Generalized Koszul duality applied to complete intersection rings

Jesse Burke, UCLA

Dave Benson's Birthday
Isle of Skye
June 26, 2015

BGG (Bernstein-Gelfand-Gelfand) Correspondence

Q commutative ring

$S = Q[T_1, \ldots, T_c]$ with $|T_i| = -2$

$\Lambda = \Lambda_Q(\bigoplus_{i=1}^{c} Qe_i)$ with $|e_i| = 1$

$$\mathrm{D}^{\mathrm{f}}_{\mathrm{dg}}(S) \underset{R}{\overset{L}{\underset{\longrightarrow}{\overset{\longleftarrow}{\cong}}}} \mathrm{D}^{\mathrm{f}}_{\mathrm{dg}}(\Lambda)$$

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Morphisms: when P, P' are graded *free* S-modules, a morphism is a homotopy class of a morphism of S-modules that commutes with the given derivations. Every object is isomorphic to an object with underlying free module; thus $\mathrm{D}^{\mathrm{f}}_{\mathrm{cdg}}(S) \cong [\text{gr-mf}(S, W)]$.

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$$P = \left(S \oplus S(2)^6\right) \oplus \left(S(1)^4 \oplus S(3)^3\right) \quad d = \begin{pmatrix} 0 & d_1 \\ d_0 & 0 \end{pmatrix}$$

where

$$P^{\text{ev}} \xrightarrow{\quad d_0 \quad} P^{\text{odd}}(1)$$

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$$d0 = \begin{pmatrix} 0 \\ -2 \\ -2 \\ -2 \\ -2 \\ -2 \\ -2 \end{pmatrix} \quad \begin{pmatrix} -T_1x - T_3z & -y^2 - z^2 & 0 & 0 & z^2 & xz - z^2 & yz \\ -T_2y & x^2 & -xz + z^2 & 0 & -z^2 & 0 & 0 \\ T_3x + T_3z & 0 & y^2 + z^2 & -yz & -xz - z^2 & -x^2 & -xy - yz \\ T_2 & 0 & 0 & x - z & y & 0 & -z \\ 0 & T_3x^2 + T_2yz - T_3z^2 & -T_1x^2 + T_1xz - T_3xz + T_3z^2 & 0 & -T_1xz - T_2yz - T_3z^2 & -T_2xy + T_2yz & -T_2y^2 \\ -T_3^2x - T_3^2z & T_2xy - T_3xz + T_2yz - T_3z^2 & -T_3y^2 + T_1xz - T_2yz & T_2y^2 + T_3yz & -T_1x^2 - T_1xz + T_3xz + T_3z^2 & T_3x^2 + T_2yz & T_3xy + T_3yz \\ 0 & -T_3xy - T_2xz - T_3yz - T_2z^2 & T_1xy + T_3yz + T_2z^2 & -T_1xz - T_2yz - T_3z^2 & 0 & T_2y^2 - T_2z^2 & -T_1x^2 - T_1xz \end{pmatrix}$$

$$d1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -2 \\ -2 \\ -2 \end{pmatrix} \quad \begin{pmatrix} x^2 & y^2 + z^2 & xz - z^2 & yz^2 & 0 & 0 & 0 \\ T_2y & -T_1x - T_3z & 0 & 0 & z & 0 & 0 \\ 0 & -T_3x - T_3z & -T_2y - T_3z & 0 & x + z & -z & 0 \\ -T_2x - T_2z & 0 & -T_3y - T_2z & -T_1x^2 - T_1xz & 0 & -y & z \\ 0 & -T_2y & T_3x - T_3z & -T_2y^2 & z & x - z & 0 \\ T_3x + T_3z & 0 & T_1x & 0 & 0 & -z & -y \\ T_2z & T_2z & 0 & T_1xz + T_2yz + T_3z^2 & y & 0 & x - z \end{pmatrix}$$

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$$\Lambda \rightsquigarrow A$$

c-parameter 1st order deformation; corresponds to element

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Deformation theoretic proof seems out of reach; instead use Koszul duality to check equivalence.

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degree -1 Q-linear map defined by $\tau(X_i) = e_i$ and zero elsewhere.
τ is a *twisting cochain* and so gives an adjoint pair

$$\mathrm{D}^{\mathrm{co}}(C_W) \underset{C \otimes^{\tau} -}{\overset{A \otimes^{\tau} -}{\rightleftarrows}} \mathrm{D}^{\infty}(A)$$

Main Theorem of Koszul Duality (-)

Let A be an A_∞-algebra, C_W a curved dg-coalgebra and $\tau : C \to A$ a twisting cochain. The adjoint above is an equivalence if and only if the counit

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In the case A is the Koszul complex, C divided powers coalgebra the map

$$A \otimes^\tau C \otimes^\tau A \to A$$

was shown to be a quasi-isomorphism by Avramov and Buchweitz in 2000.

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This version generalizes Positselski to A_∞-algebras (messy, but mostly formal) and to commutative base ring (real work)

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For non-Golod rings, deform Koszul dual of algebra underlying minimal model?

3) $\mathfrak{g}$ restricted Lie algebra with k-basis $(x_1, \dots, x_n)$; set

$$y_i = x_i^{[p]} - x_i^p \in U(\mathfrak{g})$$

$$O(\mathfrak{g}) := \mathrm{Sym}_k(\mathfrak{g}^{(1)}) \cong k[y_1, \dots, y_n] \subseteq U(\mathfrak{g})$$

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$$u(\mathfrak{g}) = U(\mathfrak{g}) \otimes_O \frac{O(\mathfrak{g})}{(y_1, \ldots, y_n)}$$

restricted enveloping algebra; set $A = \mathrm{Kos}(y_1, \ldots, y_n)$ so

$$U \otimes_O A \xrightarrow{\simeq} u(\mathfrak{g})$$

is quasi-isomorphism.

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More generally, are trying to study the family of algebras

$$A_\chi = U \otimes_O \frac{O(\mathfrak{g})}{(y_1 - \chi(y_1), \ldots, y_n - \chi(y_n))}$$

for character $\chi : g^{(1)} \to k$.

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$$\begin{array}{ccc} \mathrm{D}^{\mathrm{f}}_{\mathrm{cdg}}(S_W) & \underset{L}{\overset{R}{\rightleftarrows}}\ (\cong) & \mathrm{D}^{\mathrm{f}}_{\mathrm{dg}}(A) \\ & {\scriptstyle\cong}\searrow \quad \swarrow{\scriptstyle\cong} & \\ & \mathrm{D}^{\mathrm{f}}(R) & \end{array}$$

If M is an R-module, what are representatives in these categories?

Fix Q and R free resolutions:

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Proposition (Eisenbud, 1980)

There exists a system of higher homotopies $\{\sigma_a | a \in \mathbb{N}^c\}$ on G, with $\sigma_a : G \to G$ a degree $2|a| - 1$ endomorphism. These determine a differential d on $S \otimes G$ such that $(S \otimes G, d) \in \mathrm{D}^{\mathrm{f}}_{\mathrm{cdg}}(S_W)$.

Example of higher homotopies

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$$M = Q/(xz + yz, y^2 + z^2, x^2, y^3, z^3)$$
$$G = 0 \to Q^3 \to Q^6 \to Q^4 \to Q^1 \to 0$$

$$P = S \otimes G \cong \left(S \oplus S(2)^6\right) \oplus \left(S(1)^4 \oplus S(3)^3\right)$$

$$S \oplus S(2)^6 \xrightarrow{d_0} S(2)^4 \oplus S(4)^3$$

$$S(1)^4 \oplus S(3)^3 \xrightarrow{d_1} S(1) \oplus S(3)^6$$

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$$d1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -2 \\ -2 \\ -2 \end{pmatrix} \quad \begin{pmatrix} x^2 & y^2 + z^2 & xz - z^2 & yz^2 & 0 & 0 & 0 \\ T_2y & -T_1x - T_3z & 0 & 0 & z & 0 & 0 \\ 0 & -T_3x - T_3z & -T_2y - T_3z & 0 & x + z & -z & 0 \\ -T_2x - T_2z & 0 & -T_3y - T_2z & -T_1x^2 - T_1xz & 0 & -y & z \\ 0 & -T_2y & T_3x - T_3z & -T_2y^2 & z & x - z & 0 \\ T_3x + T_3z & 0 & T_1x & 0 & 0 & -z & -y \\ T_2z & T_2z & 0 & T_1xz + T_2yz + T_3z^2 & y & 0 & x - z \end{pmatrix}$$

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$$S(2)^6 \cong S(2) \otimes_Q G_2 \xrightarrow{1 \otimes d_2^G} S(2) \otimes_Q G_1 \cong S(2)^4$$

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$$T_1 \begin{pmatrix} -x \\ 0 \\ 0 \\ 0 \end{pmatrix} + T_2 \begin{pmatrix} 0 \\ -y \\ 1 \\ 0 \end{pmatrix} + T_3 \begin{pmatrix} -z \\ 0 \\ x + z \\ 0 \end{pmatrix}$$

$$= \sum T_i \otimes \sigma_i : S \otimes G_0 \to S(1) \otimes G_1$$

$$\sigma_i : G_0 \to G_1$$

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$\sigma_{(0,0,2)}$ only nonzero σ_J with $|J| \geq 2$

R-free resolution from higher homotopies

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$$0 \leftarrow G_0 \leftarrow G_1 \leftarrow \begin{matrix} (S^2)^* \otimes G_0 \\ G_2 \end{matrix} \leftarrow \begin{matrix} (S^2)^* \otimes G_1 \\ G_3 \end{matrix} \leftarrow \begin{matrix} (S^4)^* \otimes G_0 \\ (S^2)^* \otimes G_2 \end{matrix} \leftarrow \dots$$

with $- \otimes R$ applied to above.

Explanation for higher homotopies: we can transfer the R-module structure on M to an A_∞ A-module structure on $G \xrightarrow{\simeq} M$.

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This is encoded in an extended $\operatorname{Bar} A$-comodule structure on $\operatorname{Bar} A \otimes G$. But by Koszul duality,

$$\operatorname{Bar} A \simeq C_W$$

is a homotopy equivalence, and so $\operatorname{Bar} A \otimes G \simeq C_W \otimes G$. Now dualize C to S.

Proposition (-, Eisenbud, Schreyer)

There exists a system of higher operators $\{t^{i_1,\ldots,i_j} \,|1 \leq i_1 < \ldots < i_j \leq c\}$, *with* $t^{i_1,\ldots,i_j} : F \to F$ *a degree* j *endomorphism. These determine a derivation* d *on* $A \otimes F$ *such that* $(A \otimes F, d)$ *is a dg* A*-module quasi-isomorphic to* M.

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These are dual to the higher homotopies, via the generalized BGG correspondence.

Representatives of M

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$$
\begin{array}{ccc}
S \otimes G \in \mathsf{D}^{\mathsf{f}}_{\mathsf{cdg}}(S_W) & \underset{L}{\overset{R}{\underset{\cong}{\rightleftarrows}}} & \mathsf{D}^{\mathsf{f}}_{\mathsf{dg}}(A) \ni A \otimes F \\
& {\scriptstyle\cong}\searrow \quad \swarrow{\scriptstyle\cong} & \\
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Want to use this BGG to study numerical invariants of M.

Assume $(Q, \mathfrak{n}, k)$ is local and the resolutions $G, R \otimes F$ are *minimal*.

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Guiding questions: what are the shapes and sizes of G and F? How are they related?

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$$\beta_M^Q(i) = \dim_k G_i \otimes k = \dim_k \operatorname{Tor}_i^Q(M, k)$$

$$\beta_M^R(i) = \dim_k F_i \otimes k = \dim_k \operatorname{Ext}_R^n(M, k)$$

$$\mathrm{P}_M^Q(t) := \sum_{n \geq 0} \beta_M^Q(n) t^n$$

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Apply $- \otimes_Q k$ to BGG diagram:

$$\bar{S} \otimes \bar{G} \in \mathrm{D}^{\mathrm{f}}_{\mathrm{dg}}(\bar{S}) \underset{L}{\overset{R}{\underset{\cong}{\rightleftarrows}}} \mathrm{D}^{\mathrm{f}}_{\mathrm{dg}}(\bar{\Lambda}) \ni \bar{\Lambda} \otimes \bar{F}$$

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Eilenberg-Moore spectral sequence

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For dg-modules M, N over dg-algebra B, have Eilenberg-Moore spectral sequence:

$$E^2 = \mathrm{Ext}^*_{H(B)}(H(M), H(N)) \Rightarrow H(\mathbf{R}\,\mathrm{Hom}_B(M, N))$$

and analogous for Tor.

Applying to:

$$\mathbf{R}\operatorname{Hom}_{\bar{\Lambda}}(k, M \otimes^{\mathbf{L}}_{Q} k) \cong \bar{S} \otimes \bar{G} \cong \mathbf{R}\operatorname{Hom}_{R}(M, k)$$

$$k \otimes^{\mathbf{L}}_{\bar{S}} \mathbf{R}\operatorname{Hom}_{R}(M, k) \cong \bar{\Lambda} \otimes \bar{F} \cong M \otimes^{\mathbf{L}}_{Q} k$$

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gives

$$E^2 = \operatorname{Ext}^*_{\bar{\Lambda}}(k, \operatorname{Tor}^Q_*(M, k)) \Rightarrow \operatorname{Ext}^*_R(M, k)$$

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gives

$$E^2 = \operatorname{Ext}^*_{\bar{\Lambda}}(k, \operatorname{Tor}^Q_*(M, k)) \Rightarrow \operatorname{Ext}^*_R(M, k)$$

$$E_2 = \operatorname{Tor}^{\bar{S}}_*(\operatorname{Ext}^*_R(M, k), k) \Rightarrow \operatorname{Tor}^Q_*(M, k)$$

These were previously known by Avramov-Buchweitz, and Avramov-Gasharov-Peeva, respectively. The second was inspired by spectral sequence of Benson-Carlson (TAMS ’94).

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In particular, gives (from first page) well known inequalities:

$$\mathrm{P}_M^R(t) \leq \frac{\mathrm{P}_M^Q(t)}{(1-t^2)^c}$$

$$\mathrm{P}_M^Q(t) \leq \mathrm{P}_M^R(t)(1+t)^c$$

with equality if and only if the corresponding spectral sequences collapse on the first page if and only if higher homotopies (resp. operators) are minimal.

Putting these together:

$$\mathsf{P}_M^Q(t) \leq \mathsf{P}_M^R(t)(1+t)^c \leq \frac{\mathsf{P}_M^Q(t)}{(1-t)^c}$$

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so we see that both cannot collapse at once. What's happening?

Analogy with equivariant cohomology

Analogy with equivariant cohomology

X is a smooth manifold, T torus acting smoothly on X

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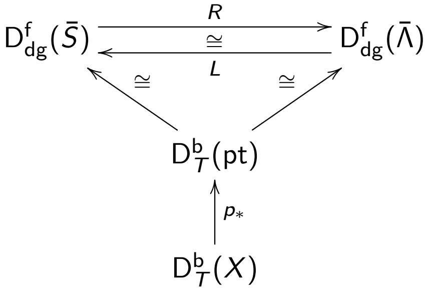
X is a smooth manifold, T torus acting smoothly on X

Goresky, Kottwitz and MacPherson (GKM) show that there is a commutative diagram



$\bar{S} = H^*_T(\mathrm{pt}) \cong \mathbb{R}[T_1, \ldots, T_c] \quad \bar{\Lambda} = H_*(T)$
$\mathrm{D}^{\mathrm{b}}_{T}(X)$ equivariant derived category of X.

So we have

$$\mathrm{D}^{\mathrm{b}}_{T}(X) \cong \mathrm{D}^{\mathrm{f}}(R) \xrightarrow{-\otimes_R k} \mathrm{D}^{\mathrm{b}}_{T}(\mathrm{pt})$$

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Considering numerical invariants of M as above is analogous to pushing forward an object of the equivariant derived category of X to that of a point; roughly this results in a T-vector bundle on a point, i.e. a vector space with T-action.

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Considering numerical invariants of M as above is analogous to pushing forward an object of the equivariant derived category of X to that of a point; roughly this results in a T-vector bundle on a point, i.e. a vector space with T-action. In the inequalities,

$$\mathsf{P}^Q_M(t) \le \mathsf{P}^R_M(t)(1+t)^c \le \frac{\mathsf{P}^Q_M(t)}{(1-t)^c}$$

equality in first corresponds to free action, equality in second is trivial action.

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Use this to compute invariants of $\mathrm{Ext}^*_R(M, k)$ or $\mathrm{Tor}^R_*(M, k)$ using BGG for graded modules? This is related to, and motivated by, current work of Eisenbud, Peeva, and Schreyer.

Future Directions

Localization theorem - ring structure on equivariant cohomology determined by fixed points and "extra data", e.g. moment graph

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Can we use this intuition for any Koszul duality situation?