colinear map h with respect to the decomposition (5.1.9) by

$$h = \begin{bmatrix} 0 & \varphi^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It follows that

$$d(h) = 1 - \eta\epsilon \quad \text{and} \quad \epsilon\eta = 1,$$

thus ϵ is a homotopy equivalence.

The diagram below is commutative by (4.3.15),

$$\begin{tikzcd} \underline{\mathrm{mod}}^\infty(\Omega(C)) \arrow[r, "R_{\tau_{\Omega(C)}}"] \arrow[dr, "R_{\tau^C}"'] & \underline{\mathrm{comod}}^{\mathrm{ext}}(\operatorname{Bar}\Omega(C)) \arrow[d, "\phi^*"] \\ & \underline{\mathrm{comod}}^{\mathrm{ext}}(C). \end{tikzcd}$$

The functor $R_{\tau_{\Omega(C)}}$ is an equivalence by definition. Since R_{τ^C} is fully faithful by the previous part, ϕ^* is fully faithful. To see it is essentially surjective, let $C \otimes X$ be an extended cdg comodule. By 5.1.7, this corresponds to a twisting cochain $C \to \operatorname{Hom}(X, X)$, and by 5.1.4 this corresponds to a dg-algebra map $\Omega(C) \to \operatorname{Hom}(X, X)$. We can compose this dg-algebra map with universal twisting cochain $\operatorname{Bar}\Omega(C) \to \Omega(C)$ to get a twisting cochain $\operatorname{Bar}\Omega(C) \to \operatorname{Hom}(X, X)$. This corresponds to the extended comodule $\operatorname{Bar}\Omega(C) \otimes X$. Since the diagram

$$\begin{tikzcd} \operatorname{Bar}\Omega(C) \arrow[r] & \operatorname{Hom}(X, X) \\ C \arrow[u, "\phi"] \arrow[ur] & \end{tikzcd}$$

is commutative, $\phi^*(\mathrm{Bar}\,\Omega(C) \otimes X) \cong C \otimes X$, the original extended comodule. Thus ϕ^* is essentially surjective. □

Corollary 5.1.10. *If X is a complex with an extended C-comodule structure, then there exists an $A_\infty\,\Omega(C)$-module structure on X and a homotopy equivalence*

$$\phi_C^*(\mathrm{Bar}\,\Omega(C) \otimes X) \to C \otimes X$$

whose first component is 1_X.

Proof. By 5.1.8.(2), there exists an $A_\infty\,\Omega(C)$-module $\mathrm{Bar}\,\Omega(C) \otimes Y$ such that $\phi_C^*(\mathrm{Bar}\,\Omega(C) \otimes Y) \cong C \otimes Y$ is homotopic to $C \otimes X$. This implies that the complexes Y and X are homotopic, and thus by 3.4.1 there is an $A_\infty\,\Omega(C)$-module structure on X and a homotopy equivalence of $A_\infty\,\Omega(C)$-modules $Y \to X$. Applying 5.1.8.(2) again finishes the proof. □

why is the first component 1_X? why the last line of the proof?

Applying 5.1.5 to $\mathrm{Bar}\,A$, there is a morphism of cdg coalgebras

$$\phi_{\mathrm{Bar}\,A} : \mathrm{Bar}\,A \to \mathrm{Bar}\,\Omega(\mathrm{Bar}\,A).$$

Taking primitives of this morphism, we have a morphism of complexes

$$\varphi_A : A \to \Omega(\mathrm{Bar}\,A).$$

Corollary 5.1.11. *Let A be a semiprojective A_∞-algebra. Let*

$$\tau_A : \mathrm{Bar}\,A \to A[1] \qquad \tau^{\mathrm{Bar}\,A} : \mathrm{Bar}\,A \to \Omega(\mathrm{Bar}\,A)[1]$$

be the universal twisting cochains. The map

$$1 \otimes \varphi_A \otimes 1 \otimes 1 : \mathrm{Bar}\,A \otimes^{\tau_A} A \otimes^{\tau_A} \mathrm{Bar}\,A \otimes^{\tau_A} M$$
$$\to \mathrm{Bar}\,A \otimes^{\tau^{\mathrm{Bar}\,A}} \Omega(\mathrm{Bar}\,A) \otimes^{\tau^{\mathrm{Bar}\,A}} \mathrm{Bar}\,A \otimes^{\tau_A} M$$

is a natural isomorphism in $\underline{\mathrm{comod}}(\mathrm{Bar}\,A)$ for all A_∞ A-modules M. In particular, it gives an isomorphism of functors

$$R_{\tau_A} L_{\tau_A} \cong R_{\tau^{\mathrm{Bar}\,A}} L_{\tau^{\mathrm{Bar}\,A}}.$$

Proof. Consider the object $\mathrm{Bar}\,A \otimes^{\tau_A} M$ in $\underline{\mathrm{comod}}(\mathrm{Bar}\,A)$. The unit of the adjunction $(L_{\tau^{\mathrm{Bar}\,A}}, R_{\tau^{\mathrm{Bar}\,A}})$ is an isomorphism

$$B(A) \otimes^{\tau_A} M \xrightarrow{\cong} \mathrm{Bar}\,A \otimes^{\tau^{\mathrm{Bar}\,A}} \Omega(\mathrm{Bar}\,A) \otimes^{\tau^{\mathrm{Bar}\,A}} \mathrm{Bar}\,A \otimes^{\tau_A} M$$

since $L_{\tau^{\mathrm{Bar}\,A}}$ is fully faithful on the image of $R_{\tau^{\mathrm{Bar}\,A}}$ by 5.1.8. Also, the counit of the adjunction (L_{τ_A}, R_{τ_A}) is an isomorphism

$$\mathrm{Bar}\,A \otimes^{\tau_A} M \xrightarrow{\cong} \mathrm{Bar}\,A \otimes^{\tau_A} A \otimes^{\tau_A} \mathrm{Bar}\,A \otimes^{\tau_A} M.$$

The map $1 \otimes \varphi_A \otimes 1 \otimes 1$, which is clearly $\mathrm{Bar}\,A$-colinear and natural, gives a commutative diagram
(5.1.12)

$$\begin{tikzcd}
 & \mathrm{Bar}\,A \otimes^{\tau^{\mathrm{Bar}\,A}} \Omega(\mathrm{Bar}\,A) \otimes^{\tau^{\mathrm{Bar}\,A}} \mathrm{Bar}\,A \otimes^{\tau_A} M \\
\mathrm{Bar}\,A \otimes^{\tau_A} M \arrow[ur, "\cong"'] \arrow[dr, "\cong"] & \\
 & \mathrm{Bar}\,A \otimes^{\tau_A} A \otimes^{\tau_A} \mathrm{Bar}\,A \otimes^{\tau_A} M \arrow[uu, "1\otimes\varphi_A\otimes1\otimes1"']
\end{tikzcd}$$

and so $1 \otimes \varphi_A \otimes 1 \otimes 1$ is an isomorphism in $\underline{\text{comod}}(\operatorname{Bar} A)$. $\square$

Corollary 5.1.13. *Let A be a semiprojective A_∞-algebra. The map of complexes*

$$\varphi_A : A \to \Omega(\operatorname{Bar} A)$$

is a quasi-isomorphism.

Proof. Consider the diagram (5.1.12) above, with $M = A$. If we take primitives, then there is a diagram in the homotopy category of complexes

$$\begin{tikzcd} & \Omega(\operatorname{Bar} A) \otimes^{\tau^{\operatorname{Bar} A}} \operatorname{Bar} A \otimes^{\tau_A} A \\ A \arrow[r, "\cong"'] & A \otimes^{\tau_A} \operatorname{Bar} A \otimes^{\tau_A} A \arrow[u, "\cong"'] \end{tikzcd}$$

and it is clear that there is a map of complexes $\Omega(\operatorname{Bar} A) \to \Omega(\operatorname{Bar} A) \otimes^{\tau^{\operatorname{Bar} A}} \operatorname{Bar} A \otimes^{\tau_A} A$ such that the following is commutative

$$\begin{tikzcd} \Omega(\operatorname{Bar} A) \arrow[r, dashed] & \Omega(\operatorname{Bar} A) \otimes^{\tau^{\operatorname{Bar} A}} \operatorname{Bar} A \otimes^{\tau_A} A \\ A \arrow[u, "\varphi_A"] \arrow[r, "\cong"'] & A \otimes^{\tau_A} \operatorname{Bar} A \otimes^{\tau_A} A \arrow[u, "\cong"'] \end{tikzcd}$$

and so φ_A must be a homotopy equivalence, and in particular a quasi-isomorphism. $\square$

5.2. **Semiprojective cdgcs and comodules.** Before constructing the coderived category of a cdgc, we need some preliminary material on semiprojective comodules and coalgebras. Recall from 4.1.6, that if a cdgc (C, d_C, h) is flat over k, then d_C induces a differential on $C_{[n]}/C_{[n-1]}$ that makes it a complex.

Definition 5.2.1. A cdgc C is *semiprojective* if C is cocomplete, projective as a graded k-module, and $C_{[n]}/C_{[n-1]}$ is a semiprojective complex for all n.

Lemma 5.2.2. *Let C be a semiprojective cdgc.*

(1) $C_{[n-1]} \to C_{[n]}$ *is a split inclusion of graded modules for all n;*
(2) $\Omega(C)$ *is a semiprojective complex.*

Proof. The first part holds, since the module underlying $C_{[n]}/C_{[n-1]}$ is projective. For the second part, define a bounded below filtration of the complex $\Omega(C)$ by setting $F_n\Omega(C) := \oplus_{j\geq 0}(C_{[n]})^{\otimes j}$. There are isomorphisms of complexes

$$F_n\Omega(C)/F_{n-1}\Omega(C) \cong \oplus_{j\geq 0}(C_{[n]}/C_{[n-1]})^{\otimes j}$$

and so the subquotients are semiprojective complexes. Thus $\Omega(C)$ is semiprojective, by e.g. [1, Chapter 2, Lemma 4.4.3]. $\square$

Example 5.2.3. If C is a cocomplete cdgc such that C is projective as a graded k-module, $C_n = 0$ for $n \ll 0$, and $C_{[n]} \hookrightarrow C$ is split for all n, then C is semiprojective. This follows since $C_{[n]}/C_{[n-1]}$ is a bounded below complex of projective modules, and thus is semi-projective.

The above conditions always hold if e.g. C is the free graded cocommutative coalgebra on a positively graded free k-module. So (C, d, h) is semi-projective for any d, h that make it a cdgc.

Example 5.2.4. If A is an A_∞-algebra with split unit such that the complex (A, m_1) is semiprojective, then $\operatorname{Bar} A$ is a semiprojective cdgc. This follows since $(\overline{A}, \overline{m}_1)$ is also semiprojective, and thus so is $(\overline{A}[1])^{\otimes n}$ for all n, and we have isomorphisms of complexes

$$(\operatorname{Bar} A_{[n]})/(\operatorname{Bar} A_{[n-1]}) \cong (\overline{A}[1])^{\otimes n}.$$

Lemma 5.2.5. *Let C be a semiprojective cdgc, X a complex with an extended cdg C-comodule structure, and $pX \to X$ a semiprojective resolution of X. There exists an extended cdg comodule structure on pX and a morphism of extended comodules $\pi : C \otimes pX \to C \otimes X$ whose first component is $pX \to X$.*

Proof. By 3.4.2 there exists an $A_\infty\, \Omega(C)$-module structure on X. By 3.3.2, pX has an $A_\infty\, \Omega(C)$-module structure such that $1 \otimes \pi$ is an $A_\infty\, \Omega(C)$-module morphism. Consider the composition

$$\phi_C^*(\operatorname{Bar}\Omega(C) \otimes pX) = C \otimes pX \xrightarrow{\phi_C^*(1\otimes\pi)} \phi_C^*(\operatorname{Bar}\Omega(C) \otimes X) \to C \otimes X,$$

where the second arrow is due 5.1.10. The first component is $pX \to X$. □

Definition 5.2.6. An extended cdg comodule $C \otimes X$ is *semiprojective* (*acyclic*) if the complex (X, d^X), defined in 5.1.6, is semiprojective (acyclic).

Let $\underline{\text{comod}}^{\text{sp}}(C)$ and $\underline{\text{comod}}^{\text{ac}}(C)$ be the full subcategories of $\underline{\text{comod}}^{\text{ext}}(C)$ with objects semiprojective, respectively acyclic, extended comodules.

5.3. **Semiorthogonal decompositions.**

We first collect some facts about semiorthogonal decompositions. Unproven assertions can be proven quickly using [16, §9].

Definition 5.3.1. Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ a triangulated subcategory.

(1) Define $\mathcal{S}^\perp$ and ${}^\perp\mathcal{S}$ to be the full subcategories with objects

$$\mathcal{S}^\perp = \{X \mid \operatorname{Hom}_{\mathcal{T}}(\mathcal{S}, X) = 0\} \quad {}^\perp\mathcal{S} = \{X \mid \operatorname{Hom}_{\mathcal{T}}(X, \mathcal{S}) = 0\}.$$

(2) A pair of fully faithful functors $\mathcal{A} \to \mathcal{T}, \mathcal{B} \to \mathcal{T}$ forms a *semiorthogonal decomposition* of $\mathcal{T}$ if $\mathcal{B} \subseteq {}^\perp\mathcal{A}$ and for every X in $\mathcal{T}$ there is a triangle

$$X' \to X \to X'' \to$$

with X' in $\mathcal{B}$ and X'' in $\mathcal{A}$. If this holds, we write $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$.

If $\langle \mathcal{A}, \mathcal{B} \rangle$ is a semiorthogonal decomposition of $\mathcal{T}$, a *localization triangle* for X is a triangle

$$X' \to X \to X'' \to$$

with X' in $\mathcal{B}$ and X'' in $\mathcal{A}$.

5.3.2. If B is a triangulated subcategory of $\mathcal{T}$ and Y is in $B^\perp$, then the canonical map

$$\operatorname{Hom}_{\mathcal{T}}(Y, X) \to \operatorname{Hom}_{\mathcal{T}/B}(Y, X)$$

is an isomorphism for all X in $\mathcal{T}$ by [16, 9.1.6], and the dual statement holds. It follows from this that if $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semiorthogonal decomposition, then the compositions

$$\mathcal{A} \to \mathcal{T} \to \mathcal{T}/\mathcal{B}$$
$$\mathcal{B} \to \mathcal{T} \to \mathcal{T}/\mathcal{A}$$

are equivalences. If for every X in $\mathcal{T}$ we fix a triangle

$$X' \to X \to X'' \to$$

with X' in $\mathcal{B}$ and X'' in $\mathcal{A}$, then the inverse equivalence

$$\mathcal{T}/\mathcal{B} \to \mathcal{A}$$

sends the image of X to X'', and the inverse equivalence

$$\mathcal{T}/\mathcal{A} \to \mathcal{B}$$

sends the image of X to X'. In particular, this shows localization triangles are unique up to isomorphism.

Example 5.3.3. Let R be an associative ring and $\mathcal{T} = \underline{\mathrm{mod}}(R)$ the homotopy category of complexes of R-modules. Define full subcategories $\mathcal{A} = \underline{\mathrm{mod}}^{\mathrm{ac}}(R)$ and $\mathcal{B} = \underline{\mathrm{mod}}^{\mathrm{proj}}(R)$ with objects the acyclic complexes and semiprojective complexes (the definition and properties given in 2.4.1 also hold for noncommutative rings), respectively. By 2.4.2.(2), $\mathcal{B} \subseteq {}^{\perp}\mathcal{A}$, and by 2.4.2.(4), for every complex X there is a triangle

$$pX \to X \to aX \to$$

with $pX \in \mathcal{B}$ and $aX \in \mathcal{A}$ ($pX \to X$ is a semi-projective resolution and aX is the cone). Thus

$$\underline{\mathrm{mod}}(R) = \langle \underline{\mathrm{mod}}^{\mathrm{ac}}(R), \underline{\mathrm{mod}}^{\mathrm{proj}}(R)\rangle \tag{5.3.4}$$

is a semiorthogonal decomposition.

Semi-injective complexes are defined dually to semi-projectives, and they satisfy the dual properties of 2.4.2. Thus there is also a semi-orthogonal decomposition

$$\underline{\mathrm{mod}}(R) = \langle \underline{\mathrm{mod}}^{\mathrm{inj}}(R), \underline{\mathrm{mod}}^{\mathrm{ac}}(R)\rangle. \tag{5.3.5}$$

The functor

$$\underline{\mathrm{mod}}^{\mathrm{proj}}(R) \to \underline{\mathrm{mod}}(R)/\underline{\mathrm{mod}}^{\mathrm{ac}}(R) = D(R)$$

is an equivalence and the inverse sends a complex to the homotopy class of a semi-projective resolution. Dually,

$$\underline{\mathrm{mod}}^{\mathrm{inj}}(R) \to \underline{\mathrm{mod}}(R)/\underline{\mathrm{mod}}^{\mathrm{ac}}(R) = D(R)$$

is an equivalence and the inverse sends a complex to the homotopy class of a semi-injective resolution.

5.4. **Definition of the coderived category.** Throughout C is a cocomplete cdgc.

Definition 5.4.1. We let dg-coacyc(C) be the localizing dg-subcategory of dg-comod(C) generated by the totalizations of short exact sequences of cdg C-comodules, and we assume these sequences are split over k. An object of dg-coacyc(C) is a *coacyclic* comodule. We let $\underline{\mathrm{coacyl}}(C)$ be the homotopy category of dg-comod(C).

Lemma 5.4.2. *Let C be a cocomplete cdgc and M a cdg C-comodule. The following are equivalent.*

(1) *M is coacyclic.*
(2) *There is a degree -1 endomorphism h of M such that $d_{\mathrm{Hom}}(h) = 1_M$.*
(3) *$L_{\tau^C}(M) \cong 0$, where ...*

Proof. The argument in [17, Theorem 6.3] shows that coacyclic comodules in our sense are exactly $\ker L_{\tau^C}$. □

Remark 5.4.3. We have modified the definition given in [17] by the extra assumption that the sequence is split over k. We note that whenever *loc. cit.* works with coalgebras, k is assumed to be a field, so our definitions agree in that case.

Lemma 5.4.4. *Let C be a cocomplete cdgc.*

(1) *There is a semiorthogonal decomposition*

$$\underline{\mathrm{comod}}(C) = \langle \underline{\mathrm{comod}}^{\mathrm{ext}}(C), \underline{\mathrm{coacyl}}(C)\rangle.$$

A localization triangle for N in $\underline{\mathrm{comod}}(C)$ is

$$\mathrm{cone}(\eta_N^{\tau^C})[-1] \to N \xrightarrow{\eta_N^{\tau^C}} R_{\tau^C} L_{\tau^C} N \to,$$

where $\eta_N^{\tau^C}$ is defined in 4.4.2.

(2) *If C is semiprojective, there is a semiorthogonal decomposition*

$$\underline{\mathrm{comod}}^{\mathrm{ext}}(C) = \langle \underline{\mathrm{comod}}^{\mathrm{ac}}(C), \underline{\mathrm{comod}}^{\mathrm{sp}}(C)\rangle.$$

A localization triangle for $C \otimes X$ in $\underline{\mathrm{comod}}^{\mathrm{ext}}(C)$ is

$$C \otimes pX \xrightarrow{\pi} C \otimes X \to \mathrm{cone}(\pi) \to,$$

where $pX \xrightarrow{\simeq} X$ is a semiprojective resolution of X, and the C-comodule structure on $pX \otimes C$ and morphism π are due to 5.2.5.

Proof. By 5.1.8.(1) R_{τ^C} is fully faithful with image $\underline{\mathrm{comod}}^{\mathrm{ext}}(C)$. By adjointness $\ker L_{\tau^C} \subseteq {}^{\perp}(\mathrm{image}\, R_{\tau^C}) = {}^{\perp}\big(\underline{\mathrm{comod}}^{\mathrm{ext}}(C)\big)$. Since R_{τ^C} is fully faithful, $\mathrm{cone}(\eta_N^{\tau^C})[-1]$ is in $\ker L_{\tau^C} = \underline{\mathrm{coacyl}}(C)$.

For part 2, since π is a quasi-isomorphism, $\mathrm{cone}(\pi)$ is an acyclic A_∞ A-module. Thus it is enough to show there are no nonzero maps from $\underline{\mathrm{comod}}^{\mathrm{sp}}(C)$ to $\underline{\mathrm{comod}}^{\mathrm{ac}}(C)$. Let $C \otimes P$ be semiprojective and $C \otimes X$ acyclic. By 5.1.10 there exist A_∞ $\Omega(C)$-module structures on P and X that we denote $\mathrm{Bar}\,\Omega(C) \otimes P$ and $\mathrm{Bar}\,\Omega(C) \otimes X$. We claim there are no maps between these A_∞-modules. Let $\alpha : \mathrm{Bar}\,\Omega(C) \otimes P \to \mathrm{Bar}\,\Omega(C) \otimes X$ be an A_∞-module morphism and consider the diagram

$$\begin{tikzcd} & \mathrm{Bar}\,\Omega(C) \otimes X \arrow[d, "\simeq"] \\ \mathrm{Bar}\,\Omega(C) \otimes P \arrow[r] & 0. \end{tikzcd}$$

Both $\alpha, 0 : \mathrm{Bar}\,\Omega(C) \otimes P \to \mathrm{Bar}\,\Omega(C) \otimes X$ make the diagram commute and for all $n \geq 2$ we have

$$H_* \operatorname{Hom}(\overline{\Omega(C)}^{\otimes n-1} \otimes P, X) = 0,$$

since P and $\overline{\Omega(C)}$ are semiprojective and X is acyclic. It follows from 3.3.2.(2) that α and the zero map are homotopic. Since ϕ_C^* is fully faithful and $\phi_C^*(\mathrm{Bar}\,\Omega(C) \otimes P) \cong C \otimes P$ and similarly for X, it follows that there are no non-zero maps in $\underline{\mathrm{comod}}(C)$ from $C \otimes P$ to $C \otimes X$. $\square$

Definition 5.4.5. The *coderived category* of a cocomplete cdgc C is the Verdier quotient

$$\mathsf{D}^{\mathrm{co}}(C) = \frac{\underline{\mathrm{comod}}(C)}{\mathrm{thick}(\underline{\mathrm{comod}}^{\mathrm{ac}}(C), \underline{\mathrm{coacyl}}(C))}.$$

When C is semiprojective, two applications of 5.3.2 to the decompositions of 5.4.4 show the coderived category of C is the homotopy category of semiprojective extended comodules. In more detail, it says the following.

Proposition 5.4.6. *Let C be a semiprojective cdgc. The composition*

$$\underline{\mathrm{comod}}^{\mathrm{sp}}(C) \hookrightarrow \underline{\mathrm{comod}}(C) \twoheadrightarrow \mathsf{D}^{\mathrm{co}}(C)$$

is an equivalence (that is the identity on objects). The inverse sends a comodule N to the homotopy class of

$$C \otimes^{\tau^C} p(\Omega(C) \otimes^{\tau^C} N),$$

where $p(\Omega(C) \otimes^{\tau^C} N) \to \Omega(C) \otimes^{\tau^C} N$ is a semiprojective resolution.

Example 5.4.7. Let $C = k$ be the trivial coalgebra. Then $\mathsf{D}^{\mathrm{co}}(k)$ is the usual derived category of k-modules.

Example 5.4.8. Relation to derived category (or homotopy category of injectives) of C^*?

Example 5.4.9. Lie algebra example? What is Positselski saying about this example showing the necessity of coderived categories?

Remark 5.4.10. When k is a field, this agrees with the definition of coderived category given in [17]. The proposition above gives another proof of ... in *loc. cit.* ...

From our definition, it is a definite possibility that $\mathsf{D}^{\mathrm{co}}(C)$ depends on the base ring k. In fact this often is not the case, in the following sense. If $k \to k'$ is a map of commutative rings, and C is a cdgc over k, then $C' = C \otimes_k k'$ is naturally a cdgc over k'. (Note that restriction of base rings is more delicate: we need to transfer the curvature $C' \to k'$ to a k-linear map $C' \to k$).

Proposition 5.4.11. *Let $k \to k'$ be a map of commutative rings. Let C' be a cdgc over k', and let C be C' considered as a cdgc over k via restriction. If C and C' are semiprojective, then there is a canonical equivalence*

$$\mathsf{D}^{\mathrm{co}}(C) \xrightarrow{\cong} \mathsf{D}^{\mathrm{co}}(C')$$

where the coderived categories are taken over k and k', respectively.

Proof. Let $C \otimes_k M$ be a semiprojective extended C-comodule. We have

$$C \otimes_k M \cong (C \otimes_{k'} k') \otimes_k M \cong C' \otimes_{k'} M'$$

where $M' = k' \otimes_k M$ is a semiprojective k'-complex. Check this is dg-functor from dg-category of semiprojectives...Taking homotopy and using 5.4.6, this defines a functor as claimed. Clear it's canonical? What does it do to morphisms? Should be clearly fully faithful...To see essentially surjective, let N be any cdg C'-comodule that is projective as a graded k'-module, e.g. a semiprojective extended C'-comodule. Then consider

$$k' \otimes_k \mathrm{Bar}_k(k') \otimes_k N \xrightarrow{\simeq} N.$$

...this might not work; what if we assume that N is semiprojective extended? □

5.5. **Defintion of the derived category of an A_∞-algebra.** Let A be an A_∞-algebra with split unit. To define the derived category of A, recall that dg-mod$^\infty(A)$ is a full dg-subcategory of dg-comod(Bar A). The shift $M[1]$ of an A_∞-module M was defined in 4.3.9. We have an isomorphism of cdg Bar A-comodules

$$(\mathrm{Bar}\, A \otimes M)[1] \xrightarrow{(1 \otimes s^{-1})s} \mathrm{Bar}\, A \otimes M[1]$$

and thus dg-mod$^\infty(A)$ is closed under shifts in dg-comod(Bar A).

Definition 5.5.1. Let A be an A_∞-algebra with split unit, let L, M be A_∞ A-modules and let $f : \operatorname{Bar} A \otimes L \to M$ be a morphism of A_∞ A-modules. The *cone of* f is the A_∞ A-module with underlying module $L[1] \oplus M$ and structure morphism

$$m^{\mathrm{cone}(f)} : \operatorname{Bar} A \otimes (L[1] \oplus M) \xrightarrow{\cong} (\operatorname{Bar} A \otimes L[1]) \oplus (\operatorname{Bar} A \otimes M)$$
$$\xrightarrow{\begin{pmatrix} m^{L[1]} & 0 \\ f(1 \otimes s^{-1}) & m^M \end{pmatrix}} L[1] \oplus M.$$

If $g : \operatorname{Bar} A \otimes L \to \operatorname{Bar} A \otimes M$ is the map of cdg $\operatorname{Bar} A$-comodules corresponding to an f as above, then one checks that $\mathrm{cone}(f)$ and the cone of g are isomorphic. Thus $\text{dg-mod}^\infty(A)$ is closed under cones in $\text{dg-comod}(A)$, and so is pretriangulated.

Definition 5.5.2. An A_∞ A-module M is *acyclic* if the complex (M, m_1^M) is acyclic; M is *semiprojective* if the complex (M, m_1^M) is semiprojective. We denote by $\underline{\mathrm{mod}}^\infty_{\mathrm{ac}}(A)$ full subcategory of acyclic modules in $\underline{\mathrm{mod}}^\infty(A)$ and $\underline{\mathrm{mod}}^\infty_{\mathrm{sp}}(A)$ the full subcategory of semiprojective modules.

The following is 5.4.4.(2) applied to the cdgc $\operatorname{Bar} A$.

Lemma 5.5.3. *Let A be a semiprojective A_∞-algebra with split unit. There is a semiorthogonal decomposition*

$$\underline{\mathrm{mod}}^\infty(A) = \langle \underline{\mathrm{mod}}^\infty_{\mathrm{ac}}(A), \underline{\mathrm{mod}}^\infty_{\mathrm{sp}}(A) \rangle.$$

The localization triangle for A_∞ A-module M is

$$pM \xrightarrow{\pi} M \to \mathrm{cone}(\pi) \to,$$

where $\pi : pM \to M$ is a semiprojective resolution of the complex (M, m_1^M).

Definition 5.5.4. Let A be an A_∞-algebra with split unit. The *derived category* of A is the Verdier quotient

$$\mathsf{D}^\infty(A) = \underline{\mathrm{mod}}^\infty(A) / \underline{\mathrm{mod}}^\infty_{\mathrm{ac}}(A).$$

Applying 5.3.2 to 5.5.3, gives the following.

Proposition 5.5.5. *Let A be an A_∞-algebra with split unit. The composition*

$$\underline{\mathrm{mod}}^\infty_{\mathrm{sp}}(A) \hookrightarrow \underline{\mathrm{mod}}^\infty(A) \twoheadrightarrow \mathsf{D}^\infty(A)$$

is an equivalence. The inverse sends an A_∞ A-module to the image of a semiprojective resolution.

If A is a dg-algebra, i.e. $m_n^A = 0$ for all $n \geq 3$, with a split unit, then the above derived category agrees with the usual derived category of A.

Definition 5.5.6. Let A be a dg-algebra with split unit. We let $\underline{\mathrm{mod}}^{\mathrm{dg}}(A)$ be the subcategory of $\underline{\mathrm{mod}}^\infty(A)$ with objects dg A-modules, i.e. A_∞ A-modules M with $m_n^M = 0$ for $n \geq 3$ and morphisms of dg-modules.

It follows from 4.3.8 and 4.3.14 that if $\tau : C \to A[1]$ is a twisting cochain, with A a dg-algebra with split unit, then the functor $L_\tau : \underline{\mathrm{comod}}(C) \to \underline{\mathrm{mod}}^\infty(A)$ factors through the functor $\underline{\mathrm{mod}}^{\mathrm{dg}}(A) \to \underline{\mathrm{mod}}^\infty(A)$.

Lemma 5.5.7. *Let A be a dg-algebra with split unit. The canonical functor $\underline{\mathrm{mod}}^{\mathrm{dg}}(A) \to \underline{\mathrm{mod}}^{\infty}(A)$ is an equivalence. In particular, there is an equivalence*

$$\mathsf{D}(A) \to \mathsf{D}^{\infty}(A)$$

of the classical derived category of dg A-modules with the derived category of A_∞ A-modules.

Proof. Consider the universal twisting cochain $\tau_A : \mathrm{Bar}\, A \to A[1]$. The functor L_{τ_A} is full and essentially surjective. It also factors through $\underline{\mathrm{mod}}^{\mathrm{dg}}(A) \to \underline{\mathrm{mod}}^{\infty}(A)$. Since this functor is faithful, it must be an equivalence. This equivalence clearly takes acyclic dg A-modules to acyclic A_∞ A-modules, so we also have an equivalence of derived categories. □

5.6. **Compact generation of the (co)derived category.**

Lemma 5.6.1. *Let A be an A_∞-algebra with split unit and let X be an A_∞ A-module. The map*

$$\mathrm{Hom}^{\infty}_A(A, X) \to X$$
$$\phi \mapsto \phi_1(1_A)$$

is a quasi-isomorphism of complexes.

Definition 5.6.2. Let $\mathcal{T}$ be a triangulated category. An object X is *compact* if $\mathrm{Hom}_{\mathcal{T}}(X, -)$ commutes with coproducts. A triangulated subcategory of $\mathcal{T}$ is *localizing* if it is closed under coproducts. The *localizing subcategory generated by* X is the smallest localizing subcategory that contains X. An object X is a *compact generator* of $\mathcal{T}$ if it is compact and the localizing category generated by it is $\mathcal{T}$.

Proposition 5.6.3. *Let A be a semiprojective A_∞-algebra with split unit and C a semiprojective cdgc.*

1. *The A_∞ A-module A is a compact generator of $\underline{\mathrm{mod}}^{\infty}_{\mathrm{sp}}(A)$, and the image of A in $\mathsf{D}^{\infty}(A)$ is a compact generator.*
2. *The cdg C-comodule $C \otimes^{\tau^C} \Omega(C)$ is a compact generator of $\underline{\mathrm{comod}}^{\mathrm{sp}}(C)$ and the image of k in $\mathsf{D}^{\mathrm{co}}(C)$ is a compact generator.*

Proof. Let X be an object of $\underline{\mathrm{mod}}^{\infty}_{\mathrm{sp}}(A)$. By 5.6.1

$$\mathrm{Hom}_{\underline{\mathrm{mod}}^{\infty}_{\mathrm{sp}}(A)}(A, X) = H_0\left(\mathrm{Hom}^{\infty}_A(A, X)\right) \cong H_0(X),$$

and it follows that A is compact. If $\mathrm{Hom}_{\underline{\mathrm{mod}}^{\infty}_{\mathrm{sp}}(A)}(A[i], X) = 0$ for all i, then X is acyclic and so $X \cong 0$ in $\underline{\mathrm{mod}}^{\infty}_{\mathrm{sp}}(A)$ by 5.5.3. It now follows from [15, 2.1.2] that the localizing subcategory generated by A is $\underline{\mathrm{mod}}^{\infty}_{\mathrm{sp}}(A)$.

For the second part, we have the following commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{comod}}^{\text{sp-ext}}(\mathrm{Bar}\,\Omega(C)) \arrow[d, "\phi_C^*"', "\cong"] & \underline{\mathrm{mod}}^{\infty}_{\mathrm{sp}}(\Omega(C)) \arrow[l, "R_{\tau_{\Omega(C)}}"', "="] \arrow[dl, "R_{\tau^C}"] \\
\underline{\mathrm{comod}}^{\text{sp-ext}}(C) &
\end{tikzcd},$$

see 4.3.15. By the previous part, $R_{\tau_{\Omega(C)}}(\Omega(C)) = \mathrm{Bar}\,\Omega(C) \otimes^{\tau_{\Omega(C)}} \Omega(C)$ is a compact generator of $\underline{\mathrm{comod}}^{\text{sp-ext}}(\mathrm{Bar}\,\Omega(C))$. Thus $\phi_C^*(\mathrm{Bar}\,\Omega(C) \otimes^{\tau_{\Omega(C)}} \Omega(C)) \cong C \otimes^{\tau^C} \Omega(C)$ is a compact generator of $\underline{\mathrm{comod}}^{\text{sp-ext}}(C)$. By 5.4.6, the equivalence $\mathsf{D}^{\mathrm{co}}(C) \to$

$\underline{\text{comod}}^{\text{sp-ext}}(C)$ sends k to $C \otimes^{\tau^C} \Omega(C)$ since $L_{\tau^C} k = \Omega(C) \otimes^{\tau^C} k \cong \Omega(C)$ and $\Omega(C)$ is a semiprojective complex. Thus k is a compact generator of $\mathsf{D}^{\text{co}}(C)$. □

6. Derived functors between (co)derived categories

Given a twisting cochain $\tau : C[1] \to A$, we want to define an adjoint pair of functors between $\mathsf{D}^{\text{co}}(C)$ and $\mathsf{D}^{\infty}(A)$. In the adjoint pair 4.4.9, the functor R_τ does not necessarily send $\underline{\text{comod}}^{\text{ac}}(C)$, nor $\underline{\text{coacyl}}(C)$, to $\underline{\text{mod}}^{\infty}_{\text{ac}}(A)$ and thus does not define a functor between (co)derived categories. To remedy this, we use the semiorthogonal decompositions of the (co)module categories introduced above.

We will introduce derived functors given by a morphism of cdg coalgebras, a morphism of A_∞-algebras, and a twisting cochain.

6.1. **Derived functors and semiorthogonal decompositions.** In this subsection, we recall Deligne's definition of derived functor between triangulated categories, and observe that in special cases they may be computed using semiorthogonal decompositions. This simple result will used constantly in the sequel.

Definition 6.1.1. Cohomological functors of triangulated categories...Yoneda embedding...representable...dual of all above?

where $\text{mod}(\mathcal{S})$ is the category of cohomological functors $\mathcal{S} \to \text{mod}(k)$,

Definition 6.1.2 (Ref SGA4 and Drinfeld 5.1)**.** Let $G : \mathcal{T} \to \mathcal{S}$ be a k-linear triangulated functor and $\mathcal{B}$ a triangulated subcategory of $\mathcal{T}$. The *right derived functor of G (with respect to $\mathcal{B}$)* is the functor

$$\mathbf{R}G : \mathcal{T}/\mathcal{B} \to \text{mod}(\mathcal{S}),$$

defined for $Y \in \mathcal{T}$ and $X \in \mathcal{S}$ by

$$\mathbf{R}G(Y)(X) = \text{colim}_{Y \to Z \in Q_Y} \text{Hom}_{\mathcal{S}}(X, G(Z)),$$

where Q_Y is the full subcategory of the comma category under Y with objects $f : Y \to Z$ such that $\text{cone}(f) \in \mathcal{B}$. We say $\mathbf{R}G$ is *defined* at $Y \in \mathcal{T}$ if $\mathbf{R}G(Y)$ is representable.

Lemma 6.1.3. *Let $G : \mathcal{T} \to \mathcal{S}$ be a k-linear triangulated functor and $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ a semiorthogonal decomposition. Then $\mathbf{R}G : \mathcal{T}/\mathcal{B} \to \mathcal{S}$ is defined on all objects and is given by $\mathbf{R}G(Y) = G(Y'')$, where $Y' \to Y \to Y'' \to$ is the localization triangle of Y.*

Proof. The category $Q_Y = \{f : Y \to Z \mid \text{cone}(f) \in \mathcal{B}\}$ has terminal object $Y \to Y''$, thus the colimit $\mathbf{R}G(Y)$ over $Q(Y)$ is $G(Y'')$. □

Example 6.1.4. Let R be an associative ring and consider the semiorthogonal decompositions

$$\mathcal{T} = \underline{\text{mod}}(R) = \langle \underline{\text{mod}}^{\text{inj}}(R), \underline{\text{mod}}^{\text{ac}}(R) \rangle = \langle \mathcal{A}, \mathcal{B} \rangle,$$

of Example 5.3.3. Fix an object M in $\underline{\text{mod}}(R)$, and consider

$$G = \text{Hom}_R(M, -) : \underline{\text{mod}}(R) \to \mathsf{D}(\mathbb{Z}).$$

The right derived functor $\mathbf{R}G$ on an object N is $\mathbf{R}G(N) = iN$, where $N \to iN$ is a semi-injective resolution. Thus $\mathbf{R}G = \mathbf{R}\,\text{Hom}_R(M, -)$.

Dually, we have left derived functors.

Definition 6.1.5 (Ref SGA4 and Drinfeld 5.1)**.** Let $F : \mathcal{T} \to \mathcal{S}$ be a k-linear triangulated functor and $\mathcal{A}$ a triangulated subcategory of $\mathcal{T}$. The *left derived functor of T with respect to $\mathcal{A}$* is the functor

$$\mathbf{L}F : \mathcal{T}/\mathcal{A} \to \mathrm{mod}(\mathcal{S})^{\mathrm{op}},$$

defined for $Y \in \mathcal{T}$ and $X \in \mathcal{S}$ by

$$\mathbf{L}F(Y)(X) = \lim_{W \to Y \in P_Y} \mathrm{Hom}_{\mathcal{S}}(F(W), X),$$

where P_Y is the full subcategory of the comma category over Y with objects $f : W \to Y$ such that $\mathrm{cone}(f) \in \mathcal{A}$. We say $\mathbf{L}F$ is *defined* at $Y \in \mathcal{T}$ if $\mathbf{L}F(Y)$ is representable.

Lemma 6.1.6. *Let $F : \mathcal{T} \to \mathcal{S}$ be a k-linear triangulated functor and $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ a semiorthogonal decomposition. Then $\mathbf{L}F : \mathcal{T}/\mathcal{A} \to \mathcal{S}$ is defined on all objects and is given by $\mathbf{L}F(Y) = F(Y')$, where $Y' \to Y \to Y'' \to$ is the localization triangle of Y.*

Example 6.1.7. Let R be an associative ring and consider the semiorthogonal decomposition

$$\mathcal{T} = \underline{\mathrm{mod}}(R) = \langle \underline{\mathrm{mod}}^{\mathrm{ac}}(R), \underline{\mathrm{mod}}^{\mathrm{proj}}(R) \rangle = \langle \mathcal{A}, \mathcal{B} \rangle$$

of Example 5.3.3. Fix an object $M \in \underline{\mathrm{mod}}(R^{\mathrm{op}})$ and define $F : \underline{\mathrm{mod}}(R) \to \mathsf{D}(\mathbb{Z})$ to be $F(N) = M \otimes_R N$. Then

$$\mathbf{L}F(N) = M \otimes_R pN$$

where $pN \to N$ is a semi-projective resolution; thus $\mathbf{L}F = M \otimes^{\mathbf{L}}_R -$.

Lemma 6.1.8. *Consider semi-orthogonal decompositions of k-linear triangulated categories*

$$\mathcal{T} = \langle \mathcal{A}_1, \mathcal{B}_1 \rangle \qquad \mathcal{S} = \langle \mathcal{A}_2, \mathcal{B}_2, \rangle$$

and let

$$\begin{tikzcd} \mathcal{T} \arrow[r, shift left, "F"] & \mathcal{S} \arrow[l, shift left, "G"] \end{tikzcd}$$

be an adjoint pair of triangulated functors such that

$$F(\mathcal{B}_1) \subseteq \mathcal{B}_2 \quad \textit{and} \quad G(\mathcal{A}_2) \subseteq \mathcal{A}_1. \tag{6.1.9}$$

There are two adjoint pairs:

$$\begin{tikzcd} \mathcal{T}/\mathcal{B}_1 \arrow[r, shift left, "\overline{F}"] & \mathcal{S}/\mathcal{B}_2 \arrow[l, shift left, "\mathbf{R}G"] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{T}/\mathcal{A}_1 \arrow[r, shift left, "\mathbf{L}F"] & \mathcal{S}/\mathcal{A}_2 \arrow[l, shift left, "\overline{G}"] \end{tikzcd}$$

where $\overline{F}, \overline{G}$ are the functors induced by F, G, using (6.1.9).

Remark 6.1.10. It is easy to convince oneself that such a result need not hold if we replace the condition (6.1.9) with $F(\mathcal{A}_1) \subseteq \mathcal{A}_2$ and $G(\mathcal{B}_2) \subseteq \mathcal{B}_1$.

Remark 6.1.11. For later use, let us record the (co)unit maps of the adjoint $(\overline{F}, \mathbf{R}G)$; those of $(\mathbf{L}F, \overline{G})$ are dual. Let $X \in \mathcal{T}$ and $Y \in \mathcal{S}$ with localization sequences

$$X' \to X \xrightarrow{i} X'' \to \quad \text{and} \quad Y' \to Y \xrightarrow{i} Y'' \to .$$

We write η, ϵ for the unit, counit of the adjoint pair (F, G). The unit of $(\overline{F}, \mathbf{R}G)$ is the image in $\mathcal{T}/\mathcal{B}_1$ of

$$X \xrightarrow{i} X'' \xrightarrow{\eta_{X''}} G(F(X'')) = G(F(X)'') = \mathbf{R}GF(X).$$

The counit is

$$\overline{F}\mathbf{R}G(\overline{Y}) = \overline{F}(G(Y'')) \xrightarrow{\overline{F}(i)^{-1}} \overline{F}(G(Y)) = \overline{FG(Y)} \xrightarrow{\overline{\eta_Y}} \overline{Y},$$

where $\overline{Y}$ is the image of Y in $\mathcal{S}/\mathcal{B}_2$.

6.2. **Derived adjoint given by a morphism of cdg coalgebras.** Let $(\phi, a) : C \to D$ be a morphism of cdg coalgebras and consider the semi orthogonal decompositions given by 5.4.4.(1).

$$\underline{\mathrm{comod}}(C) = \langle \underline{\mathrm{comod}}^{\mathrm{ext}}(C), \ker L_{\tau^C} \rangle$$

$$\underline{\mathrm{comod}}(D) = \langle \underline{\mathrm{comod}}^{\mathrm{ext}}(D), \ker L_{\tau^D} \rangle$$

By Lemma **??**, there are functors

$$\begin{tikzcd}
\underline{\mathrm{comod}}(C) \arrow[r, "\phi_*"] & \underline{\mathrm{comod}}(D) \\
\underline{\mathrm{comod}}^{\mathrm{ext}}(C) \arrow[u] & \underline{\mathrm{comod}}^{\mathrm{ext}}(D) \arrow[u] \arrow[l, "\phi^*"]
\end{tikzcd}$$

that satisfy the adjoint condition of Definition **??**. We now show that $\phi_*(\underline{\mathrm{coacyl}}(C)) \subseteq \ker L_{\tau^D}$, and thus these functors satisfy the other condition of Definition **??**.

Define a twisting cochain $\tau : C \to \Omega(D)$ to be the composition of τ^D and (ϕ, a), as in 4.3.5, so we have a commutative diagram

$$\begin{tikzcd}
D \arrow[r, "\tau^D"] & \Omega(D). \\
C \arrow[u, "(\phi,a)"] \arrow[ur, "\tau"'] &
\end{tikzcd}$$

For a cdg C-comodule N, by 4.4.7 we have

$$L_\tau N = L_{\tau^D}(\phi_* N)$$

and by 6.2.1.(1) below, we have

$$L_\tau N \cong \Omega(D) \otimes_{\Omega(C)} L_{\tau^C} N.$$

Thus if N is in $\ker L_{\tau^C}$, $\phi_* N$ is in $\underline{\mathrm{coacyl}}(D)$.

Lemma 6.2.1. *Let $\tau : C \to A[1]$ be a twisting cochain, with A a dg-algebra. Let $\varphi : \Omega(C) \to A$ be the unique map of dg-algebras given by 5.1.4 such that $\tau = \varphi\tau^C$.*

(1) *For a cdg C-comodule N, we have*

$$L_\tau N \cong A \otimes_{\Omega(C)} L_{\tau^C} N.$$

(2) *For a dg A-module M, we have*

$$R_\tau M \cong R_{\tau^C}(\varphi^* M),$$

where φ^* is restriction along φ.

Applying **??** we have the following.

Proposition 6.2.2. *Let $(\phi, a) : C \to D$ be a morphism of cocomplete cdg coalgebras. There is a diagram*

$$\begin{tikzcd}
\underline{\mathrm{comod}}(C)/\ker L_{\tau^C} \arrow[r, shift left, "\overline{\phi_*}"] \arrow[d, shift right] \arrow[d, phantom, "\cong"] & \underline{\mathrm{comod}}(D)/\ker L_{\tau^D} \arrow[l, shift left, "\mathbf{R}\phi^*"] \arrow[d, shift right] \arrow[d, phantom, "\cong"] \\
\underline{\mathrm{comod}}^{\mathrm{ext}}(C) \arrow[r, shift left, "\mathbf{L}\phi_*"] \arrow[u, shift right] & \underline{\mathrm{comod}}^{\mathrm{ext}}(D) \arrow[l, shift left, "\phi^*"] \arrow[u, shift right]
\end{tikzcd}$$

with the rows adjoint pairs and all squares commutative, up to the (co)units of the vertical equivalences.

To describe the functors, let $\tau : C \to \Omega(D)$ be the twisting cochain given by the composition of τ^D and (ϕ, a), as in 4.3.5. Then, using 4.4.7.(1), we have

$$\mathbf{L}\phi_*(C \otimes X) \cong D \otimes^{\tau^D} \Omega(D) \otimes^{\tau} (C \otimes X).$$

Also, using 4.4.7.(2), we have

$$\mathbf{R}\phi^*(N) \cong C \otimes^{\tau} \Omega(D) \otimes^{\tau^D} N.$$

We now assume that C, D are semiprojective and consider the decompositions given by 5.4.4.(2)

$$\underline{\mathrm{comod}}^{\mathrm{ext}}(C) = \langle \underline{\mathrm{comod}}^{\mathrm{ac}}(C), \underline{\mathrm{comod}}^{\mathrm{sp}}(C) \rangle$$

$$\underline{\mathrm{comod}}^{\mathrm{ext}}(D) = \langle \underline{\mathrm{comod}}^{\mathrm{ac}}(D), \underline{\mathrm{comod}}^{\mathrm{sp}}(D) \rangle.$$

One checks that $\mathbf{L}\phi_*$ sends sends $\underline{\mathrm{comod}}^{\mathrm{sp}}(C)$ to $\underline{\mathrm{comod}}^{\mathrm{sp}}(D)$, so we have the following diagram

$$\begin{tikzcd}
\underline{\mathrm{comod}}(D)/\underline{\mathrm{coacyl}}(D) \arrow[r, "\mathbf{R}\phi^*"'] & \underline{\mathrm{comod}}(C)/\underline{\mathrm{coacyl}}(C) \\
\underline{\mathrm{comod}}^{\mathrm{ext}}(D) \arrow[r, "\phi^*"] \arrow[u] & \underline{\mathrm{comod}}^{\mathrm{ext}}(C) \arrow[u] \\
\underline{\mathrm{comod}}^{\mathrm{sp}}(D) \arrow[u] & \underline{\mathrm{comod}}^{\mathrm{sp}}(C) \arrow[l, "\mathbf{L}\phi_*"'] \arrow[u]
\end{tikzcd} \tag{6.2.3}$$

with the top rectangle commutative, up to isomorphism. Since the lower rectangle satisfies the adjoint condition by 6.2.2, the outside rectangle satisfies the adjoint condition of **??**. Also note that since $\phi^*(\underline{\mathrm{comod}}^{\mathrm{ac}}(D)) \subseteq \underline{\mathrm{comod}}^{\mathrm{ac}}(C)$, $R\phi^*$ takes the image of $\underline{\mathrm{comod}}^{\mathrm{ac}}(D)$ to the image of $\underline{\mathrm{comod}}^{\mathrm{ac}}(C)$. Thus we may apply **??** (twice) to get the following.

Theorem 6.2.4. *Let C, D be semiprojective cdg coalgebras and $(\phi, a) : C \to D$ a morphism. There is a diagram*

$$\begin{tikzcd}
\mathsf{D}^{\mathrm{co}}(D) \arrow[d, shift right, "\cong"'] \arrow[rr, shift right, "\mathbf{R}\phi^*"'] & & \mathsf{D}^{\mathrm{co}}(C) \arrow[ll, shift right, "\mathbf{LL}\phi_*"'] \arrow[d, shift right, "\cong"'] \\
\underline{\mathrm{comod}}^{\mathrm{ext}}(D)/\underline{\mathrm{comod}}^{\mathrm{ac}}(D) \arrow[u, shift right] \arrow[d, shift right, "\cong"'] \arrow[rr, shift right, "\phi^*"'] & & \underline{\mathrm{comod}}^{\mathrm{ext}}(C)/\underline{\mathrm{comod}}^{\mathrm{ac}}(C) \arrow[ll, shift right, "(\mathbf{LL}\phi_*)'"'] \arrow[u, shift right] \arrow[d, shift right, "\cong"'] \\
\underline{\mathrm{comod}}^{\mathrm{sp}}(D) \arrow[u, shift right] \arrow[rr, shift right, "\mathbf{RR}\phi^* \cong \phi^*"'] & & \underline{\mathrm{comod}}^{\mathrm{sp}}(C) \arrow[ll, shift right, "\mathbf{L}\phi_*"'] \arrow[u, shift right]
\end{tikzcd}$$

with the rows adjoint pairs and all squares commutative, up to the (co)units of the vertical equivalences.

Let $\tau : C \to \Omega(D)$ be the twisting cochain given by the composition of τ^D and (ϕ, a), as in 4.3.5. We have

$$\mathbf{LL}\phi_*(N) \cong D \otimes^{\tau^D} p(\Omega(D) \otimes^{\tau} N),$$

$$(\mathbf{LL}\phi_*)'(C \otimes X) \cong D \otimes^{\tau^D} \Omega(D) \otimes^{\tau} (C \otimes pX)$$

$$\mathbf{RR}\phi^*(D \otimes Y) \cong \phi^*(D \otimes Y) \cong C \otimes Y,$$

where $p(\Omega(D) \otimes^{\tau} N) \to \Omega(D) \otimes^{\tau} N$ is a semiprojective resolution over k with A_∞ $\Omega(D)$-module structure given by 3.3.2, $pX \to X$ is a semiprojective resolution and $C \otimes pX$ is the extended comodule structure given by 5.2.5.

Proof. We only need to show the formulas for the functor. By definition we have

$$\mathbf{LL}\phi_*(N) = D \otimes^{\tau^D} \Omega(D) \otimes^{\tau} C \otimes^{\tau^C} p(\Omega(C) \otimes N).$$

Since $\mathbf{L}\phi_*$ preserves semiprojective extended comodules, this is isomorphic to

$$D \otimes^{\tau^D} \Omega(D) \otimes^{\tau^D} D \otimes^{\tau^D} p(\Omega(D) \otimes^{\tau} N).$$

Finally, the above is isomorphic to $D \otimes^{\tau^D} p(\Omega(D) \otimes^{\tau} N)$ by 4.4.5. The formula for $(\mathbf{LL}\phi_*)'$ is by definition, and the third isomorphism follows from the diagram (6.2.3) and the fact that ϕ^* preserves semiprojective extended comodules. $\square$

Definition 6.2.5. Let $\varphi : A \to B$ be a morphism of semiprojective A_∞-algebras and $\phi : \mathrm{Bar}\, A \to \mathrm{Bar}\, B$ the corresponding morphism of cdg coalgebras. Define

$$\mathbf{L}\varphi_* := (\mathbf{LL}\phi_*)' : \mathsf{D}^\infty(A) = \frac{\underline{\mathrm{comod}}^{\mathrm{ext}}(\mathrm{Bar}\, A)}{\underline{\mathrm{comod}}^{\mathrm{ac}}(\mathrm{Bar}\, A)} \to \frac{\underline{\mathrm{comod}}^{\mathrm{ext}}(\mathrm{Bar}\, B)}{\underline{\mathrm{comod}}^{\mathrm{ac}}(\mathrm{Bar}\, B)} = \mathsf{D}^\infty(B)$$

$$\varphi^* := \phi^* : \mathsf{D}^\infty(B) = \frac{\underline{\mathrm{comod}}^{\mathrm{ext}}(\mathrm{Bar}\, B)}{\underline{\mathrm{comod}}^{\mathrm{ac}}(\mathrm{Bar}\, B)} \to \frac{\underline{\mathrm{comod}}^{\mathrm{ext}}(\mathrm{Bar}\, A)}{\underline{\mathrm{comod}}^{\mathrm{ac}}(\mathrm{Bar}\, A)} = \mathsf{D}^\infty(A)$$

This is the middle row in the diagram of 6.2.4 applied to ϕ. Thus we have the following.

Corollary 6.2.6. *Let $\varphi : A \to B$ be a morphism of semiprojective A_∞-algebras and $\phi : \operatorname{Bar} A \to \operatorname{Bar} B$ the corresponding morphism of cdg coalgebras. There is a diagram*

$$\begin{tikzcd}
\mathsf{D}^{\mathrm{co}}(\operatorname{Bar} B) \arrow[r, shift left, "\mathbf{R}\phi^*"'] \arrow[d, shift right, "\cong"] & \mathsf{D}^{\mathrm{co}}(\operatorname{Bar} A) \arrow[l, shift left, "\mathbf{L}\mathbf{L}\phi_*"'] \arrow[d, shift right, "\cong"] \\
\mathsf{D}^{\infty}(B) \arrow[u, shift right] \arrow[r, shift right, "\varphi^*"'] & \mathsf{D}^{\infty}(A) \arrow[u, shift right] \arrow[l, shift right, "\mathbf{L}\varphi_*"']
\end{tikzcd}$$

with the rows adjoint pairs and all squares commutative, up to the (co)units of the vertical equivalences. We have

$$\mathbf{L}\varphi_*(N) \cong B \otimes^{\tau} \operatorname{Bar} A \otimes^{\tau_A} pN$$

where $\tau : \operatorname{Bar} A \to B[1]$ is the twisting cochain corresponding to φ. The functor φ^* is restriction of A_∞-modules along the morphism φ.

Finally, we use Theorem 6.2.4 to define an adjoint pair of functors between (co)derived categories given by a twisting cochain.

Definition 6.2.7. Let C be a semiprojective cocomplete cdgc, A a semiprojective A_∞-algebra with split unit, and $\tau : C \to A[1]$ a twisting cochain. Let $(\phi, a) : C \to \operatorname{Bar} A$ be the morphism of cdg coalgebras corresponding to τ. In this case, the diagram of 6.2.4 is the following.

(6.2.8)
$$\begin{tikzcd}
\mathsf{D}^{\mathrm{co}}(\operatorname{Bar} A) \arrow[r, shift right, "\mathbf{R}\phi^*"'] \arrow[d, shift right, "\cong"] & \mathsf{D}^{\mathrm{co}}(C) \arrow[l, shift right, "\mathbf{L}\mathbf{L}\phi_*"'] \arrow[d, shift right, "\cong"] \\
\mathsf{D}^{\infty}(A) = \dfrac{\underline{\mathrm{comod}}^{\mathrm{ext}}(\operatorname{Bar} A)}{\underline{\mathrm{comod}}^{\mathrm{ac}}(\operatorname{Bar} A)} \arrow[u, shift right] \arrow[r, shift right, "\phi^*"'] \arrow[d, shift right, "\cong"] & \underline{\mathrm{comod}}^{\mathrm{ext}}(C)/\underline{\mathrm{comod}}^{\mathrm{ac}}(C) \arrow[u, shift right] \arrow[l, shift right, "(\mathbf{L}\mathbf{L}\phi_*)'"'] \arrow[d, shift right, "\cong"] \\
\underline{\mathrm{comod}}^{\mathrm{sp}}(\operatorname{Bar} A) \arrow[u, shift right] \arrow[r, shift right, "\mathbf{R}\mathbf{R}\phi^* \cong \phi^*"'] & \underline{\mathrm{comod}}^{\mathrm{sp}}(C) \arrow[u, shift right] \arrow[l, shift right, "\mathbf{L}\phi_*"']
\end{tikzcd}$$

Define an adjoint pair $(\mathbf{L}_\tau, \mathbf{R}_\tau)$ via the following diagram.

$$\begin{tikzcd}
 & \mathsf{D}^{\mathrm{co}}(\operatorname{Bar} A) \arrow[dl, shift right] \arrow[dr, shift left, "\mathbf{R}\phi^*"'] & \\
\mathsf{D}^{\infty}(A) \arrow[ur, shift right, "\cong"] \arrow[rr, dashed, shift right, "\mathbf{R}_\tau"'] & & \mathsf{D}^{\mathrm{co}}(C) \arrow[ul, shift left, "\mathbf{L}\mathbf{L}\phi^*"'] \arrow[ll, dashed, shift right, "\mathbf{L}_\tau"']
\end{tikzcd}$$

Example 6.2.9. Let A be a semiprojective A_∞-algebra and $\tau_A : \operatorname{Bar} A \to A[1]$ the universal twisting cochain. Then $\phi : \operatorname{Bar} A \to \operatorname{Bar} A$ is the identity map, and the pair $(\mathbf{L}_{\tau_A}, \mathbf{R}_{\tau_A})$ is the usual equivalence

$$\begin{tikzcd}
\mathsf{D}^{\infty}(A) \arrow[r, shift left] \arrow[r, phantom, "\cong"] & \mathsf{D}^{\mathrm{co}}(\operatorname{Bar} A) \arrow[l, shift left]
\end{tikzcd}.$$

Using the properties above, we have the following.

Proposition 6.2.10. *Let $\tau : C \to A[1]$ be a twisting cochain from a semiprojective cdgc to a semiprojective A_∞-algebra. In the following diagram,*

$$
\begin{tikzcd}
\mathsf{D}^{\mathrm{co}}(C) \arrow[r, shift left, "\mathbf{L}_\tau"] \arrow[d, shift right, "\cong"'] & \mathsf{D}^{\infty}(A) \arrow[l, shift left, "\mathbf{R}_\tau"] \arrow[d, shift right, "\cong"'] \\
\underline{\mathrm{comod}}^{\mathrm{sp}}(C) \arrow[r, shift left, "L_\tau"] \arrow[u, shift right] & \underline{\mathrm{mod}}^{\infty}_{\mathrm{sp}}(A), \arrow[l, shift left, "R_\tau"] \arrow[u, shift right]
\end{tikzcd}
\tag{6.2.11}
$$

the rows are adjoint pairs and all squares are commutative, up to the (co)units of the vertical equivalences.

We have

$$\mathbf{L}_\tau(N) \cong \operatorname{Bar} A \otimes^{\tau_A} p(A \otimes^\tau N)$$
$$\mathbf{R}_\tau(M) \cong C \otimes^\tau pM,$$

where $p(A \otimes^\tau N) \to A \otimes^\tau N$ and $pM \to M$ are semiprojective resolutions with A_∞ A-module structures given by 3.3.2. The functors L_τ, R_τ are those defined in 4.3.10.

Proof. We only need to show the description of the functors. For $\mathbf{L}_\tau$, this follows from the description of $\mathbf{LL}\phi^*$, where $\phi : C \to \operatorname{Bar} A$ is the morphism determined by τ, 4.4.7, and 5.1.11. The description of $\mathbf{R}_\tau$ follows from the definition. In the bottom row, we have the adjoint pair $(\mathbf{L}\phi_*, \mathbf{RR}\phi^* \cong \phi^*)$ by 6.2.7. But $\phi^* = R_\tau$ by definition, and $\mathbf{L}\phi^* \cong L_\tau$ by the description given in 6.2.2. □

6.3. **Weak equivalences.**

Definition 6.3.1. A morphism $(\phi, a) : C \to D$ of semiprojective cdg coalgebras is a *weak equivalence* if the adjoint pair $(\mathbf{LL}\phi_*, \mathbf{R}\phi^*)$, defined in 6.2.4,

$$
\begin{tikzcd}
\mathsf{D}^{\mathrm{co}}(D) \arrow[r, shift right, "\mathbf{R}\phi^*"'] & \mathsf{D}^{\mathrm{co}}(C), \arrow[l, shift right, "\mathbf{LL}\phi_*"']
\end{tikzcd}
$$

is an equivalence.

Example 6.3.2. Let C be a semiprojective cdgc. The morphism $\phi_C : C \to \operatorname{Bar}\Omega(C)$, defined in 5.1.5, is a weak equivalence. Indeed, ϕ_C^* is an equivalence by 5.1.8.(2); this and the diagram (6.2.3) show that $(\mathbf{LL}(\phi_C)_*, \mathbf{R}\phi_C^*)$ is an equivalence.

Theorem 6.3.3. *A morphism $(\phi, a) : C \to D$ of semiprojective cdg coalgebras is a weak equivalence if and only if the corresponding morphism of dg-algebras*

$$\Omega(C) \to \Omega(D)$$

is a quasi-isomorphism.

Proof. Let $\tau : C \to \Omega(D)$ be the twisting cochain given by the composition of τ^D and (ϕ, a), as in 4.3.5. There is a diagram

$$
\begin{tikzcd}
\underline{\mathrm{comod}}^{\mathrm{sp}}(D) \arrow[r, shift right, "\phi^*"'] \arrow[d, shift left, "L_{\tau^D}"] \arrow[d, phantom, shift right=3, "\cong"] & \underline{\mathrm{comod}}^{\mathrm{sp}}(C) \arrow[l, shift right, "\mathbf{L}\phi_*"'] \arrow[d, equal] \\
\underline{\mathrm{mod}}^{\infty}_{\mathrm{sp}}(\Omega(D)) \arrow[r, shift right, "R_\tau"'] \arrow[u, shift left=3, "R_{\tau^D}"] & \underline{\mathrm{comod}}^{\mathrm{sp}}(C) \arrow[l, shift right, "L_\tau"']
\end{tikzcd}
$$

with the rows adjoint pairs, and the top row is the adjoint pair given in 6.2.4. We have

$$\phi^* R_{\tau^D} = R_\tau \quad R_{\tau^D} L_\tau = \mathbf{L}\phi^*$$

and thus the other squares are commutative, up to isomorphism. In particular, the top row is an equivalence, if and only if the bottom row is.

Consider the unit of the adjoint pair (L_τ, R_τ) for the object $C \otimes^{\tau^C} \Omega(C)$ in $\underline{\text{comod}}^{\text{sp}}(C)$. This is defined in 4.4.9. It factors as

$$C \otimes^{\tau^C} \Omega(C) \xrightarrow[\simeq]{\eta_C} C \otimes^{\tau^C} \Omega(C) \otimes^{\tau^C} C \otimes^{\tau^C} \Omega(C)$$

$$\xrightarrow{1\otimes\varphi\otimes 1\otimes 1} C \otimes^{\tau} \Omega(D) \otimes^{\tau} C \otimes^{\tau^C} \Omega(C),$$

where $\varphi : \Omega(C) \to \Omega(D)$ is the morphism of dg-algebras induced by ϕ. By 6.2.1.(2), there is an isomorphism

$$C \otimes^{\tau} \Omega(D) \otimes^{\tau} C \otimes^{\tau^C} \Omega(C) \cong C \otimes^{\tau^C} \varphi^*(\Omega(D)) \otimes^{\tau^C} C \otimes^{\tau^C} \Omega(C).$$

Since R_{τ^C} is fully faithful, the unit is an isomorphism if and only if

$$\varphi \otimes 1 \otimes 1 : \Omega(C) \otimes^{\tau^C} C \otimes^{\tau^C} \Omega(C) \to \varphi^*(\Omega(D) \otimes^{\tau} C \otimes^{\tau^C} \Omega(C))$$

is a homotopy equivalence in $\underline{\text{mod}}^{\text{dg}}(\Omega(C))$. Moreover there is a commutative diagram of complexes

$$\begin{tikzcd}
\varphi^*(\Omega(D)) \arrow[r] & \varphi^*(\Omega(D) \otimes^{\tau} C \otimes^{\tau^C} \Omega(C)) \\
\Omega(C) \arrow[u, "\varphi"] \arrow[r, "\eta_{\Omega(C)}", "\simeq"'] & \Omega(C) \otimes^{\tau^C} C \otimes^{\tau^C} \Omega(C). \arrow[u]
\end{tikzcd}$$

So φ is a homotopy equivalence of complexes if and only if the unit of (L_τ, R_τ) is an isomorphism for the object $C \otimes^{\tau^C} \Omega(C)$. Also, since $\Omega(C)$, $\Omega(D)$ are semi-projective complexes, φ is a homotopy equivalence if and only if it is a quasi-isomorphism.

If ϕ is a weak equivalence, then (L_τ, R_τ) is an equivalence, so the unit will be an isomorphism, so φ is a quasi-isomorphism.

Conversely, since L_τ, R_τ both preserve coproducts, the set of objects in $\underline{\text{comod}}^{\text{sp}}(C)$ is an isomorphism is a localizing subcategory. If φ is a quasi-isomorphism, then this localizing subcategory contains $C \otimes^{\tau^C} \Omega(C)$, and so by 5.6.3.(2), must be the whole category. Thus L_τ is fully faithful. We claim that it is also essentially surjective. By 5.6.3.(1), it is enough to show that $\Omega(D)$ is in the image. The counit of the adjunction for $\Omega(D)$ is of the form

$$\Omega(D) \otimes^{\tau} C \otimes^{\tau} D \cong \Omega(D) \otimes_{\Omega(C)} (\Omega(C) \otimes^{\tau^C} C \otimes^{\tau^C} \varphi^*\Omega(D)) \to \Omega(D).$$

This is the image of the map

$$\Omega(C) \otimes^{\tau^C} C \otimes^{\tau^C} \varphi^*\Omega(D) \to \varphi^*\Omega(D)$$

under the adjunction $(\Omega(D) \otimes_{\Omega(C)} -, \varphi^*)$. The above map is a quasi-isomorphism by 4.4.5, and since φ is a quasi-isomorphism, the image under the adjunction is also a quasi-isomorphism. $\square$

Corollary 6.3.4. *Let $\varphi : A \to B$ be a morphism of semiprojective A_∞-algebras. The adjoint pair*

$$\begin{tikzcd} \mathsf{D}^\infty(A) \arrow[r, shift left, "\mathbf{L}\varphi_*"] & \mathsf{D}^\infty(B) \arrow[l, shift left, "\varphi^*"] \end{tikzcd},$$

defined in 6.2.6, is an equivalence if and only if φ_1 is a quasi-isomorphism.

Proof. Let $\phi = \operatorname{Bar}\varphi : \operatorname{Bar} A \to \operatorname{Bar} B$ be the morphism of cdg coalgebras corresponding to φ. By 6.2.6, $(\mathbf{L}\varphi_*, \varphi^*)$ is an equivalence if and only if $(\mathbf{LL}\phi_*, \mathbf{R}\phi^*)$ is an equivalence. By 6.3.3, this pair is an equivalence if and only if the induced map $\operatorname{Bar}\Omega(\varphi) : \operatorname{Bar}\Omega(A) \to \operatorname{Bar}\Omega(B)$ is a quasi-isomorphism. We are done by considering the commutative diagram of complexes

$$\begin{tikzcd} \operatorname{Bar}\Omega(A) \arrow[r, "\operatorname{Bar}\Omega(\varphi)"] & \operatorname{Bar}\Omega(B) \\ A \arrow[u, "\varphi_A", "\simeq"'] \arrow[r, "\varphi_1"'] & B \arrow[u, "\simeq", "\varphi_B"'] \end{tikzcd}$$

where the vertical arrows are quasi-isomorphisms by (5.1.12). □

6.4. **Acyclic twisting cochains.** Above, we determined when a morphism of cdg coalgebras, or A_∞-algebras, gives a derived equivalence. Here we do the same for twisting cochains. This is inspired by [13, §2.2.4], although the arguments here are completely different.

Definition 6.4.1. Let A be a semiprojective A_∞-algebra with split unit and C a semiprojective cdgc. A twisting cochain $\tau : C \to A[1]$ is *acyclic* if the adjoint pair $(\mathbf{L}_\tau, \mathbf{R}_\tau)$, defined in 6.2.7, is an equivalence.

Lemma 6.4.2. *The universal twisting cochains $\tau_A : \operatorname{Bar} A \to A[1]$ and $\tau^C : C \to \Omega(C)[1]$ are acyclic.*

Theorem 6.4.3. *Let A be a semiprojective A_∞-algebra with split unit, C a semiprojective cdgc, and*

$$\tau : C \to A[1]$$

a twisting cochain.

The following conditions are equivalent.

1. *The twisting cochain τ is acyclic.*
2. *The counit of the pair (L_τ, R_τ), defined in 4.4.9,*

$$\epsilon^\tau_X : \operatorname{Bar} A \otimes^{\tau_A} A \otimes^\tau C \otimes^\tau X \to \operatorname{Bar} A \otimes^{\tau_A} X,$$

is a homotopy equivalence for all semiprojective A_∞ A-modules X.

3. *The counit of the pair $(\mathbf{L}_\tau, \mathbf{R}_\tau)$ for the object A,*

$$A \otimes^\tau C \otimes^\tau C \to A,$$

is an isomorphism in $\mathsf{D}^\infty(A)$. (Equivalently, the first component of ϵ^τ_A,

$$A \otimes^\tau C \otimes^\tau A \to A,$$

is a quasi-isomorphism.)

(4) *The unit of the pair* (L_τ, R_τ)*, defined in 4.4.8,*

$$\eta^\tau_{C\otimes X} : C \otimes X \to C \otimes^\tau A \otimes^\tau C \otimes X,$$

is a homotopy equivalence for all semiprojective extended comodules.

(5) *The unit of the pair* $(\mathbf{L}_\tau, \mathbf{R}_\tau)$ *for the object* k*,*

$$k \to C \otimes^\tau A$$

is an isomorphism in $\mathsf{D}^{\mathrm{co}}(C)$*.*

(6) *The morphism of cdg coalgebras*

$$(\phi, a) : C \to \operatorname{Bar} A$$

induced by τ *is a weak-equivalence.*

(7) *The morphism of dg-algebras*

$$\Omega(C) \to \Omega(\operatorname{Bar} A)$$

is a quasi-isomorphism.

Proof. We will use the diagram (6.2.11) implicitly throughout the proof.

The subcategory of $\mathsf{D}^\infty(A)$ with objects M such that the counit $A \otimes^\tau C \otimes^\tau M \to M$ is an isomorphism is localizing. Thus if $A \otimes^\tau C \otimes^\tau A \to A$ is an isomorphism in $\mathsf{D}^\infty(A)$, then by 5.6.3.(1), the counit is an isomorphism on all of $\mathsf{D}^\infty(A)$. Thus 2 and 3 are equivalent. Analogously, 4 and 5 are equivalent.

The functor L_τ is faithful on objects: if $L_\tau(N) = 0$, then $N \cong 0$, and so is R_τ. Thus, since they are triangulated, they are conservative (a functor F is conservative if $F(f)$ being an isomorphism implies f is an isomorphism). It is a formal property that if one functor in an adjoint pair of conservative functors is fully faithful, then the pair is an equivalence. This shows that 2 and 5 are equivalent. By the diagram (6.2.11), 1 is equivalent to 2 and 5. Again by this diagram, 1 is equivalent to 6. And finally, 6 and 7 are equivalent by 6.3.3. □

Condition 3 is often the easiest to check, as in the following.

Example 6.4.4. Let $\tau : C \to A$ be the generalized BGG twisting cochain of 4.3.4. Then [2, Proposition 2.6] shows that $A \otimes^\tau C \otimes^\tau A \to A$ is a quasi-isomorphism. (This is a generalization of Cartan's resolution of the simple module over an exterior algebra.) Thus by Theorem 6.4.3, there is an equivalence

$$\mathsf{D}^{\mathrm{co}}(C) \xrightarrow{\cong} \mathsf{D}^\infty(A).$$

We explore this further, and make a connection to commutative algebra, in §8.

Corollary 6.4.5. *Let* $\tau : C \to A[1]$ *be an acyclic twisting cochain. There is an equivalence*

$$\underline{\mathrm{comod}}^{\mathrm{sp}}(C) \underset{R_\tau}{\overset{L_\tau}{\underset{\cong}{\rightleftarrows}}} \underline{\mathrm{mod}}^\infty_{\mathrm{sp}}(A).$$

In particular, if M *is a semiprojective complex of* k*-modules and* $C \otimes M$ *is an extended comodule structure on* M*, then there is* A_∞ A*-module structure on* M*, unique up to homotopy with the property that* $R_\tau(M) \cong C \otimes M$*.*

Proof. Since τ is acyclic, the diagram (6.2.11) shows that $R_\tau : \underline{\mathrm{mod}}^\infty_{\mathrm{sp}}(A) \to \underline{\mathrm{comod}}^{\mathrm{sp}}(C)$ is an equivalence. Thus there exists a unique object in $\underline{\mathrm{mod}}^\infty_{\mathrm{sp}}(A)$, represented by e.g. N, with $R_\tau(N) = C \otimes N \cong C \otimes M$. In particular, N is homotopic to M as complexes. Thus by 3.4.1 there is an A_∞ A-module structure on M such that N and M are homotopic as A_∞ A-modules. Thus $R_\tau(M) \cong C \otimes M$. □

7. Applications to (dg) k-algebras

A motivation of all of the above machinery is to study homological algebra of representations of a k-algebra, even a cyclic k-algebra. But since a k-algebra need not have a split unit or be semiprojective, much of the machinery does not apply directly to these objects. Instead, we study them through a semi-projective resolution.

In this section we start to develop these applications. The arguments apply without change to dg-algebras, so we work in this level of generality. We will use without further comment that a dg-algebra is an A_∞-algebra with $m_n^B = 0$ for $n \geq 3$.

7.1. **Derived equivalences.**

Definition 7.1.1. Let B be a dg-algebra. If M, N are dg B-modules, the set of B-linear maps $\mathrm{Hom}_B(M, N)$ is a subcomplex of $\mathrm{Hom}(M, N)$. The *homotopy category of dg B-modules*, denoted $\underline{\mathrm{mod}}^{\mathrm{dg}}(B)$, has objects dg B-modules and morphisms $H^0 \mathrm{Hom}_B(M, N)$.

If $\tau : C \to B[1]$ is a twisting cochain, and B does not have a split unit, we have not defined a functor from representations of B to C-comodules. In the case B is a dg-algebra, we can do this easily in the following way.

Definition 7.1.2. Let $\tau : C \to B[1]$ be a twisting cochain with C a cocomplete cdgc and B a dg-algebra. Let $\varphi : \Omega(C) \to B$ be the unique map of dg-algebras given by 5.1.4. Define a pair of adjoint functors

$$\underline{\mathrm{comod}}(C) \underset{R_\tau}{\overset{L_\tau}{\rightleftarrows}} \underline{\mathrm{mod}}^{\mathrm{dg}}(B)$$

as the composition of the adjoint pairs

$$\underline{\mathrm{comod}}(C) \underset{R_{\tau C}}{\overset{L_{\tau C}}{\rightleftarrows}} \underline{\mathrm{mod}}^{\mathrm{dg}}(\Omega(C)) \underset{\varphi^*}{\overset{\varphi_*}{\rightleftarrows}} \underline{\mathrm{mod}}^{\mathrm{dg}}(B)$$

where φ_* is induction and φ^* is restriction, along the map of dg-algebras φ. By 6.2.1 L_τ is the functor defined in 4.3.10, and we noted above 5.5.7, the image of $R_{\tau C}$ is contained in $\underline{\mathrm{mod}}^{\mathrm{dg}}(\Omega(C))$.

can we give a formula for the differential of R_τ?

Definition 7.1.3. Let B be a dg-algebra. Semiprojective dg B-modules are defined exactly as in 2.4.1 and all of the properties listed below 2.4.1 hold for dg-modules as well. We let $\underline{\mathrm{mod}}^{\mathrm{dg}}_{\mathrm{sp}}(B)$ be the full subcategory of $\underline{\mathrm{mod}}^{\mathrm{dg}}(B)$ with objects semiprojective dg B-modules and $\underline{\mathrm{mod}}^{\mathrm{dg}}_{\mathrm{ac}}(B)$ the full subcategory with objects acyclic dg B-modules.

Lemma 7.1.4. *Let B be a dg-algebras. There is a semiorthogonal decomposition*

$$\underline{\mathrm{mod}}^{\mathrm{dg}}(B) = \langle \underline{\mathrm{mod}}^{\mathrm{dg}}_{\mathrm{ac}}(B), \underline{\mathrm{mod}}^{\mathrm{dg}}_{\mathrm{sp}}(B)\rangle.$$

In particular, by, 5.3.2, the composition

$$\underline{\mathrm{mod}}^{\mathrm{dg}}_{\mathrm{sp}}(B) \to \underline{\mathrm{mod}}^{\mathrm{dg}}(B) \to \underline{\mathrm{mod}}^{\mathrm{dg}}(B)/\underline{\mathrm{mod}}^{\mathrm{dg}}_{\mathrm{ac}}(B) = D(B)$$

is an equivalence.

Proof. Since semiprojective resolutions of dg B-modules exist, the same reasoning as 5.5.3 shows that the semiorthogonal decomposition exists. □

Lemma 7.1.5. *Let C be a semiprojective cdgc, B a dg-algebra and $\tau : C \to B[1]$ a twisting cochain. Consider the semiorthogonal decomposition*

$$\underline{\mathrm{comod}}^{\mathrm{ext}}(C) = \langle \underline{\mathrm{comod}}^{\mathrm{ac}}(C), \underline{\mathrm{comod}}^{\mathrm{sp}}(C)\rangle$$

of 5.4.4.(2). The functor L_τ takes $\underline{\mathrm{comod}}^{\mathrm{sp}}(C)$ to $\underline{\mathrm{mod}}^{\mathrm{dg}}_{\mathrm{sp}}(B)$ and R_τ takes $\underline{\mathrm{mod}}^{\mathrm{dg}}_{\mathrm{ac}}(B)$ to $\underline{\mathrm{comod}}^{\mathrm{ac}}(C)$.

Proof. Let $C \otimes X$ be a semiprojective extended comodule. Consider the filtration of the dg B-module $B \otimes^\tau C \otimes X$ induced by the primitive filtration of C, so the subquotients are

$$(B \otimes C_{[n]}/C_{[n-1]} \otimes X, d_B \otimes 1 \otimes 1 + 1 \otimes d)$$

where d is the differential of the complex $C_{[n]}/C_{[n-1]} \otimes X$. Since $C_{[n]}/C_{[n-1]} \otimes X$ is semiprojective over k, it is easy to see each subquotient is a semiprojective over B. Thus the original dg B-module is semiprojective.

Let M be a dg B-module. Then $R_\tau(M) = C \otimes^{\tau^C} \varphi^*(M)$, where $\varphi : \Omega(C) \to B$ is the dg-algebra morphism induced by τ. If M is acyclic, then so is $\varphi^*(M)$, and so $R_\tau(M)$ is an acyclic extended comodule. □

Consider the following diagram

$$\begin{tikzcd}
\underline{\mathrm{mod}}^{\mathrm{dg}}(B) \arrow[r, "\pi R_\tau"'] \arrow[d, equal] & \underline{\mathrm{comod}}(C)/\underline{\mathrm{coacyl}}(C) \\
\underline{\mathrm{mod}}^{\mathrm{dg}}(B) \arrow[r, "R_\tau"'] & \underline{\mathrm{comod}}^{\mathrm{ext}}(C) \arrow[u, "\pi"'] \\
\underline{\mathrm{mod}}^{\mathrm{dg}}_{\mathrm{sp}}(B) \arrow[u] & \underline{\mathrm{comod}}^{\mathrm{sp}}(C) \arrow[l, "L_\tau"'] \arrow[u]
\end{tikzcd}$$

with the top square commutative. This satisfies the adjoint condition of **??**, and the above lemma shows that it satisfies the other conditions of **??**. Applying 6.1.8 gives the following.

Theorem 7.1.6. *Let $\tau : C \to B[1]$ be a twisting cochain with B a dg-algebra and C a semiprojective cdgc. There is a diagram*

$$\begin{tikzcd}
D(B) \arrow[r, shift right, "R_\tau"'] \arrow[d, equal] & \mathsf{D}^{\mathrm{co}}(C) \arrow[l, shift right, "\mathbf{L}_\tau"'] \arrow[d, shift right, "\cong"'] \\
D(B) \arrow[r, shift right, "R_\tau"'] \arrow[d, shift right, "\cong"'] & \frac{\underline{\mathrm{comod}}^{\mathrm{ext}}(C)}{\underline{\mathrm{comod}}^{\mathrm{ac}}(C)} \arrow[l, shift right, "\mathbf{L}'_\tau"'] \arrow[u, shift right] \arrow[d, shift right, "\cong"'] \\
\underline{\mathrm{mod}}^{\mathrm{dg}}_{\mathrm{sp}}(B) \arrow[r, shift right, "\mathbf{R}_\tau"'] \arrow[u, shift right] & \underline{\mathrm{comod}}^{\mathrm{sp}}(C) \arrow[l, shift right, "L_\tau"'] \arrow[u, shift right]
\end{tikzcd}$$

with rows adjoint pairs and all squares commutative up to the (co)units of the vertical equivalences. We have

$$\mathbf{L}_\tau(N) = L_\tau(C \otimes p(\Omega(C) \otimes N)) \cong B \otimes^\tau C \otimes p(\Omega(C) \otimes N)$$
$$\mathbf{L}'_\tau(C \otimes X) = B \otimes^\tau (C \otimes pX)$$
$$\mathbf{R}_\tau(M) = C \otimes pM,$$

where $p(\Omega(C) \otimes N) \to \Omega(C) \otimes N$, $pX \to X$ and $pM \to M$ are semiprojective resolutions over k, and $C \otimes pX, C \otimes pM$ have the extended comodule structure given by 5.2.5.

Corollary 7.1.7. *The three pairs of adjoints in 7.1.6 are an equivalence if and only if $\varphi : \Omega(C) \to B$ is a quasi-isomorphism.*

Proof. The functor $L_\tau : \underline{\mathrm{comod}}^{\mathrm{sp}}(C) \to \underline{\mathrm{mod}}^{\mathrm{dg}}_{\mathrm{sp}}(B)$ factors as

$$\underline{\mathrm{comod}}^{\mathrm{sp}}(C) \xrightarrow[\cong]{L_{\tau C}} \underline{\mathrm{mod}}^{\mathrm{dg}}_{\mathrm{sp}}(\Omega(C)) \xrightarrow{\varphi_*} \underline{\mathrm{mod}}^{\mathrm{dg}}_{\mathrm{sp}}(B).$$

Thus L_τ is an equivalence if and only if φ_* is an equivalence, and by e.g. [17, Theorem 1.7], this happens if and only if φ is a quasi-isomorphism. $\square$

In the proof of 4.3.6, we showed that strictly unital morphisms $A \to B$ of A_∞-algebras, where A has a split unit, correspond to twisting cochains $\mathrm{Bar}\, A \to B[1]$. We will use this implicitly.

Corollary 7.1.8. *Let A be a semiprojective A_∞-algebra, B a dg-algebra and $\varphi : A \to B$ a morphism of A_∞-algebras. Let $\tau : \mathrm{Bar}\, A \to B[1]$ be the corresponding twisting cochain. There is an adjoint pair of functors*

$$\mathsf{D}(B) \underset{\varphi^*}{\overset{\mathbf{L}\varphi_*}{\leftrightarrows}} \mathsf{D}^\infty(A)$$

with φ^* restriction along φ and $\mathbf{L}\varphi_*(M) = B \otimes^\tau (\mathrm{Bar}\, A \otimes pM)$, where $pM \to M$ is a semiprojective resolution of M with A_∞ A-module structure given by 3.3.2.

This pair is an equivalence if and only if φ_1 is a quasi-isomorphism.

Proof. This is simply the adjoint

$$D(B) \underset{R_\tau}{\overset{\mathbf{L}'_\tau}{\leftrightarrows}} \frac{\underline{\mathrm{comod}}^{\mathrm{ext}}(\mathrm{Bar}\, A)}{\underline{\mathrm{comod}}^{\mathrm{ac}}(\mathrm{Bar}\, A)} = \mathsf{D}^\infty(A)$$

applied to the twisting cochain $\tau : \operatorname{Bar} A \to B$.

By the previous theorem, it is an equivalence if and only if the induced map $\Omega(\operatorname{Bar} A) \to B$ is a quasi-isomorphism. But the map φ_1 factors as

$$A \xrightarrow[\varphi_A]{\simeq} \Omega(\operatorname{Bar} A) \longrightarrow B,$$

where the quasi-isomorphism φ_A is defined in (5.1.12). Thus φ is a quasi-isomorphism if and only if the functors are an equivalence. $\square$

In particular, we have the following.

Corollary 7.1.9. *Let B be a dg-algebra, $A \to B$ a semiprojective resolution over k, and equip A with an A_∞-algebra structure and a morphism of A_∞-algebras*

$$\varphi : A \to B,$$

using 2.5.1. There is an equivalence

$$\begin{tikzcd} \mathsf{D}(B) \arrow[r, shift right, "\varphi^*"'] \arrow[r, phantom, "\cong"] & \mathsf{D}^\infty(A) \arrow[l, shift right, "\mathbf{L}\varphi_*"'] \end{tikzcd} .$$

7.2. **Resolutions.** Let B be a dg-algebra and $\tau : C \to B[1]$ a twisting cochain. Consider the adjunction

$$\begin{tikzcd} D(B) \arrow[r, shift right, "R_\tau"'] & \dfrac{\underline{\mathrm{comod}}^{\mathrm{ext}}(C)}{\underline{\mathrm{comod}}^{\mathrm{ac}}(C)} \arrow[l, shift right, "\mathbf{L}'_\tau"'] \end{tikzcd}$$

that is the middle row of 7.1.6. If M is a dg B-module, then using (the dual of) **??**, the counit of the above adjunction is the image in $\mathsf{D}(B)$ of the map

$$B \otimes^\tau (C \otimes pM) \to \Omega(C) \otimes^{\tau^C} (C \otimes pM) \twoheadrightarrow pM \to M.$$

In particular if the above adjoint is an equivalence, the counit is a quasi-isomorphism.

One of the main ways that we can use these higher homotopies is to construct B-semiprojective resolutions using semiprojective resolutions over k and the bar construction.

Corollary 7.2.1. *Let $A \xrightarrow{\simeq} B$ be a semiprojective resolution over k of the complex underlying B, and equip A with an A_∞-algebra structure such that $A \to B$ is a strict morphism of A_∞-algebras. Let M be a dg B-module and $pM \to M$ a semiprojective resolution over k, with pM given a A_∞ A-module structure via 3.3.2. Then*

$$B \otimes^\tau \operatorname{Bar} A \otimes^{\tau_A} pM \to M$$

is a semiprojective resolution of M over B.

Proof. Since $\operatorname{Bar} A \otimes^{\tau_A} pM$ is in $\mathrm{comod}^{\text{sp-ext}}(\operatorname{Bar} A)$ and L_τ preserves semiprojectives, $B \otimes^\tau \operatorname{Bar} A \otimes^{\tau_A} pM$ is semirprojective over B. Since $A \to B$ is a quasi-isomorphism, the above adjoint is an equivalence. Thus the counit is an isomorphism in $\mathsf{D}(B)$, i.e. a quasi-isomorphism. $\square$

Heuristically, semiprojective resolutions over k are much easier to construct than over B. A motivating case is when k is a regular ring, or even a field. The case that B is a cyclic k-algebra is studied in [6].

We can potentially reduce the size of the resolution 7.2.1 by finding an acyclic twisting cochain for the semiprojective resolution A.

Corollary 7.2.2. *Let B be a dg-algebra, $A \xrightarrow{\simeq} B$ a semiprojective resolution over k with A_∞-structure, and let $\tau' : C \to A[1]$ be an acyclic twisting cochain with corresponding morphism of cdg coalgebras $\phi : C \to \operatorname{Bar} A$. Let $\tau'' : \operatorname{Bar} A \to B[1]$ be the twisting cochain corresponding to $A \to B$, and let $\tau : C \to B[1]$ be the composition of ϕ and τ'', in the sense of 4.3.5.*

Let M be a dg B-module and $pM \to M$ a semiprojective k-resolution with A_∞ A-module structure. The counit of the adjunction $(\mathbf{L}_\tau, R_\tau)$

$$B \otimes^\tau (C \otimes pM) \to M$$

is a B-semiprojective resolution of M.

In the next section we give an example where this construction lets us considerably reduce the size of the resolutions.

Remark. The B-linear endomorphism $B \otimes (C \otimes^\tau pM)$ given by

$$d(b \otimes c \otimes g) = b \otimes d_C(c) \otimes g + b \otimes \sum_{n \geq 1} (-1)^{|c_{(1)}|} c_{(1)} \otimes m_n^G([c_{(2)}| \dots |c_{(n+1)}] \otimes g)$$

makes $B \otimes (C \otimes^\tau pM)$ into a complex of semiprojective B-modules.

7.3. **Finite (co)derived categories and curved algebras.** Often in applications we are interested in the finite derived category of an algebra. Our definition of (co)derived category of a semiprojective (co)algebra gives an easy definition of finite (co)derived category. We define this here and show how it behaves under various functors.

Also, the finite coderived category of a cdg coalgebra is easily dualized, so that we may consider modules over a cdg algebra, and these are often easier to work with in practice. We describe this here. (This is in fact a special case of Positselski's co/contra correspondence [17, §5].)

Definition 7.3.1. Let C be a semiprojective cdgc and A a semiprojective A_∞-algebra.

(1) Set $\underline{\text{comod}}_{\mathsf{f}}^{\text{sp}}(C)$ to be the full subcategory of $\underline{\text{comod}}^{\text{sp}}(C)$ with objects that are isomorphic to $C \otimes X$, with X a finitely generated graded projective k-module. Let $\mathsf{D}_{\mathsf{f}}^{\text{co}}(C)$ be the image of $\underline{\text{comod}}_{\mathsf{f}}^{\text{sp}}(C)$ in $\mathsf{D}^{\text{co}}(C)$ under the equivalence given in 5.4.6:

$$\underline{\text{comod}}^{\text{sp}}(C) \xrightarrow{\cong} \mathsf{D}^{\text{co}}(C).$$

We call $\mathsf{D}_{\mathsf{f}}^{\text{co}}(C)$ the *finite coderived category of C*.

(2) The *finite derived category of A*, written $\mathsf{D}_{\mathsf{f}}^{\infty}(A)$, is the image in $\mathsf{D}^{\infty}(A)$ of $\underline{\text{comod}}_{\mathsf{f}}^{\text{sp}}(\operatorname{Bar} A)$ under the equivalence

$$\underline{\text{comod}}^{\text{sp}}(\operatorname{Bar} A) = \underline{\text{mod}}_{\text{sp}}^{\infty}(A) \to \mathsf{D}^{\infty}(A).$$

Lemma 7.3.2. (1) *Let $(\phi, a) : C \to D$ be a morphism of semiprojective coalgebras. The functor $\mathbf{R}\phi^* : \mathsf{D}^{\text{co}}(D) \to \mathsf{D}^{\text{co}}(C)$ restricts to a functor*

$$\mathbf{R}\phi^* : \mathsf{D}_{\mathsf{f}}^{\text{co}}(D) \to \mathsf{D}_{\mathsf{f}}^{\text{co}}(C).$$

(2) *Let $\varphi : A \to B$ be a morphism of semiprojective A_∞-algebras. The functor $\varphi^* : \mathsf{D}^{\infty}(B) \to \mathsf{D}^{\infty}(A)$ restricts to a functor*

$$\varphi^* : \mathsf{D}_{\mathsf{f}}^{\infty}(B) \to \mathsf{D}_{\mathsf{f}}^{\infty}(A).$$

(3) *Let $\tau : C \to A[1]$ be a twisting cochain from a semiprojective cdgc to a semiprojective A_∞-algebra. The functor $\mathbf{R}_\tau$ restricts to a functor*

$$\mathbf{R}_\tau : \mathsf{D}^\infty_\mathsf{f}(A) \to \mathsf{D}^{\mathrm{co}}_\mathsf{f}(C).$$

Dualizing the definition of curved dg coalgebra, we have the following.

Definition 7.3.3. [17] A *curved differential graded algebra* is a triple (S, d, h) with S a graded k-algebra, d a derivation, and h an element of S_{-2} such that $d^2 = [h, -]$, commutation by h. A *curved differential graded module* is a pair (M, d^M) with M a graded S-module and d^M a derivation with respect to d such that $(d^M)^2 = h \cdot (-)$, multiplication by h.

Given M, N curved dg-modules, the standard derivation on $\mathrm{Hom}_S(M, N)$ makes it a complex. We let dg-mod(S) be the dg-category with objects curved dg modules and morphism complexes $\mathrm{Hom}_S(M, N)$. The homotopy category is denoted $\underline{\mathrm{mod}}^{\mathrm{dg}}(S)$.

We say a curved dg-module is *extended* if the underlying module is isomorphic to $S \otimes X$ for some graded module X. We denote by $\underline{\mathrm{mod}}^{\mathrm{ext}}(S)$ the full subcategory of $\underline{\mathrm{mod}}^{\mathrm{dg}}(S)$ with objects extended modules. If $S \otimes X$ is an extended module, then there is an induced differential on X. We say $S \otimes X$ is *semiprojective* as an extended S-module if the complex X is semiprojective over k. We denote by $\underline{\mathrm{mod}}^{\mathrm{sp}}(S)$ the full subcategory of $\underline{\mathrm{mod}}^{\mathrm{ext}}(S)$ with objects semiprojective modules.

We let $\underline{\mathrm{mod}}^{\mathrm{ext}}_{\mathrm{f}}(S)$ be the full subcategory of $\underline{\mathrm{mod}}^{\mathrm{ext}}(S)$ with objects those isomorphic to $S \otimes X$ with X a finitely generated k-module and $\underline{\mathrm{mod}}^{\mathrm{sp}}_{\mathrm{f}}(S)$ to be the intersection of $\underline{\mathrm{mod}}^{\mathrm{ext}}_{\mathrm{f}}(S)$ and $\underline{\mathrm{mod}}^{\mathrm{sp}}(S)$.

Positselski has defined two "exotic" derived categories associated to a curved dg-algebra in [17] and studied this situation extensively. We will work with the naive definition $\underline{\mathrm{mod}}^{\mathrm{sp}}_{\mathrm{f}}(S)$ as the "finite derived category" of a curved dg-algebra.

If (C, d, h) be a cdgc, then $(C^* = \mathrm{Hom}(C, k), d^*, h \in (C^*)_{-2})$ is a curved dg-algebra that we denote C^*.

do we need any assumptions on C for this? should check change from sp extended to just extended?

Proposition 7.3.4. *Let (C, d, h) be a cdg coalgebra and $S = C^*$ the dual curved dg-algebra. There is an equivalence*

$$\underline{\mathrm{comod}}^{\mathrm{sp}}_{\mathrm{f}}(C) \xrightarrow{\cong} \underline{\mathrm{mod}}^{\mathrm{sp}}_{\mathrm{f}}(S)$$

that sends an extended C-comodule with structure map $C \otimes X \to X$ to the S-module determined by the corresponding map $X \to S \otimes X$ under the isomorphism

$$\mathrm{Hom}(C \otimes X, X) \cong \mathrm{Hom}(X, C^* \otimes X).$$

Proof. One checks that a map $C \otimes X \to X$ gives X the structure of an extended C-comodule if and only if the corresponding map $X \to C^* \otimes X$ gives X an extended C^*-module structure. Similarly, if $C \otimes X, C \otimes Y$ are cdg C-comodules, then a map $C \otimes X \to Y$ determines a morphism of cdg C-comodules if and only if the corresponding map $X \to C^* \otimes Y$ determines a morphism of C^*-modules. □

Remark. This is related to Positselski's co/contra comodule correspondence. Indeed, finite extended C^*-modules embed in the category of C contramodules...

We note that if $(S, 0, h)$ is a curved dg-algebra with zero differential, then an object of $\underline{\mathrm{mod}}^{\mathrm{sp}}_{\mathrm{f}}(S)$ is exactly a "graded matrix factorization" of the element $h \in S_2$.

8. Complete intersection rings

Let Q be a commutative ring and $\mathbf{f} = f_1, \dots, f_c$ a finite sequence of elements. Let A be the Koszul complex on $\mathbf{f}$. To match up with earlier notation 4.3.4, A is the Koszul complex on the linear map $V = Q^c \xrightarrow{l} Q$, where $l = [f_1 \dots f_c]$ and V is in homological degree 1. Let C be the divided powers coalgebra on $V[1]$, and let

$$\tau : C \to A[1]$$

be the map that is the identity on V and zero else. By Example 6.4.4, τ is an acyclic twisting cochain between A and the cdgc $(C, 0, h)$, with $h = l \circ p_2$ where $p_2 : C \twoheadrightarrow C_2 = V[1]$ is projection.

Let $R = Q/(\mathbf{f})$. The sequence $\mathbf{f}$ is *Koszul-regular* if A is a Q-free resolution of R. (If the sequence is regular in the usual sense, it is Koszul regular, and if Q is local and Noetherian the converse holds.) For the rest of the section we assume that $\mathbf{f}$ is Koszul-regular. By definition the canonical map

$$A \xrightarrow{\simeq} R$$

is a quasi-isomorphism. Thus A is a semiprojective Q-resolution of R with an A_∞-structure (in this case a dg-algebra structure) and we can study R via the twisting cochain τ. We first show how this approach recovers and extends some classical tools for studying complete intersections.

8.1. **Higher homotopies (as defined by Eisenbud).** Let M be an R-module and $G \xrightarrow{\simeq} M$ a Q-projective resolution. A *system of higher homotopies*, as defined by Eisenbud in [7, §7], is a set $\{\sigma_a | a \in \mathbb{N}^c\}$ with $\sigma_a : G \to G$ a degree $2|a| - 1$ map (if $a = (a_1, \dots, a_c)$, $|a| = \sum a_i$) such that

(1) $\sigma_0 = d^G$;
(2) σ_{e_i} is a homotopy for multiplication by f_i, where $e_i = (0, \dots, 0, 1, 0, \dots, 0)$;
(3) for $a \in \mathbb{N}^c$ with $|a| \geq 2$,

$$\sum_{b+c=a} \sigma_b \sigma_c = 0.$$

Lemma 8.1.1. *Let $(C, 0, h)$ be the cdgc defined above and let G be a complex of Q-modules. A system of higher homotopies in the sense of* [7, §7] *is equivalent to an extended C-comodule structure on G.*

Proof. The correspondence is given as follows. An extended comodule structure on C is determined by a map $C \otimes G \to G$. Let $X_1, \dots, X_c$ be a basis for C_2 corresponding to $f_1, \dots, f_c$. Given $a \in \mathbb{N}^c$, define σ_a to be $m : G \cong X^a \otimes G \to G$, and conversely if one starts with a system of higher homotopies. The lemma is now an unwinding of definitions. □

This characterization lets us easily define morphism between higher homotopies and homotopies between such morphisms.

Definition 8.1.2. A *morphism between complexes with higher homotopies $G \to H$* is a morphism between the corresponding cdg comodules $C \otimes G \to C \otimes H$. A *homotopy between morphisms* is a homotopy of morphisms of cdg C-comodules.

Theorem 8.1.3. *Let $R = Q/(\boldsymbol{f})$, A the Koszul complex on $\boldsymbol{f}$ and C the cdg coalgebra defined above. Let M be a complex of R-modules and $G \xrightarrow{\simeq} M$ a semiprojective Q-resolution. (If M is a module, this is just a Q-projective resolution.)*

(1) *There exists a system of higher homotopies on G that is unique up to homotopy. Given a morphism $M \to N$ of complexes of R-modules, there is a corresponding morphism of higher homotopies that is unique up to homotopy.*
(2) *A system of higher homotopies on G is equivalent, up to homotopy, to an A_∞ A-module structure on G such that $G \to M$ is a morphism of A_∞ A-modules (where we view M as an A_∞ A-module via restriction).*

Proof. By 3.3.2, G has a unique up to homotopy A_∞ A-module structure. The rest of the statements now follow from 6.4.5 applied to the acyclic twisting cochain $C \to A[1]$. $\square$

Eisenbud defined these higher homotopies to construct an R-free resolution from a Q-free resolution. We recover this from 7.2.2.

Corollary 8.1.4. *Let $R = Q/(\boldsymbol{f})$ be as above...*

8.2. **Equivalences of categories.** Recover graded matrix factorization result by Sections 7.1 and 7.3...

9. Proofs of some results of Sections 2-4

9.1 *Proof of Theorem 2.2.17.* Throughout, the proof we consider the split short exact sequences of graded modules

$$\begin{tikzcd} 0 \arrow[r] & k[1] \arrow[r, shift right, "\eta"'] & A[1] \arrow[l, shift right, "v"'] \arrow[r, shift right, "p"'] & \overline{A}[1] \arrow[l, shift right, "b"'] \arrow[r] & 0, \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & K_A \arrow[r, shift right, "\widetilde{\eta}"'] & T^c(A[1]) \arrow[l, shift right, "\widetilde{v}"'] \arrow[r, shift right, "\widetilde{p}"'] & T^c(\overline{A}[1]) \arrow[l, shift right, "\widetilde{b}"'] \arrow[r] & 0, \end{tikzcd}$$

where $\widetilde{p} = T^{\mathsf{co}}(p), \widetilde{b} = T^{\mathsf{co}}(b)$, $K_A = \ker \widetilde{p}$, $\widetilde{\eta}$ is inclusion, and $\widetilde{v}$ is induced by $\widetilde{b}$. We fix a map $m : T^{\mathsf{co}}(A[1]) \to A[1]$. If f is any map from a tensor coalgebra to a module, we write f_n for the restriction of the map to the nth tensor power. In particular, $m_n = m|_{A[1]^{\otimes n}} : A[1]^{\otimes n} \to A[1]$. Define a map

$$\varphi = \varphi' + \varphi'' = bpm\widetilde{\eta}\widetilde{v} + \eta v m \widetilde{b}\widetilde{p} : T^{\mathsf{co}}(A[1]) \to A[1].$$

Recall, that $\overline{m} = pm\widetilde{b}$. Both squares in the following diagram are commutative:

$$\begin{tikzcd} T^{\mathsf{co}}(A[1]) \arrow[r, shift right, "\widetilde{p}"'] \arrow[d, "m-\varphi"'] & T^{\mathsf{co}}(\overline{A}[1]) \arrow[l, shift right, "\widetilde{b}"'] \arrow[d, "\overline{m}"] \\ A[1] \arrow[r, shift right, "p"'] & \overline{A}[1], \arrow[l, shift right, "b"'] \end{tikzcd} \tag{9.1.1}$$

Define coderivations

$$\partial : T^{\mathsf{co}}(A[1]) \to T^{\mathsf{co}}(A[1]) \quad \partial^{\varphi} : T^{\mathsf{co}}(A[1]) \to T^{\mathsf{co}}(A[1]) \quad \overline{\partial} : T^{\mathsf{co}}(\overline{A}[1]) \to T^{\mathsf{co}}(\overline{A}[1])$$

to be those determined by

$$m : T^{\mathsf{co}}(A[1]) \to A[1] \quad \varphi : T^{\mathsf{co}}(A[1]) \to A[1] \quad \overline{m} : T^{\mathsf{co}}(\overline{A}[1]) \to \overline{A}[1],$$

respectively. It follows from 2.1.8.(2) that $\partial - \partial^{\varphi}$ is the coderivation determined by $m - \varphi$, and that both squares in the top half of the following diagram are commutative (so in fact, all squares in the diagram are commutative):

(9.1.2)
$$\begin{tikzcd} T^{\mathsf{co}}(A[1]) \arrow[r, shift left, "\widetilde{p}"'] \arrow[d, "\partial-\partial^{\varphi}"'] & T^{\mathsf{co}}(\overline{A}[1]) \arrow[l, shift left, "\widetilde{b}"'] \arrow[d, "\bar{\partial}"] \\ T^{\mathsf{co}}(A[1]) \arrow[r, shift left, "\widetilde{p}"'] \arrow[d, "m-\varphi"'] & T^{\mathsf{co}}(\overline{A}[1]) \arrow[l, shift left, "\widetilde{b}"'] \arrow[d, "\overline{m}"] \\ A[1] \arrow[r, shift left, "p"'] & \overline{A}[1]. \arrow[l, shift left, "b"'] \end{tikzcd}$$

Additionally, let $\partial^{\varphi'}, \partial^{\varphi''}$ be the coderivations determined by φ', φ'', so $\partial^{\varphi} = \partial^{\varphi'} + \partial^{\varphi''}$. We have

(9.1.3) $$\overline{m}\bar{\partial} = p(m-\varphi)(\partial - \partial^{\varphi})b^{\otimes n} = p(m-\varphi')(\partial - \partial^{\varphi''})\widetilde{b}$$

by the commutivity of the above squares, and the equalities $p\varphi'' = 0, \varphi'\widetilde{b} = 0$.

Assume now that m is an A_∞-algebra structure on A with v a split unit. We want to show that $(\bar{\partial})^2 = [h,-]$ and $h\bar{\partial} = 0$. Since both $(\bar{\partial})^2$ and $[h,-]$ are coderivations, by 2.1.8.(2), it is enough to show, for all $n \geq 1$,

$$(\overline{m}\bar{\partial})_n = [h,-]^1_n : \overline{A}[1]^{\otimes n} \to \overline{A}[1].$$

We first simplify $(\overline{m}\bar{\partial})_n$ using 9.1.3 and the equality $\varphi'' = \eta h\widetilde{p}$ (recall from the statement of the theorem $h = vm\widetilde{b} : T^{\mathsf{co}}(\overline{A}[1]) \to k[1]$). Since v is a split unit for A, $\varphi'_n = 0$ for $n \neq 2$ and $m_{n-i+1}(1^{\otimes j} \otimes \eta h_i p^{\otimes i} \otimes 1^{\otimes n-i-j}) = 0$ for $n-i+1 \neq 2$, i.e. $i \neq n-1$. Also $m\partial = 0$ since m is an A_∞-algebra. Thus, using (9.1.3),

$$\begin{aligned}(\overline{m}\bar{\partial})_n &= (pm_2 - p\varphi'_2)(b \otimes \eta h_{n-1} + \eta h_{n-1} \otimes b) - p\varphi'_2(1 \otimes m_{n-1} + m_{n-1} \otimes 1)b^{\otimes n} \\ &= (pm_2 - pm_2\widetilde{\eta}_2\widetilde{v}_2)(b \otimes \eta h_{n-1} + \eta h_{n-1} \otimes b) - p\varphi'_2(1 \otimes m_{n-1} + m_{n-1} \otimes 1)b^{\otimes n} \\ &= pm_2 b^{\otimes 2} p^{\otimes 2}(b \otimes \eta h_{n-1} + \eta h_{n-1} \otimes b) - p\varphi'_2(1 \otimes m_{n-1} + m_{n-1} \otimes 1)b^{\otimes n} \\ &= -p\varphi'_2(1 \otimes m_{n-1} + m_{n-1} \otimes 1)b^{\otimes n}.\end{aligned}$$

We need a formula for φ'_2. Note that $K_A \cap A[1]^{\otimes 2} \cong k[1] \otimes A[1] \oplus \overline{A}[1] \otimes k[1]$, and treating this as an identity, $\widetilde{v}_2 = \begin{bmatrix} v \otimes 1 \\ p \otimes v \end{bmatrix} : A[1]^{\otimes 2} \to K_A \cap A[1]^{\otimes 2}$. Using the above string of equalities, we have

$$\begin{aligned}(\overline{m}\bar{\partial})_n &= -pm_2 \begin{bmatrix} v \otimes 1 \\ p \otimes v \end{bmatrix} (b \otimes m_{n-1}b^{\otimes n-1} + m_{n-1}b^{\otimes n-1} \otimes b) \\ &= -pm_2(b \otimes \eta h_{n-1} + \eta h_{n-1} \otimes b).\end{aligned}$$

To now show equality with $p_1[h,-]$, we evaluate both on an element of $\overline{A}[1]^{\otimes n}$ (to make explicit the signs involved with m_2, see 2.2.6). For $x_i \in \overline{A}$, we have

$$\begin{aligned}(\overline{m}\bar{\partial})_n([x_1|\dots|x_n]) &= -pm_2(b \otimes \eta h_{n-1} + \eta h_{n-1} \otimes b)[x_1|\dots|x_n] \\ &= h_{n-1}([x_1|\dots|x_{n-1}])[x_n] - h_{n-1}([x_2|\dots|x_n])[x_1].\end{aligned}$$

We calculate $p_1[h,-]$ as a series of compositions:

$$[x_1|\dots|x_n] \overset{\Delta}{\longmapsto} \textstyle\sum_i [x_1|\dots|x_i] \otimes [x_{i+1}|\dots|x_n] \longmapsto$$

$$\textstyle\sum_i h_i([x_1|\dots|x_i])[x_{i+1}|\dots|x_n] - h_{n-i}([x_{i+1}|\dots|x_n])[x_1|\dots|x_i] \xmapsto{p_1}$$
$$h_{n-1}([x_1|\dots|x_{n-1}])[x_n] - h_{n-1}([x_2|\dots|x_n])[x_1] = p_1[h,[x]].$$

Thus $\overline{\partial}^2 = [h, -]$. We now show $h\overline{\partial} = 0$. Using the diagram (9.1.1), one checks that

$$h\overline{\partial} = -vm\partial^{\varphi''}\widetilde{b}.$$

As m is strictly unital, and recall $\varphi'' = \eta vm\widetilde{b}\widetilde{p} = \eta h\widetilde{p}$,

$$\begin{aligned}(h\overline{\partial})_n = -vm_2(\partial^{\varphi''})^2_n b^{\otimes n} &= -vm_2((\varphi'')_{n-1}b^{\otimes n} \otimes b + b \otimes (\varphi'')_{n-1}b^{\otimes n-1}) \\ &= -vm_2(\eta h_{n-1} \otimes b + b \otimes \eta h_{n-1}).\end{aligned}$$

The above evaluated on an element $[x_1|\dots|x_n]$ is

$$\begin{aligned}&v\,(h_{n-1}([x_2|\dots|x_n])[x_1] - h_{n-1}([x_1|\dots|x_{n-1}])[x_n]) \\ &= h_{n-1}([x_2|\dots|x_n])v([x_1]) - h_{n-1}([x_1|\dots|x_{n-1}])v([x_2]) = 0,\end{aligned}$$

where the last equality follows since $[x_1],[x_n] \in \overline{A}[1] = \ker v$.

We now prove the converse statement. Let $(T^{\mathsf{co}}(\overline{A}[1]), \overline{\partial}, h)$ be a cdg coalgebra with $v : A[1] \to k[1]$ a splitting of $1_A \hookrightarrow A$. We wish to show there is a unique extension to a strictly unital A_∞-algebra structure on A with v a split unit. Define linear maps $\varphi', \varphi'' : A[1]^{\otimes 2} \to A[1]$, as

$$\varphi' = (A[1]^{\otimes 2} \xrightarrow{v\otimes p} k[1] \otimes \overline{A}[1] \xrightarrow{s^{-1}\otimes s^{-1}} k \otimes \overline{A} \hookrightarrow A \xrightarrow{s} A[1]) +$$
$$(A[1]^{\otimes 2} \xrightarrow{p\otimes v} \overline{A}[1] \otimes k[1] \xrightarrow{s^{-1}\otimes s^{-1}} \overline{A} \otimes k \hookrightarrow A \xrightarrow{s} A[1]);$$
$$\varphi'' = \eta h\widetilde{p}.$$

Set $\varphi = \varphi' + \varphi''$ and $m = b\overline{m}\widetilde{p} + \varphi : T^{\mathsf{co}}(A[1]) \to A[1]$. Note that 1_A is a strict unit of (A, m), i.e. satisfies the equations of Definition 2.2.5. In particular, (A, m) is an A_∞-algebra with split unit v if and only if $m\partial = 0$. We will show this.

First note the diagram (9.1.1) is commutative, so the diagram (9.1.2) is also commutative. We will show $m\partial = 0$ by showing that it's pre and post composition with any of $\widetilde{b}, \widetilde{v}, b, v$ is zero. First, by the commutativity of (9.1.2) and that m is strictly unital, we have

$$\begin{aligned}\overline{m}\overline{\partial} &= p(m - \varphi)(\partial - \partial^{\varphi})\widetilde{b} = p(m - \varphi')(\partial - \partial^{\varphi''})\widetilde{b} \\ &= pm\partial\widetilde{b} - pm\partial^{\varphi''}\widetilde{b} - p\varphi'\partial\widetilde{b} + p\varphi'\partial^{\varphi''}\widetilde{b}.\end{aligned}$$

As above, using that m is strictly unital, the sum of the second and fourth terms in the above equation is zero, and the third term is equal to $p_1[h, -]$. Since $\overline{m}\overline{\partial} = p_1[h, -]$, it follows that

$$pm\partial\widetilde{b} = 0.$$

By the commutative diagram (9.1.2), we have the following equalities:

$$0 = vb\overline{m}\overline{\partial} = v(m - \varphi)(\partial - \partial^{\varphi})\widetilde{b},$$

and from the definition of φ this reduces to

$$= vm\partial\widetilde{b} - vm\partial^{\varphi''}\widetilde{b} - v\varphi''\partial\widetilde{b} + v\varphi''\partial^{\varphi''}\widetilde{b} = 0.$$

Since $v\varphi' = 0$, we have $vm - \eta v\varphi'' = v(m - \varphi)$, and this is only nonzero on the summand $k \cdot 1_A[1]$ of $A[1]$. Thus the sum of the second and fourth terms above is

$$-vm\partial^{\varphi''}\widetilde{b} + v\varphi''\partial^{\varphi''}\widetilde{b} = v(m - \varphi)\partial^{\varphi''}\widetilde{b}.$$

Since $1_A[1]$ is not in the image of $\partial^{\varphi''}\widetilde{b}$ (since φ'' involves h, which has degree -1, so $1_A[1]$ has to be mapped to zero $\square$

9.2 *Proof of* **??**. Let $\alpha : T^c(A[1]) \to B[1]$ be a linear map such that $\alpha_1(1_A) = 1_B$ and α_n zero on any elements that contain 1_A, for $n \geq 2$. Let $\beta : T^c(A[1]) \to T^c(B[1])$ be the map of augmented coalgebras determined by $\alpha i : \overline{T}^c(A[1]) \to B[1]$. Then α induces a strictly unital map of A_∞-algebras if and only if $m^B\beta = \alpha d^A$ if and only if $m^B\beta b = \alpha d^A b$, where $b : T^c(\overline{A}[1]) \to T^c(A[1])$ is the splitting induced by v_A.

Set $\widetilde{\alpha} = \alpha b : T^c(\overline{A}[1]) \to B[1]$ and let $\widetilde{\beta} : T^c(\overline{A}[1]) \to T^c(B[1])$ be the map of coalgebras induced by $\widetilde{\alpha} i$. Using the relation between m^A and $\overline{m}^A$ given in Theorem 2.2.17, we have

$$\alpha d^A b = \alpha b \bar{d}^A + \eta_B s h^A = \widetilde{\alpha}\bar{d}^A + \eta_B s h^A.$$

Then $m^B\beta b = \alpha d^A b$ holds if and only if

$$\widetilde{\alpha}\bar{d}^A + \eta_B s h^A - \sum_{n\geq 1} m_n^B \widetilde{\alpha}^{\otimes n} \overline{\Delta}_{\overline{A}}^{(n)} = 0 \tag{9.2.1}$$

where we used that $m^B\beta b = \sum_{n\geq 1} m_n^B \widetilde{\alpha}^{\otimes n} \overline{\Delta}_{\overline{A}}^{(n)}$ using the construction 2.1.8.(2). (the formula (9.2.1) holds if and only if $\widetilde{\alpha} : T^c(\overline{A}[1]) \to B$ is a twisting cochain, defined in 4.3.1.)

Define maps c and $\bar{\alpha}$ such that the following diagram is commutative:

$$\begin{tikzcd} & T^c(\overline{A}[1]) \arrow[dl, "c"'] \arrow[d, "\widetilde{\alpha}"] \arrow[dr, "\bar{\alpha}"] & \\ k[1] & B[1] \arrow[l, "v"] \arrow[r, "p"'] & \overline{B}[1]. \end{tikzcd}$$

Let $\overline{\beta} : T^c(\overline{A}[1]) \to T^c(\overline{B}[1])$ be the map of coalgebras induced by $\bar{\alpha} i$ and let $a = s^{-1}c : T^c(\overline{A}[1]) \to k$. The equation (9.2.1) holds if and only if it holds after applying each p and v on the left, which gives the equations:

$$\bar{\alpha}\bar{d}^A - \overline{m}^B\overline{\beta} + s^{-1}p(v \otimes 1 - 1 \otimes v)\widetilde{\alpha}^{\otimes 2}\overline{\Delta} = 0$$
$$c\bar{d}^A + sh^A - sh^B\overline{\beta} + (v \otimes v)(\widetilde{\alpha} \otimes \widetilde{\alpha})\overline{\Delta} = 0.$$

In the first equation we have used that $pm^u = \overline{m}p$ in the notation of 9.1, which translates to $pm_n^B = \overline{m}_n^B p^{\otimes n}$ for all $n \neq 2$ and $pm_2^B = \overline{m}_2^B p^{\otimes 2} - s^{-1}p(v \otimes 1 - 1 \otimes v)$. For the second equation we have used that $vm_n^B = vm_n^B((\eta v + bp)^{\otimes n})$ which is $vm_n^B((bp)^{\otimes n}) = sh_n^B p^{\otimes n}$ for $n \neq 2$ and is $sh_2^B p^{\otimes 2} - v \otimes v$ when $n = 2$. Using that $(v \otimes 1 - 1 \otimes v)\widetilde{\alpha}^{\otimes 2} = \widetilde{\alpha}(v\widetilde{\alpha} \otimes 1 - 1 \otimes v\widetilde{\alpha})$, and applying s to the second, we rewrite these as

$$\bar{\alpha}\bar{d}^A - \overline{m}^B\overline{\beta} - \bar{\alpha}[a, -] = 0$$
$$a\bar{d}^A + h^A - h^B\overline{\beta} - a^2 = 0$$

and these hold if and only if $(\overline{\beta}, a) : T^c(\overline{A}[1]) \to T^c(\overline{B}[1])$ is a map of cdg coalgebras, noting that it is enough to show the first condition of **??** after applying $\epsilon_{\mathrm{Bar}\,B}$. $\square$

9.3 *Proof of 2.3.3.* We have the following split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & K_A^{n+1} \arrow[r, shift right, "a"'] & A[1]^{\otimes n+1} \arrow[l, shift right] \arrow[r, shift right, "p^{\otimes n+1}"'] & \overline{A}[1]^{\otimes n+1} \arrow[l, shift right, "b^{\otimes n+1}"'] \arrow[r] & 0 \end{tikzcd}$$

where K_A^{n+1} is the kernel of the projection $p^{\otimes n+1} : A[1]^{\otimes n+1} \to \overline{A}^{\otimes n+1}$, and b is the splitting of p induced by v. Consider $c(m|_n) \in \mathrm{Hom}(A[1]^{\otimes n+1}, A[1])$ defined in (2.3.2). We claim that

$$c(m|_n)a = 0 : K_A^{n+1} \to A[1].$$

Since $c(m|_1) = 0$, we assume that $n \geq 2$. Fix

$$x = [x_1|\dots|x_{l-1}|1_A|x_{l+1}|\dots|x_{n+1}] \in K_A^{n+1}.$$

Since A is strictly unital, m_i is zero on elements of K_A, for $i \neq 2$. This gives

$$c(m|_n)(x) = \sum_{i=2}^{n} \sum_{j=0}^{n-i+1} m^A_{n-i+2}(1^{\otimes j} \otimes m^A_i \otimes 1^{\otimes n-i-j+1})[x_1|\dots|x_{l-1}|1_A|x_{l+1}|\dots|x_{n+1}]$$

$$= m_2^A(m_n^A \otimes 1 + 1 \otimes m_n^A)(x) + \sum_{j=0}^{n-1} m_n^A(1^{\otimes j} \otimes m_2^A \otimes 1^{\otimes n-j-1})(x),$$

and one checks the above is zero (there are two cases: if $1 < l < n+1$, in which case $(m_n^A \otimes 1 + 1 \otimes m_n^A)(x) = 0$, or if $l = 1$ or $n+1$).

Since $c(m|_n)a = 0$, there is an induced map $\bar{c} \in \mathrm{Hom}(\overline{A}[1]^{\otimes n+1}, A[1])$ with

$$c = \bar{c}p.$$

Since c is a cycle, i.e. a chain map, $\bar{c}$ is also. Applying $b^{\otimes n+1}$ to the right hand side of the above gives

$$cb^{\otimes n+1} = \bar{c}pb^{\otimes n+1} = \bar{c}.$$

We calculate

$$\bar{c} = cb^{\otimes n+1} = \sum_{i=2}^{n} \sum_{j=0}^{n-i+1} m^A_{n-i+2}(1^{\otimes j} \otimes m^A_i \otimes 1^{\otimes n-i-j+1})b^{\otimes n+1}$$

$$= \sum_{i=2}^{n} \sum_{j=0}^{n-i+1} m^A_{n-i+2}b^{\otimes n-i+2}(1^{\otimes j} \otimes \overline{m}^A_i \otimes 1^{\otimes n-i-j+1}) + m_2^A(\eta s h_n \otimes b + b \otimes \eta s h_n)$$

where we have used that A is strictly unital and that $m_n b^{\otimes n} = b\overline{m}_n^A + \eta s h_n$,

$$= \sum_{i=2}^{n} \sum_{j=0}^{n-i+1} m^A_{n-i+2}b^{\otimes n-i+2}(1^{\otimes j} \otimes \overline{m}^A_i \otimes 1^{\otimes n-i-j+1}) - h_n \otimes b + b \otimes h_n.$$

This agrees with the definition of $\bar{c}$ in the statement of 2.3.3. This shows the first half of that result when $n \geq 2$. When $n = 1$, one checks directly that $\bar{c}$ is a cycle.

Let $\widetilde{m}_{n+1} : \overline{A}[1]^{\otimes n+1} \to A[1]$ be a degree -1 map. Let $m^A_{n+1} : A[1]^{\otimes n+1} \to A[1]$ be the unique degree -1 map such that $\widetilde{m}_{n+1} = m^A_{n+1}b^{\otimes n+1}$. Then $\widetilde{m}_{n+1}$ extends $m|_n$ to a A_{n+1} structure in which v is a split unit if and only if

$$d(m^A_{n+1}) + c = 0.$$

We have $ac = 0$ and $d(m_{n+1})a = m_{n+1}m_1^{(n+1)}a + m_1 m_{n+1}a = 0$ (if $n \geq 2$, it is automatic; if $n = 1$, then a quick check shows it is true), thus the above equation holds if and only if

$$d(m^A_{n+1})b + cb = m_{n+1}m_1^{(n+1)}b + m_1 m_{n+1}b + cb = 0.$$

Using 2.2.17, we can rewrite this equation as

$$m_{n+1}(b(\overline{m}_1)^{(n+1)} + \sum_{i=0}^{n} b^{\otimes i} \otimes \eta s h_1 \otimes b^{n-i}) + m_1 \widetilde{m}_{n+1} + cb = 0.$$

This simplies to

$$\begin{aligned} \widetilde{m}_{n+1}\overline{m}_1^{(n+1)} + m_1\widetilde{m}_{n+1} + cb &= 0 \quad \text{if } n \geq 2; \\ \widetilde{m}_2\overline{m}_1^{(2)} + m_1\widetilde{m}_2 - h_1 \otimes b + b \otimes h_1 &= 0, \quad \text{if } n = 1, \end{aligned} \tag{9.3.1}$$

using that the image of η is 1_A and m_n vanishes on this for $n \neq 2$. Since $\bar{c} = cb$, this shows the second half of the result. $\square$

9.4 *Proof of 3.1.8.* Assume first that $m^M : T^c(A[1]) \otimes M \to M$ is a strictly unital A_∞ A-module structure. Define $m_u^M : T^c(A[1]) \otimes M \to M$ by $(m_u^M)_n = m_n^m$ for $n \neq 1$ and $(m_u^M)_1 = m_1^M - u \otimes 1$, where $u = s^{-1}v$. Then $m_u^M(K_A \otimes M) = 0$, and so there is an induced map $\overline{m}^M : T^c(\overline{A}[1]) \otimes M \to M$ with $\overline{m}^M(p \otimes 1) = m_u^M$. This implies that $m_u^M(b \otimes 1) = \overline{m}^M$. Let $\bar{d}^M$ be the induced coderivation.

To see $\overline{m}^M$ makes $\operatorname{Bar} A \otimes M$ into a cdg $\operatorname{Bar} A$-comodule, we have to show that $(\bar{d}^M)^2 = h \cdot (-)$. Since $h \cdot (-)$ is a coderivation with respect to $[h, -] = (\bar{d}^{\operatorname{Bar} A})^2$, it is enough to show that

$$\overline{m}^M \bar{d}^M = (\epsilon_{\operatorname{Bar} A} \otimes 1) h \cdot (-)$$

by 3.1.3.(2). We have the following commutative diagram

$$\begin{tikzcd} T^c(A[1]) \otimes M \arrow[d, "d_u^M"'] & T^c(\overline{A}[1]) \otimes M \arrow[l, "b\otimes 1"'] \arrow[d, "\bar{d}^M"] \\ T^c(A[1]) \otimes M \arrow[d, "m_u^M"'] & T^c(\overline{A}[1]) \otimes M \arrow[l, "b\otimes 1"'] \arrow[dl, "\overline{m}^M"] \\ M. & \end{tikzcd}$$

We now have

$$\begin{aligned} \overline{m}^M \bar{d}^M = m_u^M(b \otimes 1)\bar{d}^M &= m^M(b \otimes 1)\bar{d}^M \\ = m^M(b \otimes 1)(\bar{d}^{\operatorname{Bar} A} \otimes 1 &+ (1 \otimes \overline{m}^M)(\Delta_{T^c(\overline{A}[1])} \otimes 1)) \end{aligned}$$

where we have used 3.1.3,

$$= m^M(b\bar{d}^{\operatorname{Bar} A} \otimes 1 + (1 \otimes m^M)(\Delta_{T^c(A[1])} \otimes 1))(b \otimes 1)$$

$$= m^M(d^{\operatorname{Bar} A} b \otimes 1 + (1 \otimes m^M)(\Delta_{T^c(A[1])} \otimes 1))(b \otimes 1) + h \otimes 1,$$

using the relation between $b\bar{d}$ and db given in 2.2.17,

$$= m^M d^M + h \otimes 1 = h \otimes 1 = (\epsilon \otimes 1) h \cdot (-)$$

using that $m^M d^M = 0$.

Conversely, if $\overline{m}^M : T^c(\overline{A}[1]) \otimes M \to M$ makes $T^c(\overline{A}[1]) \otimes M$ into a cdg $\operatorname{Bar} A$-comodule, define $m_n^M = (p^{\otimes n-1} \otimes 1)\overline{m}_n$ for $n \neq 2$, and $m_2^M = (p \otimes 1)\overline{m}_2^M + u \otimes 1$. One checks that these maps make M into a strictly unital A_∞ A-module.

Given $g : T^c(A[1]) \otimes M \to T^c(A[1]) \otimes N$ strictly unital, we have $g(\ker(T^c(A[1]) \otimes M \to T^c(\overline{A}[1] \otimes M))) \subseteq \ker(T^c(A[1] \otimes N \to T^c(\overline{A}[1]) \otimes N)$, so g induces a map

$f : T^c(\overline{A}[1]) \otimes M \to T^c(\overline{A}[1]) \otimes N$ which is a map of cdg comodules over $T^c(\overline{A}[1])$. Given such an f, define g to be bfp. $\square$

9.5 *Proof of 4.3.8.(1).* Note first that $\widetilde{m}_n^N$ is well-defined since N is cocomplete. Throughout the proof, let us now write m_n^N for $\widetilde{m}_n^N$ and $\overline{\Delta}$ for $\overline{\Delta}_N$. Recall that $\overline{d}_C^{(n)} = d_C \otimes 1_C^{\otimes n-1} + 1_C \otimes d_C \otimes 1_C^{\otimes n-2} + \ldots + 1_C^{\otimes n-1} \otimes d_C$.

We start by showing $\left(m_1^N\right)^2 = 0$. We have

(9.5.1)
$$\begin{aligned}(m_1^N)^2 = 1 \otimes d_N^2 + (1 \otimes d_N) \sum_{n \geq 1} (m_n^A \otimes 1)(1 \otimes \overline{\tau}^{\otimes n-1} \otimes 1)(1 \otimes \overline{\Delta}^{(n)})\\ + \left(\sum_{n \geq 1} (m_n^A \otimes 1)(1 \otimes \overline{\tau}^{\otimes n-1} \otimes 1)(1 \otimes \overline{\Delta}^n) \right)(1 \otimes d_N)\\ + \left(\sum_{n \geq 1} (m_n^A \otimes 1)(1 \otimes \overline{\tau}^{\otimes n-1} \otimes 1)(1 \otimes \overline{\Delta}^n) \right) \left(\sum_{j \geq 1} (m_j^A \otimes 1)(1 \otimes \overline{\tau}^{\otimes j-1} \otimes 1)(1 \otimes \overline{\Delta}^j) \right).\end{aligned}$$

Using (4.1.5), the third term is

$$\begin{aligned}&\sum_{n \geq 1} (m_n^A \otimes 1)(1 \otimes \overline{\tau}^{\otimes n-1} \otimes 1)(1 \otimes \overline{d}_C^{(n-1)} \otimes 1 + 1 \otimes 1 \otimes d_N)(1 \otimes \overline{\Delta}^{(n)})\\ =&\sum_{n \geq 1} (m_n^A \otimes 1)(1 \otimes \overline{\tau}^{\otimes n-1} \otimes 1)(1 \otimes \overline{d}_C^{(n-1)} \otimes 1)(1 \otimes \overline{\Delta}^{(n)})\\ -&\sum_{n \geq 1} (1 \otimes d_N)(m_n^A \otimes 1)(1 \otimes \overline{\tau}^{\otimes n-1} \otimes 1)(1 \otimes \overline{\Delta}^{(n)}),\end{aligned}$$

and the last term above cancels the second term of (9.5.1). The first term above is

$$\sum_{n \geq 1} \sum_{i=0}^{n-2} (m_n^A \otimes 1)(1 \otimes \overline{\tau}^{\otimes i} \otimes \overline{\tau d_C} \otimes \overline{\tau}^{n-i-2} \otimes 1)(1 \otimes \overline{\Delta}^{(n)})$$

$$\begin{aligned}(9.5.2) \quad &= \sum_{n \geq 1} \sum_{i=0}^{n-2} (m_n^A \otimes 1)(1 \otimes \overline{\tau}^{\otimes i} \otimes (\sum_{j \geq 1} m_j \overline{\tau}^{\otimes j} \overline{\Delta}_C^{(j)}) \otimes \overline{\tau}^{n-i-2} \otimes 1)(1 \otimes \overline{\Delta}^{(n)})\\ &+ \sum_{n \geq 1} \sum_{i=0}^{n-2} (m_n^A \otimes 1)(1 \otimes \overline{\tau}^{\otimes i} \otimes \eta_A sh_A \otimes \overline{\tau}^{n-i-2} \otimes 1)(1 \otimes \overline{\Delta}^n).\end{aligned}$$

The second term above, using that 1_A is a strict unit, is

$$(m_2 \otimes 1)(1 \otimes \eta_A sh_A \otimes 1)(1 \otimes \overline{\Delta}) = -(1 \otimes d_N^2)$$

and so cancels the first term in (9.5.1). The first term in 9.5.2 is

$$\sum_{n \geq 1} \sum_{j \geq 1} \sum_{i=1}^{n-1} (m_n^A \otimes 1)(1^i \otimes m_j \otimes 1^{\otimes n-i-1} \otimes 1)(1 \otimes \overline{\tau}^{n+j-2} \otimes 1)(1 \otimes \overline{\Delta}^{(n+j-1)}),$$

setting $k = n + j - 1$

$$= \sum_{k \geq 1} \sum_{n=1}^{k} \sum_{i=1}^{n-1} (m_n^A \otimes 1)(1^{\otimes i} \otimes m_{k-n+1} \otimes 1^{\otimes n-i-1} \otimes 1)(1 \otimes \overline{\tau}^{k-1} \otimes 1)(1 \otimes \overline{\Delta}^{(k)})$$

$$= -\sum_{k\geq 1}\sum_{n=1}^{k}(m_n^A \otimes 1)(m_{k-n+1}\otimes 1^{\otimes n})(1\otimes \bar{\tau}^{k-1}\otimes 1)(1\otimes \overline{\Delta}^{(k)})$$

by (2.2.3).

The fourth summand in (9.5.1) is equal to

$$\sum_{n\geq 1}\sum_{j\geq 1}(m_n^A\otimes 1)(1\otimes \bar{\tau}^{\otimes n-1}\otimes 1)(1\otimes \overline{\Delta}^n)(m_j\otimes 1)(1\otimes \bar{\tau}^{\otimes j-1}\otimes 1)(1\otimes \overline{\Delta}^j)$$

$$= \sum_{n\geq 1}\sum_{j\geq 1}(m_n^A\otimes 1)(1\otimes \bar{\tau}^{\otimes n-1}\otimes 1)(m_j\otimes 1)(1\otimes \bar{\tau}^{\otimes j-1}\otimes 1^{\otimes n-1}\otimes 1)(1\otimes \overline{\Delta}^{n+j-1})$$

$$= \sum_{n\geq 1}\sum_{j\geq 1}(m_n^A\otimes 1)(m_j\otimes 1^{\otimes n})(1\otimes \bar{\tau}^{n+j-2}\otimes 1)(1\otimes \overline{\Delta}^{n+j-1})$$

$$= \sum_{k\geq 1}\sum_{n=1}^{k}(m_n^A\otimes 1)(m_{k-n+1}\otimes 1^{\otimes n})(1\otimes \bar{\tau}^{k-1}\otimes 1)(1\otimes \overline{\Delta}^{(k)}).$$

This shows that $(m_1^N)^2 = 0$. We now show that

$$\sum_{i=1}^{n}\sum_{j=0}^{n-i} m_{n-i+1}^N(1^{\otimes j}\otimes m_i\otimes 1^{\otimes n-i-j}) = 0 \tag{9.5.3}$$

holds for $n\geq 2$. Recall that

$$m_n^N = \sum_{k\geq 1}(m_{n+k-1}\otimes 1_N)(1^{\otimes n}\otimes \bar{\tau}^{\otimes k-1}\otimes 1_N)(1^{\otimes n}\otimes \overline{\Delta}_N^k).$$

We have

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=0}^{n-i} m_{n-i+1}^N(1^{\otimes j}\otimes m_i\otimes 1^{\otimes n-i-j}) \\
&= (1\otimes d_N)m_n^N + \sum_{i=1}^{n-1}\sum_{j=0}^{n-i-1} m_{n-i+1}^N(1^{\otimes j}\otimes m_i^A\otimes 1^{\otimes n-i-j}\otimes 1_N) \\
&\quad + \sum_{i=1}^{n} m_{n-i+1}^N(1^{\otimes n-i}\otimes m_i^N) + m_n^N(1^{\otimes n}\otimes d_N) \\
&= (1\otimes d_N)\sum_{k\geq 1}(m_{n+k-1}\otimes 1_N)(1^{\otimes n}\otimes \bar{\tau}^{\otimes k-1}\otimes 1_N)(1^{\otimes n}\otimes \overline{\Delta}_N^{(k)}) \\
&\quad + \sum_{i=1}^{n-1}\sum_{j=0}^{n-i-1}\sum_{a\geq 1}(m_{n+a-i}^A\otimes 1_N)(1^{\otimes n-i+1}\otimes \bar{\tau}^{\otimes a-1}\otimes 1_N)(1^{\otimes n-i+1}\otimes \overline{\Delta}_N^{(a)})(1^{\otimes j}\otimes m_i^A\otimes 1^{\otimes n-i-j}\otimes 1_N) \\
&\quad + \sum_{i=1}^{n}\sum_{k\geq 1}(m_{n+k-i}^A\otimes 1_N)(1^{\otimes n-i+1}\otimes \bar{\tau}^{\otimes k-1}\otimes 1_N)(1^{\otimes n-i+1}\otimes \overline{\Delta}_N^{(k)})\left(1^{\otimes n-i}\otimes\left(\sum_{l\geq 1}(m_{i+l-1}^A\otimes 1_N)(1^{\otimes i}\otimes \bar{\tau}^{\otimes l-1}\otimes 1_N)(1^{\otimes i}\otimes \overline{\Delta}_N^{(l)})\right)\right) \\
&\quad + \sum_{k\geq 1}(m_{n+k-1}^A\otimes 1_N)(1^{\otimes n}\otimes \bar{\tau}^{\otimes k-1}\otimes 1_N)(1^{\otimes n}\otimes \overline{\Delta}_N^{(k)})(1^{\otimes n}\otimes d_N).
\end{aligned} \tag{9.5.4}$$

We first work with the last term of the above equation. Using (4.1.5) this is

$$= \sum_{k\geq 1}(m_{n+k-1}^A\otimes 1_N)(1^{\otimes n}\otimes \bar{\tau}^{\otimes k-1}\otimes 1_N)(1^{\otimes n}\otimes (\bar{d}_C^{(k-1)}\otimes 1_N + 1^{\otimes k-1}\otimes d_N)\overline{\Delta}_N^{(k)})$$

$$= -(1\otimes d_N)\sum_{k\geq 1}(m_{n+k-1}\otimes 1_N)(1^{\otimes n}\otimes \bar{\tau}^{\otimes k-1}\otimes 1_N)(1^{\otimes n}\otimes \overline{\Delta}_N^{(k)})$$

$$+ \sum_{k\geq 1}(m_{n+k-1}^A\otimes 1_N)(1^{\otimes n}\otimes \bar{\tau}^{\otimes k-1}\bar{d}_C^{(k-1)}\otimes 1_N)(1^{\otimes n}\otimes \Delta_N^{(k)}).$$

The first term above cancels with the first term of (9.5.4). We can expand the second term to get

$$= \sum_{k\geq 1}\sum_{j=0}^{k-1}(m^A_{n+k-1}\otimes 1_N)(1^{\otimes n}\otimes(\overline{\tau}^{\otimes j}\otimes\overline{\tau}\overline{d}_C\otimes\overline{\tau}^{\otimes k-j-1})\otimes 1_N)(1^{\otimes n}\otimes\Delta_N^{(k)}).$$

Using the definition of twisting cochain, see (4.3.2), the above is

$$\sum_{k\geq 1}\sum_{j=0}^{k-1}(m^A_{n+k-1}\otimes 1_N)(1^{\otimes n}\otimes(\overline{\tau}^{\otimes j}\otimes\left(\sum_{l\geq 1}m^A_l\overline{\tau}^{\otimes l}\overline{\Delta}_C^{(l)}+\overline{h}\cdot 1_A\right)\otimes\overline{\tau}^{\otimes k-j-1})\otimes 1_N)(1^{\otimes n}\otimes\Delta_N^{(k)}).$$

Note that since $n\geq 2$, the only time $n+k-1=2$ is when $k=1$, but in this case the term involving h doesn't appear, e.g. since $\overline{\tau}^{\otimes k-1}\overline{d}_C^{(k-1)}=1$, the h term above is always zero. So we have

$$\sum_{k\geq 1}\sum_{j=0}^{k-1}(m^A_{n+k-1}\otimes 1_N)(1^{\otimes n}\otimes(\overline{\tau}^{\otimes j}\otimes\left(\sum_{l\geq 1}m^A_l\overline{\tau}^{\otimes l}\overline{\Delta}_C^{(l)}\right)\otimes\overline{\tau}^{\otimes k-j-1})\otimes 1_N)(1^{\otimes n}\otimes\Delta_N^{(k)})$$

$$=\sum_{l\geq 1}\sum_{k\geq 1}\sum_{j=0}^{k-1}(m^A_{n+k-1}\otimes 1_N)(1^{\otimes n+j}\otimes m^A_l\otimes 1^{\otimes k-j-2}\otimes 1_N)(1^{\otimes n}\otimes\overline{\tau}^{\otimes l+k-2}\otimes 1_N)(1^{\otimes n}\otimes\Delta_N^{(k+l-1)})$$

set $a=k+l-1$

$$=\sum_{a\geq 1}\sum_{l=1}^{a}\sum_{j=0}^{k-1}(m^A_{n+a-l}\otimes 1_N)(1^{\otimes n+j}\otimes m^A_l\otimes 1^{\otimes a-l-j-1}\otimes 1_N)(1^{\otimes n}\otimes\overline{\tau}^{\otimes a-1}\otimes 1_N)(1^{\otimes n}\otimes\Delta_N^{(a)})$$

set $l=j+n$

$$=\sum_{a\geq 1}\sum_{i=1}^{a}\sum_{l=n}^{n+a-i}(m^A_{n+a-i}\otimes 1_N)(1^{\otimes l}\otimes m^A_i\otimes 1^{\otimes n+a-i-l-1}\otimes 1_N)(1^{\otimes n}\otimes\overline{\tau}^{\otimes a-1}\otimes 1_N)(1^{\otimes n}\otimes\Delta_N^{(a)}).$$

We now work on the third term of (9.5.4). We have

$$\sum_{i=1}^{n}\sum_{k\geq 1}(m^A_{n+k-i}\otimes 1_N)(1^{\otimes n-i+1}\otimes\overline{\tau}^{\otimes k-1}\otimes 1_N)(1^{\otimes n-i+1}\otimes\overline{\Delta}_N^{(k)})\left(1^{\otimes n-i}\otimes\left(\sum_{l\geq 1}(m^A_{i+l-1}\otimes 1_N)(1^{\otimes i}\otimes\overline{\tau}^{\otimes l-1}\otimes 1_N)(1^{\otimes i}\otimes\overline{\Delta}_N^{(l)})\right)\right)$$

$$=\sum_{k\geq 1}\sum_{l\geq 1}\sum_{i=1}^{n}(m^A_{n+k-i}\otimes 1_N)(1^{\otimes n-i}\otimes m^A_{l+i-1}\otimes 1^{\otimes k-1}\otimes 1_N)(1^{\otimes n}\otimes\overline{\tau}^{\otimes l+k-2}\otimes 1_N)(1^{\otimes n}\otimes\overline{\Delta}_N^{(l+k-1)}))$$

setting $a=l+k-1$

$$=\sum_{a\geq 1}\sum_{l=1}^{a}\sum_{i=1}^{n}(m^A_{n+a-i-l+1}\otimes 1_N)(1^{\otimes n-i}\otimes m^A_{l+i-1}\otimes 1^{\otimes a-l}\otimes 1_N)(1^{\otimes n}\otimes\overline{\tau}^{\otimes a-1}\otimes 1_N)(1^{\otimes n}\otimes\overline{\Delta}_N^{(a)})$$

set $j=l+i-1$

$$=\sum_{a\geq 1}\sum_{i=1}^{n}\sum_{j=i}^{a+i-1}(m^A_{n+a-j}\otimes 1_N)(1^{\otimes n-i}\otimes m^A_j\otimes 1^{\otimes a-j+i-1}\otimes 1_N)(1^{\otimes n}\otimes\overline{\tau}^{\otimes a-1}\otimes 1_N)(1^{\otimes n}\otimes\overline{\Delta}_N^{(a)})$$

replace i with $n-i$

$$=\sum_{a\geq 1}\sum_{i=0}^{n-1}\sum_{j=n-i}^{n+a-i-1}(m^A_{n+a-j}\otimes 1_N)(1^{\otimes i}\otimes m^A_j\otimes 1^{\otimes n+a-j-i-1}\otimes 1_N)(1^{\otimes n}\otimes\overline{\tau}^{\otimes a-1}\otimes 1_N)(1^{\otimes n}\otimes\overline{\Delta}_N^{(a)})$$

Finally, we work on the second term of (9.5.4). We have

$$\sum_{i=1}^{n-1}\sum_{j=0}^{n-i-1}\sum_{a\geq 1}(m^A_{n+a-i}\otimes 1_N)(1^{\otimes n-i+1}\otimes\overline{\tau}^{\otimes a-1}\otimes 1_N)(1^{\otimes n-i+1}\otimes\overline{\Delta}_N^{(a)})(1^{\otimes j}\otimes m^A_i\otimes 1^{\otimes n-i-j}\otimes 1_N)$$

$$=\sum_{a\geq 1}\sum_{i=1}^{n-1}\sum_{j=0}^{n-i-1}(m^A_{n+a-i}\otimes 1_N)(1^{\otimes j}\otimes m^A_i\otimes 1^{\otimes n+a-i-j-1}\otimes 1_N)(1^{\otimes n}\otimes\overline{\tau}^{\otimes a-1}\otimes 1_N)(1^{\otimes n}\otimes\overline{\Delta}_N^{(a)})$$

The three terms now left are

$$\sum_{a\geq 1}\sum_{i=1}^{a}\sum_{j=n}^{n+a-i}(m^A_{n+a-i}\otimes 1_N)(1^{\otimes j}\otimes m^A_i\otimes 1^{\otimes n+a-i-j-1}\otimes 1_N)(1^{\otimes n}\otimes\overline{\tau}^{\otimes a-1}\otimes 1_N)(1^{\otimes n}\otimes\Delta_N^{(a)})$$

$$+\sum_{a\geq 1}\sum_{j=0}^{n-1}\sum_{i=n-j}^{n+a-j-1}(m^A_{n+a-i}\otimes 1_N)(1^{\otimes j}\otimes m^A_i\otimes 1^{\otimes n+a-i-j-1}\otimes 1_N)(1^{\otimes n}\otimes\bar{\tau}^{\otimes a-1}\otimes 1_N)(1^{\otimes n}\otimes\overline{\Delta}^{(a)}_N)$$

$$+\sum_{a\geq 1}\sum_{i=1}^{n-1}\sum_{j=0}^{n-i-1}(m^A_{n+a-i}\otimes 1_N)(1^{\otimes j}\otimes m^A_i\otimes 1^{\otimes n+a-i-j-1}\otimes 1_N)(1^{\otimes n}\otimes\bar{\tau}^{\otimes a-1}\otimes 1_N)(1^{\otimes n}\otimes\overline{\Delta}^{(a)}_N),$$

and one checks this is zero.

□

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