GRADIENT ESTIMATES, POINCARÉ INEQUALITIES, DE GIORGI PROPERTY AND THEIR CONSEQUENCES

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Abstract. On a doubling metric measure space endowed with a "carré du champ", we consider L^p estimates (G_p) of the gradient of the heat kernel and scale-invariant L^p Poincaré inequalities (P_p) for $p \geq 2$. We show that the combination of (G_p) and (P_p) always implies the Gaussian heat kernel upper bound and the L^p boundedness of the Riesz transform (R_p). Moreover, this combination is shown to also yield the matching heat kernel lower bound if $p > \nu$, where ν is the doubling exponent. If $p \leq \nu$, the same implication holds under the additional assumption of a L^p De Giorgi type property. As a by-product, we give a shorter proof of the well-known fact that the L^2 Poincaré inequality implies Gaussian upper and lower bounds of the heat kernel as well as of the main result in [47]. Instrumental in our approach is a new notion of L^p Hölder regularity for a semigroup. Finally we improve known results on the L^p boundedness of the Riesz transform for $p > 2$.

Contents

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1. Introduction

Let M be a locally compact separable metrisable space equipped with a Borel measure μ, finite on compact sets and strictly positive on any non-empty open set. For Ω a measurable subset of M, we shall often denote $\mu(\Omega)$ by $|\Omega|$.

Let L be a non-negative self-adjoint operator on $L^2(M,\mu)$ with dense domain $\mathcal{D} \subset L^2(M,\mu)$. Denote by $\mathcal{E}$ the associated quadratic form

$$\mathcal{E}(f,g) = \int_M f L g \, d\mu,$$

for $f, g \in \mathcal{D}$, and by $\mathcal{F}$ its domain, which contains $\mathcal{D}$. Assume that $\mathcal{E}$ is a strongly local and regular Dirichlet form (see [37,45] for precise definitions). As a consequence, there exists an energy measure $d\Gamma$, that is a signed measure depending in a bilinear way on $f, g \in \mathcal{F}$ such that

$$\mathcal{E}(f,g) = \int_M d\Gamma(f,g)$$

for all $f, g \in \mathcal{F}$. A possible definition of $d\Gamma$ is through the formula

$$\int \varphi \, d\Gamma(f,f) = \mathcal{E}(\varphi f, f) - \frac{1}{2}\mathcal{E}(\varphi, f^2), \tag{1.1}$$

valid for $f \in \mathcal{F} \cap L^\infty(M,\mu)$ and $\varphi \in \mathcal{F} \cap \mathcal{C}_0(M)$. Here $\mathcal{C}_0(M)$ denotes the space of continuous functions on M that vanish at infinity. According to the Beurling-Deny-Le Jan formula, the energy measure satisfies a Leibniz rule, namely

$$d\Gamma(fg,h) = f d\Gamma(g,h) + g d\Gamma(f,h), \tag{1.2}$$

for all $f, g, h \in \mathcal{F}$, see [37, Section 3.2]. One can define a pseudo-distance d associated with $\mathcal{E}$ by

$$d(x,y) := \sup\{f(x) - f(y); f \in \mathcal{F} \cap \mathcal{C}_0(M) \text{ s.t. } d\Gamma(f,f) \le d\mu\}. \tag{1.3}$$

Throughout the whole paper, we assume that the pseudo-distance d separates points, is finite everywhere, continuous and defines the initial topology of M (see [68] and [45, Subsection 2.2.3] for details).

When we are in the above situation, we shall say that $(M, d, \mu, \mathcal{E})$ is a metric measure (strongly local and regular) Dirichlet space. Note that this terminology is slightly abusive, in the sense that in the above presentation d follows from $\mathcal{E}$.

For all $x \in M$ and all $r > 0$, denote by $B(x,r)$ the open ball for the metric d with centre x and radius r, and by $V(x,r)$ its measure $|B(x,r)|$. For a ball B of radius r and $\lambda > 0$, denote by λB the ball concentric with B and with radius λr. Finally, we will use $u \lesssim v$ to say that there exists a constant C (independent of the important parameters) such that $u \le Cv$ and $u \simeq v$ to say that $u \lesssim v$ and $v \lesssim u$.

We shall assume that (M, d, μ) satisfies the volume doubling property, that is

$$V(x,2r) \lesssim V(x,r), \quad \forall\, x \in M,\ r > 0. \tag{VD}$$

It follows that there exists $\nu > 0$ such that

$$V(x,r) \lesssim \left(\frac{r}{s}\right)^\nu V(x,s), \quad \forall\, x \in M,\ r \ge s > 0, \tag{VD_ν}$$

which implies

$$V(x,r) \lesssim \left(\frac{d(x,y)+r}{s}\right)^{\nu} V(y,s), \quad \forall\, x,y \in M,\ r \geq s > 0.$$

An easy consequence of (VD) is that balls with a non-empty intersection and comparable radii have comparable measures.

We shall say that $(M,d,\mu,\mathcal{E})$ is a doubling metric measure Dirichlet space if it is a metric measure space endowed with a strongly local and regular Dirichlet form and satisfying (VD).

The Dirichlet form $\mathcal{E}$ gives rise to a strongly continuous semigroup $(e^{-tL})_{t>0}$ of self-adjoint contractions on $L^2(M,\mu)$. In addition $(e^{-tL})_{t>0}$ is submarkovian, that is $0 \leq e^{-tL} f \leq 1$ if $0 \leq f \leq 1$. It follows that the semigroup $(e^{-tL})_{t>0}$ is uniformly bounded on $L^p(M,\mu)$ for $p \in [1,+\infty]$. Also, $(e^{-tL})_{t>0}$ is bounded analytic on $L^p(M,\mu)$ for $1 < p < +\infty$ (see [67]), which means that $(tLe^{-tL})_{t>0}$ is bounded on $L^p(M,\mu)$ uniformly in $t > 0$. Moreover, due to the doubling property (VD), the semigroup has the conservation property (see [41,68]), that is

$$e^{-tL}1 = 1, \qquad \forall\, t > 0.$$

Such a semigroup may or may not have a kernel, that is for all $t > 0$ a measurable function $p_t : M \times M \to \mathbb{R}_+$ such that

$$e^{-tL} f(x) = \int_M p_t(x,y) f(y)\, d\mu(y), \quad \text{a.e. } x \in M.$$

If it does, p_t is called the heat kernel associated with L (in fact with $(M,d,\mu,\mathcal{E})$). Then $p_t(x,y)$ is nonnegative and symmetric in x,y since e^{-tL} is positivity preserving and self-adjoint for all $t > 0$. One may naturally ask for upper and lower estimates of p_t (for upper estimates, see for instance the recent article [13] and the many relevant references therein; for lower estimates, we will give more references below). A typical upper estimate is

$$(DUE) \qquad p_t(x,y) \lesssim \frac{1}{\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}}, \quad \forall\, t > 0, \text{ a.e. } x,y \in M.$$

This estimate is called on-diagonal because if p_t happens to be continuous then (DUE) can be rewritten as

$$(1.4) \qquad p_t(x,x) \lesssim \frac{1}{V(x,\sqrt{t})}, \quad \forall\, t > 0, \forall\, x \in M.$$

Under (VD), (DUE) self-improves into a Gaussian upper estimate (see [43, Theorem 1.1] for the Riemannian case, [24, Section 4.2] for a metric measure space setting):

$$(UE) \qquad p_t(x,y) \lesssim \frac{1}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{Ct}\right), \quad \forall\, t > 0, \text{ a.e. } x,y \in M.$$

The proof of this fact in [24, Section 4.2] relies on the following Davies-Gaffney estimate which was proved in our setting in [69]: for all open subsets $E, F \subset M$,

$f \in L^2(M,\mu)$ supported in E, and $t > 0$,

$$\left(\int_F |e^{-tL} f|^2 d\mu\right)^{1/2} \leq e^{-\frac{d^2(E,F)}{4t}} \left(\int_E |f|^2 d\mu\right)^{1/2}, \tag{1.5}$$

where $d(E,F)$ denotes the distance between E and F. For more on the Davies-Gaffney estimate, see for instance [24, Section 3].

It is well-known on the contrary that the matching Gaussian lower bound

$$p_t(x,y) \gtrsim \frac{1}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{ct}\right), \quad \forall\, t > 0, \text{ a.e. } x,y \in M \tag{LE}$$

does not always follow from (DUE) (see [9]). Conversely, under (VD), (LE) implies (UE) (see [11] and [26]).

It is not too difficult to prove in our situation that the conjunction of the upper and lower bounds (UE) and (LE) (that is, (VD) and (LE)) is equivalent to a uniform parabolic Harnack inequality, see [35] as well as [8, Section 1]. One also knows ([45, Thm 2.32]) that this Harnack inequality self-improves into a Hölder regularity estimate for the heat kernel: there exists $\eta \in (0,1]$

$$|p_t(x,z) - p_t(y,z)| \lesssim \left(\frac{d(x,y)}{\sqrt{t}}\right)^{\eta} p_t(x,z), \tag{H^η}$$

for all $t > 0$ and a.e. $x,y,z \in M$ such that $d(x,y) \leq \sqrt{t}$. Note that, if (H^η) holds, p_t admits in particular a continuous version.

What is also true, but much more difficult to prove (see [42], [62], [63], [64], [65], [70], as well as [45, Theorem 2.31]) is that (UE) + (LE) is also equivalent to (VD) together with the following scale-invariant Poincaré inequality (P_2):

$$\left(\fint_B \left| f - \fint_B f d\mu\right|^2 d\mu\right)^{1/2} \lesssim r \int_B d\Gamma(f,f), \tag{P_2}$$

for every $f \in \mathcal{D}$ and every ball $B \subset M$ with radius r. Here $\fint_B f d\mu = \frac{1}{|B|}\int_B f d\mu$ denotes the average of f on B. A somewhat simplified proof of the main implication, namely the one from (VD) + (P_2) to (UE) + (LE), has been given in [47]. One of the outcomes of the present article will be to provide a further simplification (see the proof of Theorem 5.3 below) as well as a short proof of the main result in [47] (see Theorem 5.4 below).

In the present work, we shall study in particular the transition from (UE) to (LE) in the spirit of [20] and [12]. The main novelty here will be a notion of L^p η-Hölder regularity of the heat kernel: for $p \in [1,+\infty]$ and $\eta \in (0,1]$, we shall say that property ($H^\eta_{p,p}$) holds if for every $0 < r \leq \sqrt{t}$, every pair of concentric balls B_r, $B_{\sqrt{t}}$ with respective radii r and $\sqrt{t}$, and every function $f \in L^p(M,\mu)$,

$$\left(\fint_{B_r} \left| e^{-tL} f - \fint_{B_r} e^{-tL} f d\mu\right|^p d\mu\right)^{1/p} \lesssim \left(\frac{r}{\sqrt{t}}\right)^{\eta} |B_{\sqrt{t}}|^{-1/p} \|f\|_p, \tag{$H^\eta_{p,p}$}$$

with the obvious modification for $p = +\infty$.

Crucial to our approach is Theorem 3.5 below where we prove the equivalence, under (VD) and (UE), between the lower Gaussian bound (LE) and the existence of some $p \in [1,+\infty)$ and $\eta > 0$ such that $(H^{\eta}_{p,p})$ holds, a property which turns out to be independent of $p \in [1,+\infty)$.

The above scale-invariant Poincaré inequality (P_2) quantifies the control of the oscillation of functions by the Dirichlet form. As we have just seen for the Hölder regularity of the heat semigroup, it is important to have at hand a full scale of conditions for $p \in [1,+\infty]$, not just $p = 2$. This requires, beyond the notion of L^2 norm of the gradient provided by the Dirichlet form, to have a notion of L^p norm of the gradient, hence a pointwise notion of length of the gradient.

The relevant notion in our general setting is the one of "carré du champ" (see for instance [45] and the references therein). The Dirichlet form (or its energy measure) admits a "carré du champ" if for all $f, g \in \mathcal{F}$ the energy measure $d\Gamma(f,g)$ is absolutely continuous with respect to μ. Then the density $\Upsilon(f,g) \in L^1(M,\mu)$ of $d\Gamma(f,g)$ is called the "carré du champ" and satisfies the following inequality

$$|\Upsilon(f,g)|^2 \leq \Upsilon(f,f)\Upsilon(g,g). \tag{1.6}$$

In the sequel, when we assume that $(M,d,\mu,\mathcal{E})$ admits a "carré du champ", we shall abusively denote $[\Upsilon(f,f)]^{1/2}$ by $|\nabla f|$. This has the advantage to stick to the more intuitive and classical Riemannian notation, but one should not forget that one works in a much more general setting (see for instance [45] for examples), and that one never uses differential calculus in the classical sense.

We shall summarise this situation by saying that $(M,d,\mu,\mathcal{E})$ is a (doubling) metric measure Dirichlet space with a "carré du champ".

We can now formulate the L^p versions of the scale-invariant Poincaré inequalities, which may or may not be true and, contrary to the Hölder regularity conditions for the heat semigroup, do depend on $p \in [1,+\infty)$. More precisely, for $p \in [1,+\infty)$, one says that (P_p) holds if

$$\left(\fint_B \left|f - \fint_B f\,d\mu\right|^p d\mu\right)^{1/p} \lesssim r\left(\fint_B |\nabla f|^p d\mu\right)^{1/p}, \qquad \forall\, f \in \mathcal{F}, \tag{P_p}$$

where B ranges over balls in M of radius r. Recall that (P_p) is weaker and weaker as p increases, that is (P_p) implies (P_q) for $q > p$, see for instance [46], and the $p = \infty$ version is trivial in the Riemannian setting (see however interesting developments for more general metric measure spaces in [33]).

On the Euclidean space, (P_p) holds for all $p \in [1,+\infty]$. On the connected sum of two copies of $\mathbb{R}^n$, (P_p) is valid if and only if $p > n$, as one can see by adapting the proof of [46, Example 4.2]. More interesting examples follow from [48, Theorem 6.15], see also [34, Section 5]. On conical manifolds with compact basis, (P_p) holds at least for $p \geq 2$ (see [22]). A deep result from [53] states if that (P_p) holds for some $p \in (1,+\infty)$, then $(P_{p-\varepsilon})$ holds for some $\varepsilon > 0$. Finally the set $\{p \in [1,+\infty]; (P_p) \text{ holds on } M\}$ may be either $\{+\infty\}$, or $[1,+\infty]$, or of the form $(p_M,+\infty]$ for some $p_M > 1$.

We will also use estimates on the gradient (or "carré du champ") of the semigroup, which were introduced in [4]: for $p \in [1,+\infty]$, consider

$$\sup_{t>0} \|\sqrt{t}|\nabla e^{-tL}|\|_{p\to p} < +\infty, \tag{G_p}$$

which is equivalent to the interpolation inequality

$$\||\nabla f|\|_p^2 \lesssim \|Lf\|_p \|f\|_p, \qquad \forall\, f \in \mathcal{D} \tag{1.7}$$

(see [25, Proposition 3.6]). Up to an arbitrarily small loss in p, one can reformulate (G_p) in terms of integral estimates of the gradient of the heat kernel. More precisely, in presence of (VD) and (DUE), for $2 < p_0 \le +\infty$, (G_p) for all $p \in (2, p_0)$ is equivalent to

$$\||\nabla_x\, p_t(.,y)|\|_p \le \frac{C_p}{\sqrt{t}\left[V(y,\sqrt{t})\right]^{1-\frac{1}{p}}}, \text{ a.e. } y \in M,\ t > 0, \tag{1.8}$$

for all $p \in (2, p_0)$, see [4, Proposition 1.10]. Also, under (DUE), (G_∞) is equivalent to the stronger estimate

$$|\nabla_x p_t(x,y)| \lesssim \frac{1}{\sqrt{t}V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{Ct}\right), \text{ a.e. } x,y \in M,\ t > 0 \tag{$\widetilde{G_\infty}$}$$

(see [25, Section 4.4], [32, Theorem 1.1]). As far as examples are concerned, $(\widetilde{G_\infty})$ holds on manifolds with non-negative Ricci curvature ([54]), Lie groups with polynomial volume growth ([61]), and co-compact covering manifolds with polynomial growth deck transformation group ([31], [32]). On the other hand, conical manifolds with a compact basis provide a family of doubling spaces $(M, d, \mu, \mathcal{E})$ with a "carré du champ" satisfying (UE) and (LE) such that for every $p_0 > 2$ there exist examples in this family where (G_p) holds for $1 < p < p_0$ and not for $p \ge p_0$, see [55],[56],[22].

Property (G_p) is closely related to the L^p boundedness of the Riesz transform $\mathcal{R} = |\nabla L^{-1/2}|$. One says that (R_p) holds if the Riesz transform is bounded on $L^p(M,\mu)$, which means that

$$\||\nabla f|\|_p \lesssim \|\sqrt{L} f\|_p, \quad \forall\, f \in \mathcal{D}. \tag{R_p}$$

Since by definition

$$\||\nabla f|\|_2^2 = \mathcal{E}(f,f) = \|\sqrt{L} f\|_2^2, \quad \forall f \in \mathcal{F},$$

(R_2) trivially holds and (G_2) follows from the analyticity of $(e^{-tL})_{t>0}$ on $L^2(M,\mu)$. Moreover for $p \in (1,2)$, (G_p) also holds always, see [16, Proposition 2.7], and (R_p) holds due to (DUE), see [21]. Now, by interpolation, (G_p) as well as (R_p) are stronger and stronger as p increases above 2. It is easy to see that (R_p) implies (G_p) (due to the fact that the semigroup is bounded analytic on L^p) or equivalently (1.7). Conversely, [4, Theorem 1.3] (which is stated for Riemannian manifolds, but does extend to our current setting, as hinted on p.122) says that under (VD) and Poincaré inequality (P_2), for any $p_0 \in (2, +\infty)$ one has

$$(G_p) \text{ for all } 2 < p < p_0 \iff (R_p) \text{ for all } 2 < p < p_0.$$

Another outcome of the present article will be to prove the same equivalence under (P_{p_0}) instead of (P_2) (see Theorem 7.1 below).

In the present paper, we are going to look at the combination $(G_p) + (P_p)$ for $1 \le p \le +\infty$, and especially for $2 \le p < +\infty$. For $1 \le p \le 2$, $(G_p) + (P_p)$ is nothing but (P_p) and therefore is weaker and weaker as p goes from 1 to 2. On the contrary, for $2 \le p \le +\infty$, since (G_p) is stronger as p increases, whereas (P_p) is weaker, $(G_p) + (P_p)$ does not exhibit a priori any monotonicity.

At one end of the range, $(G_\infty) + (P_\infty)$, at least in the Riemannian setting, is nothing but (G_∞), which does not seem to have consequences in itself. However, it has been shown in [25, Corollary 2.2] that, in presence of (VD), the stronger version $(\widetilde{G_\infty})$ implies (UE) and (LE), therefore, by [21] and [4], (R_p) for all $1 < p < +\infty$, and finally

$$(E_p) \qquad \||\nabla f|\|_p \simeq \|\sqrt{L} f\|_p, \quad \forall\, f \in \mathcal{D}.$$

At the other end of the range, for $p = 2$, we already recalled the fundamental fact that $(G_2) + (P_2) = (P_2)$ implies $(UE) + (LE)$.

We are going to complete the picture for $2 < p < +\infty$ and by the same token simplify the proof of the case $p = 2$.

First, assuming (VD_ν), we will prove that for $\nu < p < +\infty$, $(G_p) + (P_p)$ also has strong consequences. We obtain in Proposition 2.1 that under the doubling property (VD), for any $p \in [2, +\infty)$ the combination $(G_p) + (P_p)$ implies the upper estimates (DUE) and therefore (UE). Using this step, we further show in Theorem 4.1, in the spirit of [20] and using Theorem 3.5, that under (VD_ν) and for $\nu < p < +\infty$,

$$(G_p) + (P_p) \Longrightarrow (LE). \tag{1.9}$$

Putting together these two results,

$$(G_p) + (P_p) \Longrightarrow (UE) + (LE)$$

for $\nu < p < +\infty$ (it is known anyway that $(LE) \Longrightarrow (UE)$, see [11, Theorem 1.3] and [26]). As a by-product, we shall see in Corollary 4.5 that finally $(G_p) + (P_p)$ for some $p > \nu$ implies $(G_q) + (P_q)$ for all $q \in [2, p)$.

If $\nu < 2$, the relevant range $p \in [2, +\infty)$ is covered by the above. One expects things to be easier in this case (see for instance [13, Corollary 2.3.6, Proposition 4.1.8]. This is not only a folklore case (see [23] for a discrete example), but certainly a marginal case, and we certainly have to consider the more common situation where $\nu \ge 2$.

The case $p \in [2, \nu]$ is more complicated. This is not a priori obvious, and it means that in the couple $(G_p) + (P_p)$ it is more efficient to have a stronger (G_p) at the expense of a weaker (P_p), than the opposite. In this range, we will have to introduce an extra assumption in order to ensure the validity of the implication (1.9), namely a non-local L^p-version of De Giorgi property $(\overline{DG}_p)$ (see Section 6 for details and definitions) and we shall prove in Theorem 6.4 that, again under (VD_ν), for $p \in [2, \nu]$,

$$(G_p) + (P_p) + (\overline{DG}_p) \Longrightarrow (LE).$$

It is easy to see that $(\overline{DG}_p)$ always holds for $p > \nu$, so this result is an extension of (1.9). This can be understood in the following way: for $p > \nu$, the corresponding Sobolev inequality, that is the so-called Morrey inequality (4.1), suffices to deduce from L^p gradient bounds L^∞ Hölder estimates on the heat kernel (see the short proof after Remark 4.3 below); for $p \in (2, \nu]$, however, one has to use an elliptic iterative argument to get from L^p gradient bounds up to L^∞ Hölder estimates, and the property $(\overline{DG}_p)$ precisely incorporates such an iteration. We do not know whether this new assumption $(\overline{DG}_p)$ is really necessary in this range; it may well follow from $(G_p)+(P_p)$ as in the case $p=2$ (see Appendix A), but we have not been able to prove it. Note that along the way we introduce and prove an L^p Caccioppoli inequality (6.1) which may be of independent interest.

In any case, $(\overline{DG}_p)$ is not far from being optimal since under (G_p) for some $p>2$, (P_2) (hence (LE) as we already explained) implies $(\overline{DG}_q)$ for every $q \in (2,p]$ (see Proposition 6.7 below). Again, together with Proposition 2.1 (or by [11, Theorem 1.3] adapted to our setting), we can conclude, for $p \in [2,\nu]$,

$$(G_p)+(P_p)+(\overline{DG}_p) \Longrightarrow (UE)+(LE).$$

For $p=2$ the well-known implication $(P_2) \Longrightarrow (UE)+(LE)$ then follows from the fact that (G_2) is always true and that $(P_2) \Longrightarrow (\overline{DG}_2)$. The latter can be seen by an elliptic Moser iteration, much easier than its parabolic counterpart.

Since $(UE)+(LE)$ implies (P_2) (see [62–65]), one can a posteriori summarise the above by saying that under (VD_ν) and (G_p), (P_p) self-improves to (P_2), without any further condition if $p>\nu$, and together with $(\overline{DG}_p)$ if $p \le \nu$. In particular, under (P_p) if $p>\nu$ and $(P_p)+(\overline{DG}_p)$ if $p \le \nu$, (R_p) can only hold if (P_2) holds.

Note that in the range $p>\nu$, (P_p) is particularly simple: for instance, if $V(x,r) \simeq r^\nu$, it is equivalent to the Morrey inequality

$$|f(x)-f(y)| \lesssim [d(x,y)]^{1-\frac{\nu}{p}} \||\nabla f|\|_p, \quad \forall\, f \in \mathcal{D},$$

see [19, Théorème 7.3]. In particular, it is stable under the operation of glueing, say, two manifolds with this volume growth along a compact. Now, if we glue two such manifolds satisfying (R_p) and (P_p) for $p>\nu>2$, the implication

$$(R_p)+(P_p) \Longrightarrow (G_p)+(P_p) \Longrightarrow (P_2),$$

shows that (R_p) cannot hold on the new manifold, since it is easy to see that (P_2) is false on a manifold with a least two ends having polynomial growth of exponent $\nu>2$. This remark is nothing but a systematisation of the counter-example in [21, Section 5]. This also explains why glueing is not allowed in the second assertion of [29, Theorem 1.1]. For a different statement in this direction, see [14, Corollary 7.5].

Finally, we will see in Theorem 7.7 that for any $p_0 \ge 2$, that is even when one cannot use Theorem 4.1 to rely on (LE), or (P_2), the combination $(G_{p_0})+(P_{p_0})$ implies (R_p) for $p \in (p_0, p_0+\varepsilon)$. This improves the main result of [4], which gives, up to an arbitrary small loss in p, the equivalence between (G_p) and (R_p) for $p>2$, under a L^2-Poincaré inequality (P_2), as well as the main result of [3] which treats the case $p_0=2$. It follows that, under the assumption $(G_{p_0})+(P_{p_0})$ for some $p_0 \ge 2$, the range of exponents $p \in (1,\infty)$ for which (G_p) holds coincides with the one for which (R_p) holds, and this range is an open interval of the form $(1,p_1)$, for some exponent $p_1 \in (p_0,+\infty]$ (see Theorem 7.8).

We can summarise most of our results in the following way:

Theorem. *Let $(M,d,\mu,\mathcal{E})$ be a metric measure Dirichlet space with a "carré du champ" satisfying (VD_ν). Assume (G_{p_0}) and (P_{p_0}) for some $p_0 \ge 2$. Then (DUE)*

holds. Moreover there exists $p_1 \in (p_0, +\infty]$ *such that*

$$\{p \in (1,\infty),\ (R_p)\ \textit{holds}\} = \{p \in (1,\infty),\ (G_p)\ \textit{holds}\} = (1, p_1).$$

If either $p_0 > \nu$ *or else* $2 < p_0 \le \nu$ *and* $(\overline{DG}_{p_0})$ *holds, then* (LE)*, and therefore* (P_2)*, holds.*

The plan of the paper is as follows. Section 2 is devoted to the proof of upper estimates (UE) under $(G_p) + (P_p)$ for any $p \in [2, +\infty)$. In Section 3, we obtain the crucial self-improvement property of L^p Hölder regularity estimates $(H^{\eta}_{p,p})$ for the semigroup (Proposition 3.1), and their equivalence with (LE). As an application, in Section 4, (LE) is shown to be implied by (G_p) and (P_p) for $p > \nu$ (Theorem 4.1). The counterpart $p \le \nu$ (Theorem 6.4) is investigated in Section 6, through the study of a suitable De Giorgi property called $(\overline{DG}_p)$. In Section 5, we give a simple proof of the implication from $(VD) + (P_2)$ to $(UE) + (LE)$; the only remaining non trivial part is the implication from $(VD) + (P_2)$ to the most classical De Giorgi property (DG_2), which is recalled in Appendix B. With similar arguments, we also obtain a new proof of the result from [47] that the elliptic regularity together with a scale-invariant local Sobolev inequality imply the parabolic Harnack inequality (Theorem 5.4). Finally, in Section 7, we improve the main results of [3] and [4] by proving the equivalence between the gradient estimate (G_p) and the boundedness of the Riesz transform (R_p), under the Poincaré inequality (P_p). In Appendix A, we study more closely the p-independence of property $(H^{\eta}_{p,p})$. Appendix C spells out a self-improving property of reverse Hölder estimates which is used in the proof of Theorem 7.1.

Moreover, we refer the reader to a forthcoming work of the authors [10], where these new notions (L^p Hölder regularity properties for the heat semigroup and L^p De Giorgi type estimates) will be used to establish the fact that, under certain assumptions on the heat kernel, the spaces $\{f \in L^{\infty}(M,\mu),\ L^{\alpha/2} f \in L^p(M,\mu)\}$ are algebras for the pointwise product for $\alpha \in (0,1)$ and $p \in (1, +\infty)$.

Since our results avoid parabolic Moser iteration, which is very hard to run directly in a discrete time setting (see [28]), they are well suited to an extension to random walks on discrete graphs. As a matter of fact, our Appendix B is inspired by [3], but on the other hand our approach below gives a simpler proof of the main result in [3] by avoiding the iteration step in [3, Proposition 4.5]. For the discrete version of our results on Riesz transform one can rely on [7]. We leave this for future work.

2. From Poincaré and gradient estimates to heat kernel upper bounds

In this section we shall need a version of the Davies-Gaffney estimate (1.5) which also includes the gradient, namely

$$\left(\int_F |e^{-tL} f|^2 d\mu\right)^{1/2} + \sqrt{t}\left(\int_F |\nabla e^{-tL} f|^2\, d\mu\right)^{1/2} \lesssim e^{-c\frac{d(E,F)^2}{t}} \left(\int_E |f|^2 d\mu\right)^{1/2}, \tag{2.1}$$

for some $c > 0$, all open subsets $E, F \subset M$, $f \in L^2(M,\mu)$ supported in E, and $t > 0$, $d(E,F)$ being the distance between E and F. The proof of this fact in

[4, Section 3.1] works in our setting of a Dirichlet space with a "carré du champ". Indeed, the proof relies on the following inequality: for φ a non-negative cut-off function with support S,

$$\int \varphi|\nabla e^{-tL} f|^2 d\mu \leq \left(\int |e^{-tL} f|^2 |\nabla\varphi|^2 d\mu\right)^{1/2} \left(\int_S |\nabla e^{-tL} f|^2 d\mu\right)^{1/2} + \int \varphi |e^{-tL} f||Le^{-tL} f| d\mu, \tag{2.2}$$

which follows from (1.1), (1.2), and (1.6).

Proposition 2.1. *Let $(M, d, \mu, \mathcal{E})$ be a metric measure Dirichlet space with a "carré du champ" satisfying (VD). Then the combination of (G_p) with (P_p) for some $p \in [2, +\infty)$ implies (UE).*

Proof. Assume first $2 < p < +\infty$. From the self-improving property of (P_p) (see [53]), there exists $\tilde{p} \in (2, p)$ such that $(P_{\tilde{p}})$ holds. Then, by interpolating between the L^2 Davies-Gaffney estimate for ∇e^{-tL} contained in (2.1) and (G_p), one obtains that for $t > 0$ the operator $\sqrt{t}\nabla e^{-tL}$ satisfies $L^{\tilde{p}}$-$L^{\tilde{p}}$ off-diagonal estimates at the scale $\sqrt{t}$. Similarly, by interpolating the uniform L^∞ boundedness with (2.1), one sees that the semigroup e^{-tL} also satisfies such estimates. Namely, for some $c > 0$,

$$\| \sqrt{t} |\nabla e^{-tL}| \|_{L^{\tilde{p}}(B) \to L^{\tilde{p}}(\tilde{B})} + \|e^{-tL}\|_{L^{\tilde{p}}(B) \to L^{\tilde{p}}(\tilde{B})} \lesssim \exp\left(-c \frac{d^2(B, \tilde{B})}{t}\right), \tag{2.3}$$

for every $t > 0$ and all balls $B, \tilde{B}$ of radius $\sqrt{t}$. On the other hand, the $(P_{\tilde{p}})$ Poincaré inequality self-improves into a $(P_{\tilde{p},q})$ inequality for some $q > \tilde{p}$ (given by $q^{-1} = \tilde{p}^{-1} - \nu^{-1}$ if $\tilde{p} < \nu$, $q = +\infty$ if $\tilde{p} < \nu$ and any $q > \nu$ if $\tilde{p} = \nu$, see [36]). That is, for every ball $\tilde{B}$ of radius $\sqrt{t}$, one has

$$\left(\fint_{\tilde{B}} \left| f - \fint_{\tilde{B}} f d\mu \right|^q d\mu\right)^{1/q} \lesssim \sqrt{t} \left(\fint_{\tilde{B}} |\nabla f|^{\tilde{p}} d\mu\right)^{1/\tilde{p}}.$$

Hence

$$\left(\fint_{\tilde{B}} \left| e^{-tL} f - \fint_{\tilde{B}} e^{-tL} f d\mu \right|^q d\mu\right)^{1/q} \lesssim \left(\fint_{\tilde{B}} \left| \sqrt{t} \nabla e^{-tL} f \right|^{\tilde{p}} d\mu\right)^{1/\tilde{p}},$$

for all $t > 0$ and $f \in L^{\tilde{p}}(M, \mu)$. It follows by Jensen's inequality that

$$\left(\fint_{\tilde{B}} |e^{-tL} f|^q d\mu\right)^{1/q} \lesssim \left(\fint_{\tilde{B}} |e^{-tL} f|^{\tilde{p}} d\mu\right)^{1/\tilde{p}} + \left(\fint_{\tilde{B}} \left|\sqrt{t}\nabla e^{-tL} f\right|^{\tilde{p}} d\mu\right)^{1/\tilde{p}}.$$

Then from (2.3), we deduce that for every pair of balls $B, \tilde{B}$ of radius $\sqrt{t}$ one has

$$\|e^{-tL}\|_{L^{\tilde{p}}(B) \to L^q(\tilde{B})} \lesssim \exp\left(-c \frac{d^2(B, \tilde{B})}{t}\right) |\tilde{B}|^{\frac{1}{q} - \frac{1}{\tilde{p}}}. \tag{2.4}$$

We now use [13], and refer to it for more details. Set $V_r(x) := V(x, r)$, and denote abusively by w the operator of multiplication by a function w. Using doubling, (2.4) may be written as

$$\|V_{\sqrt{t}}^{\frac{1}{\tilde{p}}-\frac{1}{q}} e^{-tL}\|_{L^{\tilde{p}}(B)\to L^q(\tilde{B})} \lesssim \exp\left(-c\frac{d^2(B,\tilde{B})}{t}\right).$$

By the doubling property, we may sum this inequality over a covering of the whole space at the scale $\sqrt{t}$ and deduce $(VE_{\tilde{p},q})$, which is

$$\sup_{t>0} \|V_{\sqrt{t}}^{\frac{1}{\tilde{p}}-\frac{1}{q}} e^{-tL}\|_{\tilde{p}\to q} < +\infty.$$

By duality, one obtains $(EV_{q',\tilde{p}'})$, that is

$$\sup_{t>0} \|e^{-tL} V_{\sqrt{t}}^{\frac{1}{\tilde{p}}-\frac{1}{q}}\|_{q'\to\tilde{p}'} < +\infty.$$

Then by interpolation [13, Proposition 2.1.5] between $(VE_{\tilde{p},q})$ and $(EV_{q',\tilde{p}'})$, one obtains $(VEV_{r,r',\frac{1}{r}-\frac{1}{2}})$, that is

$$\sup_{t>0} \|V_{\sqrt{t}}^{\frac{1}{r}-\frac{1}{2}} e^{-tL} V_{\sqrt{t}}^{\frac{1}{r}-\frac{1}{2}}\|_{r\to r'} < +\infty,$$

where $1 \le r < 2$ is given by $\frac{1}{r} = \frac{1}{2}(\frac{1}{\tilde{p}} + \frac{1}{q'}) = \frac{1}{2} + (\frac{1}{\tilde{p}} - \frac{1}{q})$. Then $(EV_{r,2})$ holds by [13, Remark 2.1.3]. Thanks to the L^1-uniform boundedness of the semigroup, the extrapolation [13, Proposition 4.1.9] yields $(EV_{1,2})$, hence (DUE) by [13, Proposition 2.1.2] and (UE) by [24, Section 4.2].

Finally, if $p = 2$, one can run the above proof by setting directly $\tilde{p} = 2$. Alternatively, one can see by [26, Section 5] that (P_2) and (VD) imply the so-called Nash inequality (N) and apply [13, Theorem 1.2.1]. □

Remark 2.2. *The case* $1 \le p < 2$ *of Proposition* 2.1 *follows trivially from the case* $p = 2$.

Remark 2.3. *One may avoid the use of the highly non-trivial result from* [53] *by assuming directly* (G_p) *and* (P_q) *for some* $q \in (2, p)$. *Note that this version does work for* $p = +\infty$.

3. L^p Hölder regularity of the heat semigroup and heat kernel lower bounds

The following statement is valid in a more general setting than the one presented in Section 1 and used in Section 2: it is enough to consider a metric measure space (M, d, μ) satisfying (VD), endowed with a semigroup $(e^{-tL})_{t>0}$ acting on $L^p(M, \mu)$, $1 \le p \le +\infty$. For $1 \le p \le +\infty$ let us write the L^p-oscillation for $u \in L^p_{loc}(M, \mu)$ and B a ball:

$$p\text{-}\operatorname{Osc}_B(f) := \left(\fint_B |f - \fint_B f\, d\mu|^p\, d\mu\right)^{1/p},$$

if $p < +\infty$ and

$$\infty\text{-}\operatorname{Osc}_B(f) := \operatorname*{ess\,sup}_B |f - \fint_B f\, d\mu|.$$

Proposition 3.1. *Let* (M, d, μ, L) *as above. Let* $p \in [1, +\infty]$ *and* $\eta \in (0, 1]$. *Then the following two conditions are equivalent:*

(a) *for all* $0 < r \le \sqrt{t}$, *every pair of concentric balls* B_r, $B_{\sqrt{t}}$ *with respective radii* r *and* $\sqrt{t}$, *and every function* $f \in L^p(M,\mu)$,

$$(H^\eta_{p,p}) \qquad p\text{-}\operatorname{Osc}_{B_r}(e^{-tL}f) \lesssim \left(\frac{r}{\sqrt{t}}\right)^\eta \left|B_{\sqrt{t}}\right|^{-1/p} \|f\|_p.$$

(b) *for all* $0 < r \le \sqrt{t}$, *every pair of concentric balls* B_r, $B_{\sqrt{t}}$ *with respective radii* r *and* $\sqrt{t}$, *and every function* $f \in L^p(M,\mu)$,

$$(H^\eta_{p,\infty}) \qquad \operatorname*{ess\,sup}_{x,y\in B_r} \left|e^{-tL}f(x) - e^{-tL}f(y)\right| \lesssim \left(\frac{r}{\sqrt{t}}\right)^\eta \left|B_{\sqrt{t}}\right|^{-1/p} \|f\|_p.$$

Remark 3.2. *It is easy to see that* $(H^\eta_{p,\infty})$ *is equivalent to the following condition, which justifies its name: for all* $0 < r \le \sqrt{t}$, *every pair of concentric balls* B_r, $B_{\sqrt{t}}$ *with respective radii* r *and* $\sqrt{t}$, *and every function* $f \in L^p(M,\mu)$,

$$(3.1) \qquad \infty\text{-}\operatorname{Osc}_{B_r}(e^{-tL}f) \lesssim \left(\frac{r}{\sqrt{t}}\right)^\eta \left|B_{\sqrt{t}}\right|^{-1/p} \|f\|_p.$$

Proposition 3.1 is an easy consequence of a well-known characterisation of Hölder continuous functions in terms of the growth of their L^p oscillations on balls. This result is due to Meyers [58] in the Euclidean space, and its proof was later simplified, see e.g. [39, III.1]. It can be formulated in terms of embeddings of Morrey-Campanato spaces into Hölder spaces. The proof goes through in a doubling metric measure space setting (see [2, Proposition 2.6] for an L^2 version). A particular case of the following lemma will be used in the proof of Proposition 6.7 below. We give a proof for the sake of completeness.

Lemma 3.3. *Let* (M,d,μ) *be a metric measure space satisfying* (VD). *Let* $1 \le p < +\infty$ *and* $\eta > 0$. *Then for every function* $f \in L^p_{loc}(M,\mu)$ *and every ball* B *in* (M,d,μ),

$$\|f\|_{C^\eta(B)} := \operatorname*{ess\,sup}_{\substack{x,y\in B\\ x\neq y}} \frac{|f(x)-f(y)|}{d^\eta(x,y)} \lesssim \|f\|_{C^{\eta,p}(B)} := \sup_{\tilde{B}\subset 6B} \frac{p\text{-}\operatorname{Osc}_{\tilde{B}}(f)}{r^\eta(\tilde{B})}.$$

Proof. Let $x,y \in B$ be Lebesgue points for f. Let $B_i(x) = B(x, 2^{-i}d(x,y))$, for $i \in \mathbb{N}$. Note that for all $i \in \mathbb{N}$, $B_i(x) \subset B_0(x) \subset 3B$. Write

$$\begin{aligned}
\left|f(x) - \fint_{B_0(x)} f\,d\mu\right| &\le \sum_{i\ge 0} \left|\fint_{B_i(x)} f\,d\mu - \fint_{B_{i+1}(x)} f\,d\mu\right| \\
&\le \sum_{i\ge 0} \fint_{B_{i+1}(x)} \left|f - \fint_{B_i(x)} f\,d\mu\right| d\mu \\
&\le \sum_{i\ge 0} \left(\fint_{B_{i+1}(x)} \left|f - \fint_{B_i(x)} f\,d\mu\right|^p d\mu\right)^{1/p} \\
&\lesssim \sum_{i\ge 0} p\text{-}\operatorname{Osc}_{B_i(x)}(f),
\end{aligned}$$

where the last inequality uses doubling. It follows that

$$\begin{aligned}\left|f(x)-\fint_{B_0(x)} f d\mu\right| &\leq \left(\sum_{i\geq 0}\left[r\left(B_i(x)\right)\right]^\eta\right)\|f\|_{C^{\eta,p}(B)}\\ &= \left(\sum_{i\geq 0} 2^{-\eta i} d^\eta(x,y)\right)\|f\|_{C^{\eta,p}(B)}\\ &\lesssim d^\eta(x,y)\|f\|_{C^{\eta,p}(\tilde{B})},\end{aligned}$$

as well as the similar estimate with the roles of x, y exchanged. Finally, since $B_0(y), B_0(x) \subset 2B_0(x)$ with comparable measures by doubling,

$$\begin{aligned}\left|\fint_{B_0(x)} f d\mu - \fint_{B_0(y)} f d\mu\right| &\leq \left|\fint_{B_0(x)} f d\mu - \fint_{2B_0(x)} f d\mu\right| + \left|\fint_{B_0(y)} f d\mu - \fint_{2B_0(x)} f d\mu\right|\\ &\lesssim \fint_{2B_0(x)}\left|f - \fint_{2B_0(x)} f d\mu\right| d\mu\\ &\leq \left(\fint_{2B_0(x)}\left|f - \fint_{2B_0(x)} f d\mu\right|^p d\mu\right)^{1/p}\\ &= p\text{-}\operatorname{Osc}_{2B_0(x)}(f),\end{aligned}$$

hence

$$\left|\fint_{B_0(x)} f d\mu - \fint_{B_0(y)} f d\mu\right| \leq d^\eta(x,y)\|f\|_{C^{\eta,p}(\tilde{B})}.$$

The claim follows by writing

$$|f(x)-f(y)| \leq \left|f(x) - \fint_{B_0(x)} f d\mu\right| + \left|\fint_{B_0(x)} f d\mu - \fint_{B_0(y)} f d\mu\right| + \left|f(y) - \fint_{B_0(y)} f d\mu\right|.$$

□

Proof of Proposition 3.1. The implication from $(H^\eta_{p,\infty})$ to $(H^\eta_{p,p})$ is obvious by integration. The case $p = +\infty$ of the converse is similar to Remark 3.2.

Assume $(H^\eta_{p,p})$ for $1 \leq p < +\infty$. Let $t > 0$, and B_r a ball of radius $0 < r \leq \sqrt{t}$. From Lemma 3.3, we deduce that for $f \in L^p(M,\mu)$, a.e. $x, y \in B_r$,

$$|e^{-tL}f(x) - e^{-tL}f(y)| \lesssim r^\eta \sup_{\tilde{B}\subset 6B_r} \frac{p\text{-}\operatorname{Osc}_{\tilde{B}}(e^{-tL}f)}{r^\eta(\tilde{B})}. \tag{3.2}$$

Now $(H^\eta_{p,p})$ yields

$$\frac{p\text{-}\operatorname{Osc}_{\tilde{B}}(e^{-tL}f)}{r^\eta(\tilde{B})} \lesssim t^{-\eta/2}|\tilde{B}_{\sqrt{t}}|^{-1/p}\|f\|_p,$$

where $\tilde{B}_{\sqrt{t}}$ is the ball concentric to $\tilde{B}$ with radius $\sqrt{t}$. The balls $\tilde{B}_{\sqrt{t}}$ and $B_{\sqrt{t}}$ have the same radius and, if $\tilde{B} \subset 6B_r$, it follows by doubling that $|\tilde{B}_{\sqrt{t}}|$ and $|B_{\sqrt{t}}|$ are comparable, hence

$$\sup_{\tilde{B}\subset 6B_r} \frac{p\text{-}\operatorname{Osc}_{\tilde{B}}(e^{-tL}f)}{r^\eta(\tilde{B})} \lesssim t^{-\eta/2}|B_{\sqrt{t}}|^{-1/p}\|f\|_p, \tag{3.3}$$

and (3.2) together with (3.3) yield $(H^\eta_{p,\infty})$. □

The previous proof relies on the fact that a pointwise Hölder regularity follows from Hölder estimates in terms of oscillation. This key observation requires a sufficiently rapid decay at 0 of the Hölder modulus of continuity under consideration. Under the additional assumption of Gaussian upper estimates for the heat kernel, we shall now give another proof that $(H^\eta_{p,p})$ implies $(H^\eta_{p,\infty})$ that does not take into account the decay of the modulus of continuity. In other words, the second proof explains how the gain of integrability due to the Gaussian estimates of the heat kernel allows to pass from a regularity in terms of L^p-oscillation to a pointwise regularity. The second proof holds for any doubling modulus of regularity instead of $\left(\frac{r}{\sqrt{t}}\right)^\eta$, for instance a logarithmic modulus of continuity $(1-\log(\frac{r}{\sqrt{t}}))^{-\alpha}$ with any $\alpha>0$, whereas the above argument fails if $\alpha\in(0,1)$.

Proposition 3.4. *Let (M,d,μ) as above and assume that $(e^{-tL})_{t>0}$ is a non-negative self-adjoint semigroup on $L^2(M,\mu)$ with a measurable kernel p_t satisfying*

$$|p_t(x,y)|\lesssim \frac{1}{V(x,\sqrt{t})}\exp\left(-\frac{d^2(x,y)}{Ct}\right),\quad \forall\, t>0,\ a.e.\ x,y\in M, \tag{3.4}$$

which we will abusively still call (UE). Assume also conservativeness: $e^{-tL}1=1$.

Let $p\in[1,+\infty]$ and ω be a doubling modulus of continuity, which is a nondecreasing function $\omega:[0,\infty)\to[0,\infty)$ with

$$\lim_{x\to 0}\omega(x)=0$$

satisfying for some $D>0$: for every $x\geq 0$ and $t\geq 1$

$$\omega(tx)\lesssim t^D\omega(x). \tag{3.5}$$

Then the following two conditions are equivalent:

(a) *for all $0<r\leq\sqrt{t}$, every pair of concentric balls B_r, $B_{\sqrt{t}}$ with respective radii r and $\sqrt{t}$, and every function $f\in L^p(M,\mu)$,*

$$p\text{-}\operatorname{Osc}_{B_r}(e^{-tL}f)\lesssim \omega\left(\frac{r}{\sqrt{t}}\right)\left|B_{\sqrt{t}}\right|^{-1/p}\|f\|_p. \tag{3.6}$$

(b) *for all $0<r\leq\sqrt{t}$, every pair of concentric balls B_r, $B_{\sqrt{t}}$ with respective radii r and $\sqrt{t}$, and every function $f\in L^p(M,\mu)$,*

$$\operatorname*{ess\,sup}_{x,y\in B_r}\left|e^{-tL}f(x)-e^{-tL}f(y)\right|\lesssim \omega\left(\frac{r}{\sqrt{t}}\right)\left|B_{\sqrt{t}}\right|^{-1/p}\|f\|_p. \tag{3.7}$$

Note that as a consequence of (3.4), $(e^{-tL})_{t>0}$ acts and is uniformly bounded on all $L^p(M,\mu)$, $1\leq p\leq+\infty$. Recall that (DUE) implies (UE) also in this setting as soon as $(e^{-tL})_{t>0}$ satisfies a Davies-Gaffney estimate (see [24, Section 4.2]).

Proof. Again, the implication from (3.7) to (3.6) is obvious by integration. In order to prove (3.7), fix a ball B_r of radius $r\leq\frac{\sqrt{t}}{4}$ (again, the remaining case follows obviously by doubling); then it is sufficient to show

$$\operatorname*{ess\,sup}_{x\in B_r}\left|e^{-tL}f(x)-\left(\fint_{B_r}e^{-(t-r^2)L}f\,d\mu\right)\right|\lesssim \omega\left(\frac{r}{\sqrt{t}}\right)\left|B_{\sqrt{t}}\right|^{-1/p}\|f\|_p, \tag{3.8}$$

which we are now going to prove.

Step 1: We claim that for every ball $\tilde{B}_r$ of radius r, we have

$$\left|\left(\fint_{B_r} e^{-tL} f d\mu\right) - \left(\fint_{\tilde{B}_r} e^{-tL} f d\mu\right)\right| \lesssim \left(1 + \frac{d(B_r, \tilde{B}_r)}{r}\right)^{D+\frac{\nu}{p}} \omega\left(\frac{r}{\sqrt{t}}\right) \left|B_{\sqrt{t}}\right|^{-1/p} \|f\|_p. \tag{3.9}$$

Indeed, consider a ball B_ρ containing both B_r and $\tilde{B}_r$ with radius $\rho \simeq r + d(B_r, \tilde{B}_r)$. We have

$$\begin{aligned}
&\left|\left(\fint_{B_r} e^{-tL} f d\mu\right) - \left(\fint_{\tilde{B}_r} e^{-tL} f d\mu\right)\right| \\
&\quad \leq \left|\left(\fint_{B_r} e^{-tL} f d\mu\right) - \left(\fint_{B_\rho} e^{-tL} f d\mu\right)\right| + \left|\left(\fint_{B_\rho} e^{-tL} f d\mu\right) - \left(\fint_{\tilde{B}_r} e^{-tL} f d\mu\right)\right|.
\end{aligned}$$

These two terms can be treated similarly, so let us focus on the first one.

$$\begin{aligned}
\left|\left(\fint_{B_r} e^{-tL} f d\mu\right) - \left(\fint_{B_\rho} e^{-tL} f d\mu\right)\right| &\leq \fint_{B_r} \left|e^{-tL} f - \left(\fint_{B_\rho} e^{-tL} f d\mu\right)\right| d\mu \\
&\leq \left(\fint_{B_r} \left|e^{-tL} f - \left(\fint_{B_\rho} e^{-tL} f d\mu\right)\right|^p d\mu\right)^{1/p} \\
&\leq \left(\frac{1}{|B_r|} \int_{B_\rho} \left|e^{-tL} f - \left(\fint_{B_\rho} e^{-tL} f d\mu\right)\right|^p d\mu\right)^{1/p} \\
&= \left(\frac{|B_\rho|}{|B_r|}\right)^{1/p} p\text{-}\operatorname{Osc}_{B_\rho}(e^{-tL} f).
\end{aligned}$$

By (VD_ν) it follows that

$$\left|\left(\fint_{B_r} e^{-tL} f d\mu\right) - \left(\fint_{B_\rho} e^{-tL} f d\mu\right)\right| \lesssim \left(\frac{\rho}{r}\right)^{\frac{\nu}{p}} p\text{-}\operatorname{Osc}_{B_\rho}(e^{-tL} f). \tag{3.10}$$

Then if $\rho \leq \sqrt{t}$, applying $(H^\eta_{p,p})$ to B_ρ and $\frac{\sqrt{t}}{\rho} B_\rho$ gives

$$p\text{-}\operatorname{Osc}_{B_\rho}(e^{-tL} f) \lesssim \omega\left(\frac{\rho}{\sqrt{t}}\right) \left|\frac{\sqrt{t}}{\rho} B_\rho\right|^{-1/p} \|f\|_p.$$

Now $\frac{\sqrt{t}}{\rho} B_\rho$ and $B_{\sqrt{t}}$ have the same radius $\sqrt{t}$ and a non-empty intersection since they both contain B_r, hence by doubling they have comparable measures. Hence

$$p\text{-}\operatorname{Osc}_{B_\rho}(e^{-tL} f) \lesssim \omega\left(\frac{\rho}{\sqrt{t}}\right) \left|B_{\sqrt{t}}\right|^{-1/p} \|f\|_p.$$

If $\rho \geq \sqrt{t}$, then, using the L^p-boundedness of the semigroup and the fact that $|B_{\sqrt{t}}| \lesssim |B_\rho|$, we can write

$$\begin{aligned} p\text{-}\operatorname{Osc}_{B_\rho}(e^{-tL}f) &\lesssim \left(\fint_{B_\rho} \left| e^{-tL} f \right|^p d\mu \right)^{1/p} \\ &\lesssim \left| B_{\sqrt{t}} \right|^{-1/p} \|f\|_p \\ &\lesssim \omega\left(\frac{\rho}{\sqrt{t}}\right) \left| B_{\sqrt{t}} \right|^{-1/p} \|f\|_p, \end{aligned}$$

where the implicit constant depends on ω through $\omega(1)$ (since ω is nondecreasing).

In all cases, one has

$$p\text{-}\operatorname{Osc}_{B_\rho}(e^{-tL}f) \lesssim \omega\left(\frac{\rho}{\sqrt{t}}\right) \left| B_{\sqrt{t}} \right|^{-1/p} \|f\|_p,$$

which with (3.10) and (3.5) yields

$$\left| \left(\fint_{B_r} e^{-tL} f d\mu \right) - \left(\fint_{B_\rho} e^{-tL} f d\mu \right) \right| \lesssim \left(\frac{\rho}{r}\right)^{D+\frac{\nu}{p}} \omega\left(\frac{r}{\sqrt{t}}\right) \left| B_{\sqrt{t}} \right|^{-1/p} \|f\|_p.$$

The claim follows.

Step 2: Conclusion of the proof of (3.8).

For $x \in B_r$ and $f \in L^p(M,\mu)$, one can write, thanks to the conservation property,

$$e^{-tL}f(x) - \left(\fint_{B_r} e^{-(t-r^2)L} f d\mu \right) = e^{-r^2L} \left(e^{-(t-r^2)L} f - \left(\fint_{B_r} e^{-(t-r^2)L} f \, d\mu \right) \right)(x).$$

Now (UE) yields, for $g \in L^p_{loc}(M,\mu)$ and $x \in B_r$,

$$e^{-r^2L} g(x) \lesssim \sum_{i\in I} e^{-c\frac{d^2(B_r, \tilde{B}_r^i)}{r^2}} \left(\fint_{\tilde{B}_r^i} |g|^p \, d\mu \right)^{1/p},$$

where $(\tilde{B}_r^i)_{i\in I}$ is a boundedly overlapping covering of the whole space by balls of radius r. Taking $g = e^{-(t-r^2)L} f - \left(\fint_{B_r} e^{-(t-r^2)L} f \, d\mu \right)$ gives

$$\begin{aligned} &\left| e^{-tL} f(x) - \left(\fint_{B_r} e^{-(t-r^2)L} f \, d\mu \right) \right| \\ &\quad \lesssim \sum_{i\in I} e^{-c\frac{d^2(B_r, \tilde{B}_r^i)}{r^2}} \left(\fint_{\tilde{B}_r^i} \left| e^{-(t-r^2)L} f - \left(\fint_{B_r} e^{-(t-r^2)L} f \, d\mu \right) \right|^p d\mu \right)^{1/p}. \end{aligned} \tag{3.11}$$

We then decompose

$$\begin{aligned} &\left(\fint_{\tilde{B}_r^i} \left| e^{-(t-r^2)L} f - \left(\fint_{B_r} e^{-(t-r^2)L} f d\mu \right) \right|^p d\mu \right)^{1/p} \\ \lesssim \; & p\text{-}\operatorname{Osc}_{\tilde{B}_r^i}(e^{-(t-r^2)L} f) + \left| \left(\fint_{\tilde{B}_r^i} e^{-(t-r^2)L} f d\mu \right) - \left(\fint_{B_r} e^{-(t-r^2)L} f d\mu \right) \right|. \end{aligned}$$

The first term is estimated by $(H^\eta_{p,p})$:

$$p\text{-}\operatorname{Osc}_{\tilde{B}^i_r}(e^{-(t-r^2)L}f) \lesssim \omega\left(\frac{r}{\sqrt{t}}\right)\left|\tilde{B}^i_{\sqrt{t}}\right|^{-1/p}\|f\|_p,$$

where $\tilde{B}^i_{\sqrt{t}}$ is dilated from $\tilde{B}^i_r$ at scale $\sqrt{t}$, and one uses the fact that $t-r^2 \simeq t$, since we have chosen $r \leq \frac{\sqrt{t}}{4}$. Now by doubling $\left|\tilde{B}^i_{\sqrt{t}}\right|^{-1/p} \lesssim \left|B_{\sqrt{t}}\right|^{-1/p}\left(1+\frac{d(B_{\sqrt{t}},\tilde{B}^i_{\sqrt{t}})}{\sqrt{t}}\right)^{\nu/p}$ and since $r \leq \sqrt{t}$ this is estimated by $\left|B_{\sqrt{t}}\right|^{-1/p}\left(1+\frac{d(B_r,\tilde{B}^i_r)}{r}\right)^{\nu/p}$.

The second term is estimated by (3.9):

$$\begin{aligned}&\left|\left(\fint_{\tilde{B}^i_r} e^{-(t-r^2)L}f\,d\mu\right)-\left(\fint_{B_r} e^{-(t-r^2)L}f\,d\mu\right)\right|\\ &\lesssim \left(1+\frac{d(B_r,\tilde{B}^i_r)}{r}\right)^{D+\frac{\nu}{p}}\omega\left(\frac{r}{\sqrt{t}}\right)\left|B_{\sqrt{t}}\right|^{-1/p}\|f\|_p\end{aligned}$$

(again one uses the fact that $t-r^2 \simeq t$). Coming back to (3.11), we obtain

$$\begin{aligned}&\left|e^{-tL}f(x)-\left(\fint_{B_r} e^{-(t-r^2)L}f\,d\mu\right)\right|\\ &\lesssim \left(\sum_{i\in I} e^{-c\frac{d^2(B_r,\tilde{B}^i_r)}{r^2}}\left(1+\frac{d(B_r,\tilde{B}^i_r)}{r}\right)^{D+\frac{\nu}{p}}\right)\omega\left(\frac{r}{\sqrt{t}}\right)\left|B_{\sqrt{t}}\right|^{-1/p}\|f\|_p.\end{aligned}$$

Since by doubling $\sum_{i\in I} e^{-c\frac{d^2(B_r,\tilde{B}^i_r)}{r^2}}\left(1+\frac{d(B_r,\tilde{B}^i_r)}{r}\right)^{D+\frac{\nu}{p}}$ is uniformly bounded, this yields (3.8). □

Following some ideas in [20, Theorem 3.1], we can now identify $(H^\eta_{p,p})$ as the property needed to pass from (UE) to (LE).

Theorem 3.5. *Let $(M,d,\mu,\mathcal{E})$ be a metric measure Dirichlet space satisfying (VD) and the upper Gaussian estimate (UE). If there exist $p\in[1,+\infty]$ and $\eta\in(0,1]$ such that $(H^\eta_{p,p})$ is satisfied, then the lower Gaussian bound (LE) holds. Conversely (LE) implies $(H^\eta_{p,p})$ for all $p\in[1,+\infty)$ and some $\eta\in(0,1]$.*

Remark 3.6. *Let us emphasise two by-products of Theorem* 3.5*:*

- *(LE) is equivalent to the existence of some $p\in[1,+\infty)$ and some $\eta\in(0,1]$ such that $(H^\eta_{p,p})$ holds;*
- *The property "there exists $\eta>0$ such that $(H^\eta_{p,p})$ holds" is independent of $p\in[1,+\infty)$. We refer the reader to Appendix A, where one proves the independence of this property on $p\in[1,+\infty]$ (including the infinite exponent) by a direct argument. In fact, we shall prove that the property $(H^\eta_{p,p})$ itself is p-independent of $p\in[1,+\infty]$, up to an arbitrarily small loss on η.*

Proof of Theorem 3.5. First assume $(H^\eta_{p,p})$ for some $p\in[1,+\infty]$ and some $\eta>0$. By Proposition 3.1, we know that this estimate self-improves into $(H^\eta_{p,\infty})$. Fix a point $z \in M$ and consider the function $f = p_t(\cdot, z)$. Then $(H^\eta_{p,\infty})$ yields

$$|p_{2t}(x,z) - p_{2t}(y,z)| \lesssim \left(\frac{d(x,y)}{\sqrt{t}}\right)^\eta \left|B_{\sqrt{t}}\right|^{-1/p} \|p_t(\cdot,z)\|_p,$$

uniformly for a.e. x, y with $d(x,y) \le \sqrt{t}$ and where $B_{\sqrt{t}}$ is any ball of radius $\sqrt{t}$ containing x, y (in particular, p_t is continuous and $p_t(x,x)$ has a meaning). It follows from (VD) and (UE) that

$$\|p_t(\cdot,z)\|_p \lesssim \left[V(z,\sqrt{t})\right]^{\frac{1}{p}-1} \tag{3.12}$$

and that

$$V^{-1}(z,\sqrt{t}) \lesssim p_{2t}(z,z).$$

For these two classical facts, see for instance [20, Theorem 3.1]. Hence

$$\begin{aligned} |p_{2t}(x,z) - p_{2t}(y,z)| &\lesssim \left(\frac{d(x,y)}{\sqrt{t}}\right)^\eta \left(\frac{V(z,\sqrt{t})}{|B_{\sqrt{t}}|}\right)^{1/p} V(z,\sqrt{t})^{-1} \\ &\lesssim \left(\frac{d(x,y)}{\sqrt{t}}\right)^\eta \left(\frac{V(z,\sqrt{t})}{|B_{\sqrt{t}}|}\right)^{1/p} p_{2t}(z,z). \end{aligned}$$

Note that this estimate is nothing but a slightly weaker form of the classical Hölder estimate (H^η) from the introduction.

In particular, for $x = z$ and every $y \in B(x, \sqrt{t})$ we deduce that

$$|p_{2t}(x,x) - p_{2t}(y,x)| \lesssim \left(\frac{d(x,y)}{\sqrt{t}}\right)^\eta p_{2t}(x,x). \tag{3.13}$$

It is well-known that (LE) follows (see for instance [20, Theorem 3.1]).

Assume now (LE). Since we have assumed (UE), it follows, through the equivalence of $(UE) + (LE)$ with the parabolic Harnack inequality (see [64, Proposition 3.2] or [45, Theorems 2.31-2.32]), that there exist $\theta \in (0,1)$ such that, for a.e. $x, y \in B_r$, $0 \le r < \sqrt{t}$, and a.e. $z \in M$

$$|p_t(x,z) - p_t(y,z)| \lesssim \frac{1}{\sqrt{V(z,\sqrt{t})V(x,\sqrt{t})}} \left(\frac{r}{\sqrt{t}}\right)^\theta \tag{3.14}$$

(this is yet another version of (H^η)). On the other hand, (UE) and doubling imply that

$$|p_t(x,z) - p_t(y,z)| \lesssim p_t(x,z) + p_t(y,z) \lesssim \frac{1}{V(x,\sqrt{t})} \exp\left(-c\frac{d^2(x,z)}{\sqrt{t}}\right). \tag{3.15}$$

If $d(x,z) \le \sqrt{t}$, $V(z,\sqrt{t}) \simeq V(x,\sqrt{t}) \simeq \left|B_{\sqrt{t}}\right|$ and (3.14) yields

$$|p_t(x,z) - p_t(y,z)| \lesssim \frac{1}{|B_{\sqrt{t}}|} \left(\frac{r}{\sqrt{t}}\right)^\theta.$$

If $d(x,z) \geq \sqrt{t}$, one multiplies the square roots of (3.14) and (3.15) to obtain

$$\begin{aligned} |p_t(x,z)-p_t(y,z)| &\lesssim \frac{1}{V^{1/4}(z,\sqrt{t})V^{3/4}(x,\sqrt{t})}\left(\frac{r}{\sqrt{t}}\right)^{\theta/2}\exp\left(-c\frac{d^2(x,z)}{\sqrt{t}}\right) \\ &= \frac{1}{V(x,\sqrt{t})}\left(\frac{r}{\sqrt{t}}\right)^{\theta/2}\left(\frac{V(x,\sqrt{t})}{V(z,\sqrt{t})}\right)^{1/4}\exp\left(-c\frac{d^2(x,z)}{\sqrt{t}}\right) \\ &\lesssim \frac{1}{|B_{\sqrt{t}}|}\left(\frac{r}{\sqrt{t}}\right)^{\theta/2}, \end{aligned}$$

where the last inequality uses again doubling.

Now we proceed as in [20, Theorem 3.1]. We have just shown that

$$\|p_t(x,.)-p_t(y,.)\|_\infty \lesssim \frac{1}{|B_{\sqrt{t}}|}\left(\frac{r}{\sqrt{t}}\right)^{\theta/2}.$$

The heat semigroup being submarkovian,

$$\|p_t(x,.)-p_t(y,.)\|_1 \leq 2.$$

It follows by Hölder inequality that for $1 \leq p < +\infty$

$$\|p_t(x,.)-p_t(y,.)\|_{p'} \lesssim \left|B_{\sqrt{t}}\right|^{-1/p}\left(\frac{r}{\sqrt{t}}\right)^{\theta/2p}, \tag{3.16}$$

for a.e. $x,y \in B_r$, $0 \leq r < \sqrt{t}$. Now

$$\begin{aligned} |e^{-tL}f(x)-e^{-tL}f(y)| &\leq \int_M |p_t(x,z)-p_t(y,z)||f(z)|\,d\mu(z) \\ &\leq \|p_t(x,.)-p_t(y,.)\|_{p'}\|f\|_p, \end{aligned}$$

which together with (3.16) yields $(H^\eta_{p,\infty})$ with $\eta = \theta/2p$, hence $(H^\eta_{p,p})$ by Proposition 3.1.

□

Remark 3.7. *In the case of a bounded space, that is diam$(M) < \infty$ (which under (VD) is equivalent to a finite measure $|M| < \infty$), to get (LE) it is sufficient to have $(H^\eta_{p,p})$ for some $p \in [1,+\infty]$ and $\eta \in (0,1]$ where we consider only scales with $\sqrt{t} \leq \delta\,$diam(M), for some $\delta \in (0,1)$.*

Indeed, let us assume this restricted $(H^\eta_{p,p})$ property. Following the above proof, we deduce the Hölder regularity (3.13) hence (LE) for $\sqrt{t} \leq \delta\,$diam(M). Then define $t_M = \delta^2\,$diam$^2(M)/2$ and consider $t \geq 2t_M$. First, the self-improvement given by Proposition 3.1 still holds in the same range where $(H^\eta_{p,p})$ is assumed, so in particular at the scale $\frac{\delta}{2}\,$diam(M). This yields $(H^\eta_{p,\infty})$, that is, for every function $f \in L^p(M,\mu)$ and $0 < r < \frac{\delta}{2}\,$diam$(M)$,

$$\operatorname*{ess\,sup}_{x,y\in B_r}\left|e^{-t_ML}f(x)-e^{-t_ML}f(y)\right| \lesssim \left(\frac{r}{\sqrt{t_M}}\right)^\eta |M|^{-1/p}\|f\|_p,$$

where we have used the fact that by doubling $|B(x,\sqrt{t_M})| \simeq |M|$ for every $x \in M$. Since $t \geq 2t_M$, we may apply this inequality to $e^{-(t-t_M)L}f$ instead of f. By the

L^p-boundedness of the semigroup with $|M| \simeq |B(x,\sqrt{t}| \simeq |B_{\sqrt{t}}|$, we get

$$\operatorname*{ess\,sup}_{x,y\in B_r} \left|e^{-tL}f(x) - e^{-tL}f(y)\right| \lesssim \left(\frac{r}{\sqrt{t_M}}\right)^{\eta} \left|B_{\sqrt{t}}\right|^{-1/p} \|f\|_p.$$

Following the above proof of (3.13), one deduces that for every $x, y \in M$ with $d(x,y) \leq \delta'$ diam(M) for some $\delta' < \delta$:

$$|p_t(x,x) - p_t(y,x)| \lesssim \left(\frac{\delta'}{\delta}\right)^{\eta} p_t(x,x) \leq \frac{1}{2} p_t(x,x),$$

where we have chosen δ' small enough such that the last inequality comes with an exact constant smaller than $1/2$.

This gives for $d(x,y) \leq \delta'$ diam(M)

$$p_t(y,x) \gtrsim |B(x,\sqrt{t})|^{-1} \simeq |M|^{-1}.$$

Then the standard iteration and the doubling property allow us to extend this inequality for every $x, y \in M$, which gives (LE):

$$p_t(y,x) \gtrsim |B(x,\sqrt{t})|^{-1} e^{-c\frac{d(x,y)^2}{t}}.$$

Remark 3.8. *Proposition 3.1 and Theorem 3.5 still hold in the context of sub-Gaussian estimates. Instead of (UE), let us assume that the heat kernel satisfies for some $m > 2$*
(UE_m)

$$p_t(x,y) \lesssim \frac{1}{V(x,t^{1/m})} \exp\left(-\left(\frac{d(x,y)^m}{Ct}\right)^{1/(m-1)}\right), \ \forall\, t > 0, \ a.e.\ x, y \in M.$$

Then one can easily check that the above remains true by replacing everywhere the scaling factor $\sqrt{t}$ by $t^{1/m}$. One could also consider the more general heat kernel estimates from [47, Section 5], where the equivalence with matching Harnack inequalities is proved, see also [8].

An alternative proof of the second statement in Theorem 3.5 can be given using [12, Theorem 6] instead of Proposition 3.1. We leave the details to the reader. Conversely, a natural follow-up of the end of the proof of Theorem 3.5 is to get the results from [12], that is the extension of [20, Theorem 4.1] to the doubling setting. There is nothing essentially new here, but we shall give a proof of [12, Proposition 10] for the sake of completeness.

We first need to introduce the notion of reverse doubling. It is known (see [44, Proposition 5.2]), that, if M is unbounded, connected, and satisfies (VD_ν), one has a so-called reverse doubling volume property, namely there exist $0 < \nu' \leq \nu$ and $c > 0$ such that, for all $r \geq s > 0$ and $x \in M$

$$c\left(\frac{r}{s}\right)^{\nu'} \leq \frac{V(x,r)}{V(x,s)}.$$

Let us say that (M,d,μ) satisfies $(VD_{\nu,\nu'})$ if, for all $r \geq s > 0$ and $x \in M$,

$$c\left(\frac{r}{s}\right)^{\nu'} \leq \frac{V(x,r)}{V(x,s)} \leq C\left(\frac{r}{s}\right)^{\nu}.$$

For the sake of simplicity we shall set ourselves in the Gaussian case ($m = 2$ in the notation of Remark 3.8), but the general case is similar.

Theorem 3.9. *Let $(M, d, \mu, \mathcal{E})$ be a metric measure Dirichlet space satisfying $(VD_{\nu,\nu'})$ and the upper Gaussian estimate (UE). Then (LE) holds if and only if, for some (all) $p \in (1, +\infty)$, some $\alpha > \frac{\nu}{p}$ and $\alpha' > \frac{\nu'}{p}$,*

$$|f(x) - f(y)| \lesssim \frac{1}{V^{1/p}(x, d(x, y))} \left(d^{\alpha}(x, y)\|L^{\alpha/2} f\|_p + d^{\alpha'}(x, y)\|L^{\alpha'/2} f\|_p \right),$$

$\forall f \in \mathcal{D}$, $x, y \in M$.

Proof. Assume (LE). Let $1 < p < +\infty$, $\alpha, \alpha' > 0$ to be chosen later, and $k \in \mathbb{N}$ such that $k > \max\left(\frac{\alpha}{2}, \frac{\alpha'}{2}\right)$. Let $f \in \mathcal{D}$. Thanks to (UE) and to the fact that by reverse doubling $V(x, r) \to +\infty$ as $r \to +\infty$, $e^{-tL} f \to 0$ in $L^2(M, \mu)$, as $t \to +\infty$ (see [17, Section 3.1.2] for details). Since $e^{-tL} f$ is bounded in $L^1(M, \mu)$, $e^{-tL} f \to 0$ in $L^p(M, \mu)$ by duality and interpolation. Thus one can write

$$f = c(k) \int_0^{+\infty} t^{k-1} L^k e^{-tL} f \, dt,$$

hence

$$\begin{aligned} |f(x) - f(y)| &\leq c(k) \int_0^{+\infty} t^{k-1} |L^k e^{-tL} f(x) - L^k e^{-tL} f(y)| \, dt \\ &= c(k) \int_0^{+\infty} t^{k-1} |e^{-(t/2)L} L^k e^{-(t/2)L} f(x) - e^{-(t/2)L} L^k e^{-(t/2)L} f(y)| \, dt. \end{aligned}$$

Now for $r = d(x, y)$ and $0 < \sqrt{t} \leq r$, we get from (3.12) and (VD_ν)

$$\begin{aligned} &|e^{-(t/2)L} L^k e^{-(t/2)L} f(x) - e^{-(t/2)L} L^k e^{-(t/2)L} f(y)| \\ &\leq \|p_t(x, .) - p_t(y, .)\|_{p'} \|L^k e^{-(t/2)L} f\|_p \\ &\leq (\|p_t(x, .)\|_{p'} + \|p_t(y, .)\|_{p'}) \|L^k e^{-(t/2)L} f\|_p \\ &\lesssim V(x, \sqrt{t})^{-1/p} t^{-(k - \frac{\alpha}{2})} \|L^{\alpha/2} f\|_p \\ &\lesssim V(x, r)^{-1/p} \left(\frac{r}{\sqrt{t}}\right)^{\nu/p} t^{-(k - \frac{\alpha}{2})} \|L^{\alpha/2} f\|_p, \end{aligned}$$

where the last inequality uses the analyticity of $(e^{-tL})_{t>0}$ on $L^p(M, \mu)$. For $0 \leq r < \sqrt{t}$, we can write as in the end of the proof of Theorem 3.5,

$$\begin{aligned} &|e^{-(t/2)L} L^k e^{-(t/2)L} f(x) - e^{-(t/2)L} L^k e^{-(t/2)L} f(y)| \\ &\leq \|p_t(x, .) - p_t(y, .)\|_{p'} \|L^k e^{-(t/2)L} f\|_p \lesssim \left|B_{\sqrt{t}}\right|^{-1/p} \left(\frac{r}{\sqrt{t}}\right)^{\theta/2p} t^{-(k - \frac{\alpha'}{2})} \|L^{\alpha'/2} f\|_p. \end{aligned}$$

Now reverse doubling yields

$$\begin{aligned} &|e^{-(t/2)L} L^k e^{-(t/2)L} f(x) - e^{-(t/2)L} L^k e^{-(t/2)L} f(y)| \\ \lesssim &V(x, r)^{-1/p} \left(\frac{r}{\sqrt{t}}\right)^{\frac{\theta}{2p} + \frac{\nu'}{p}} t^{-(k - \frac{\alpha'}{2})} \|L^{\alpha'/2} f\|_p. \end{aligned}$$

Finally

$$|f(x)-f(y)| \lesssim V(x,r)^{-1/p}\|L^{\alpha/2}f\|_p\, r^{\nu/p}\int_0^{r^2} t^{k-1}t^{-\frac{\nu}{2p}-k+\frac{\alpha}{2}}\,dt$$
$$+\,V(x,r)^{-1/p}\|L^{\alpha'/2}f\|_p\, r^{\frac{\theta}{2p}+\frac{\nu'}{p}}\int_{r^2}^{+\infty} t^{k-1}t^{-\frac{\theta}{4p}-\frac{\nu'}{2p}-k+\frac{\alpha'}{2}}\,dt.$$

The above integrals converge if $\alpha>\frac{\nu}{p}$ and $\alpha'<\frac{\theta}{2p}+\frac{\nu'}{p}$, in which case one obtains

$$|f(x)-f(y)| \lesssim V(x,r)^{-1/p}\left(r^{\alpha}\|L^{\alpha/2}f\|_p + r^{\alpha'}\|L^{\alpha'/2}f\|_p\right).$$

One can choose any $\alpha>\frac{\nu}{p}$ and some $\alpha'>\frac{\nu'}{p}$. The converse is easy, see [12, Theorem 6]. □

Remark 3.10. *One can take $\alpha=\alpha'$ if $\nu=\nu'$, recovering in particular the polynomial volume growth case from* [20, Theorem 4.1]*.*

4. The case $\nu<p<+\infty$: from Poincaré and gradient estimates to heat kernel lower bounds

In [20, Thm. 5.2], it is proved that if the volume growth is polynomial of exponent $\nu\geq 2$, then (R_p) and (P_p) for $\nu<p<+\infty$ imply (LE). Using the equivalence between (G_p) and (1.7), it is easy to see that the same proof works with (G_p) instead of (R_p). Our next theorem extends this result to the doubling case, and in addition its proof is more direct.

Theorem 4.1. *Let $(M,d,\mu,\mathcal{E})$ be a metric measure Dirichlet space with a "carré du champ" satisfying (VD_ν). Assume (G_p) and (P_p) for some $p\in(\nu,+\infty)$. Then (LE) holds.*

Proof. Replacing f with $e^{-tL}f$ in (P_p), we have, for every $t>0$ and every ball B_r of radius $r>0$,

$$p\text{-}\operatorname{Osc}_{B_r}(e^{-tL}f) \lesssim r\left(\fint_{B_r}|\nabla e^{-tL}f|^p d\mu\right)^{1/p}.$$

If $B_{\sqrt{t}}$ is concentric with B_r and $\sqrt{t}\geq r$,

$$\left(\fint_{B_r}|\nabla e^{-tL}f|^p d\mu\right)^{1/p} \lesssim \left(\frac{|B_{\sqrt{t}}|}{|B_r|}\right)^{1/p}\left(\fint_{B_{\sqrt{t}}}|\nabla e^{-tL}f|^p d\mu\right)^{1/p},$$

hence by (VD_ν)

$$\left(\fint_{B_r}|\nabla e^{-tL}f|^p d\mu\right)^{1/p} \lesssim \left(\frac{\sqrt{t}}{r}\right)^{\frac{\nu}{p}}\left|B_{\sqrt{t}}\right|^{-1/p}\||\nabla e^{-tL}f|\|_p$$
$$\lesssim \left(\frac{\sqrt{t}}{r}\right)^{\frac{\nu}{p}}\left|B_{\sqrt{t}}\right|^{-1/p}\frac{\|f\|_p}{\sqrt{t}},$$

where the last inequality follows from (G_p). Gathering the two above estimates yields

$$p\text{-}\operatorname{Osc}_{B_r}(e^{-tL}f) \lesssim \left(\frac{r}{\sqrt{t}}\right)^{1-\frac{\nu}{p}}\left|B_{\sqrt{t}}\right|^{-1/p}\|f\|_p,$$

that is $(H^\eta_{p,p})$ with $\eta = 1 - \frac{\nu}{p} \in (0,1)$ since $p > \nu$. By Proposition 2.1, (UE) also holds. We conclude by applying Theorem 3.5. $\square$

Remark 4.2. *In the above statement, (P_p) is necessary, since (LE) implies (P_2), but (G_p) is not, as the example of conical manifolds shows (see* [22]*).*

Remark 4.3. *For $p = +\infty$, the above proof with the obvious modifications shows that (G_∞) together with (UE) implies (LE). This also follows from* [25, Corollary 2.2] *and the fact that (G_∞) and (UE) imply $(\widetilde{G_\infty})$.*

Let us give an alternative proof of Theorem 4.1, which is more direct, but does not shed the same light on the range $2 \le p \le \nu$ (see Section 5 below) as the above one.

We shall start with a lemma which is close to [46, Theorem 5.1] and to several statements in [19] (for the polynomial volume growth case), but we find it useful to formulate and prove it in the following simple and natural way, which is in fact inspired by [46, Theorem 3.2].

Lemma 4.4. *Let $(M, d, \mu, \mathcal{E})$ be a metric measure Dirichlet space with a "carré du champ" satisfying (VD_ν). Then (P_p) for some $p > \nu$ implies the following Morrey inequality: for every function $f \in \mathcal{D}$ and almost every $x, y \in M$,*

$$|f(x) - f(y)| \lesssim \frac{d(x,y)}{V^{1/p}(x, d(x,y))} \||\nabla f|\|_p. \tag{4.1}$$

Proof. Let x, y be Lebesgue points for f. Let $B_i(x) = B(x, 2^{-i}d(x,y))$, for $i \in \mathbb{N}_0$. As in Lemma 3.3, one has

$$\left| f(x) - \fint_{B_0(x)} f d\mu \right| \le \sum_{i\ge 0} p\text{-}\operatorname{Osc}_{B_i(x)}(f),$$

Then using (P_p) and (VD_ν) which yields $|B_0(x)| \lesssim 2^{i\nu}|B_i(x)|$, we can write

$$\begin{aligned}
\left| f(x) - \fint_{B_0(x)} f d\mu \right| &\lesssim \sum_{i\ge 0} 2^{-i} d(x,y) \left(\fint_{B_i(x)} |\nabla f|^p \, d\mu \right)^{1/p} \\
&\lesssim \sum_{i\ge 0} 2^{-i} d(x,y) \left(\frac{2^{i\nu}}{|B_0(x)|} \right)^{1/p} \left(\int_{B_i(x)} |\nabla f|^p \, d\mu \right)^{1/p} \\
&\lesssim \left(\sum_{i\ge 0} 2^{-i(1-\frac{\nu}{p})} \right) d(x,y) |B_0(x)|^{-1/p} \||\nabla f|\|_p \\
&\lesssim d(x,y) |B_0(x)|^{-1/p} \||\nabla f|\|_p,
\end{aligned}$$

where we used $p > \nu$. Similarly we have

$$\begin{aligned}
\left| f(y) - \fint_{B_0(y)} f d\mu \right| &\lesssim d(x,y) |B_0(y)|^{-1/p} \||\nabla f|\|_p \\
&\lesssim d(x,y) |B_0(x)|^{-1/p} \||\nabla f|\|_p,
\end{aligned}$$

where $|B_0(x)| \simeq |B_0(y)|$ follows from doubling. Finally, as in Lemma 3.3,

$$\left| \fint_{B_0(x)} f d\mu - \fint_{B_0(y)} f d\mu \right| \le \left(\fint_{2B_0(x)} \left| f - \fint_{2B_0(x)} f d\mu \right|^p d\mu \right)^{1/p},$$

and by (P_p)

$$\left|\fint_{B_0(x)} f d\mu - \fint_{B_0(y)} f d\mu\right| \lesssim d(x,y)|B_0(x)|^{-1/p} \left\||\nabla f|\right\|_p .$$

The claim follows. □

We can now derive Theorem 4.1 easily. Replacing f with $e^{-tL}f$ in the conclusion of Lemma 4.4 and applying (G_p) yields

$$|e^{-tL}f(x)-e^{-tL}f(y)| \lesssim \frac{d(x,y)}{V^{1/p}(x,d(x,y))}\||\nabla e^{-tL}f|\|_p \lesssim \frac{d(x,y)}{\sqrt{t}} \,[V(x,d(x,y))]^{-1/p} \|f\|_p,$$

hence by (VD_ν)

$$|e^{-tL}f(x) - e^{-tL}f(y)| \lesssim \left(\frac{d(x,y)}{\sqrt{t}}\right)^{1-\frac{\nu}{p}} |B_{\sqrt{t}}|^{-1/p}\|f\|_p, \tag{4.2}$$

for every $f \in \mathcal{D}$ and every ball $B_{\sqrt{t}}$ with radius $\sqrt{t} \geq d(x,y)$ and containing x. Let now B_r be concentric to $B_{\sqrt{t}}$ with radius $\sqrt{t}$ such that $0 < r \leq \sqrt{t}$. Since $p > \nu$, it follows from (4.2) that

$$\operatorname*{ess\,sup}_{x,y\in B_r} |e^{-tL}f(x) - e^{-tL}f(y)| \lesssim \left(\frac{r}{\sqrt{t}}\right)^{1-\frac{\nu}{p}} |B_{\sqrt{t}}|^{-1/p}\|f\|_p.$$

This is nothing but $(H^\eta_{p,\infty})$ with $\eta = 1 - \frac{\nu}{p} \in (0,1)$, and we conclude by using (UE) as in the beginning of the proof of Theorem 3.5.

An obvious by-product of Theorem 4.1 is the following monotonicity property for $(G_p) + (P_p)$.

Corollary 4.5. *Let $(M,d,\mu,\mathcal{E})$ be a metric measure Dirichlet space with a "carré du champ" satisfying (VD_ν). Then $(G_p) + (P_p)$ for some $p > \nu$ implies (P_2), hence $(G_q) + (P_q)$ for all $q \in [2,p)$. Moreover, if (G_p) holds for some $p \in (\nu, +\infty)$, then*

$$(P_q) \Longleftrightarrow (P_2)$$

for every $q \in [2,p]$.

Proof. We have just proven that $(G_p) + (P_p)$ for $p > \nu$ implies (UE) and (LE). By [62], [70], (P_2) follows, hence (P_q) for all $q > 2$. On the other hand, (G_p) implies (G_q) for all $q \in [2,p)$ by interpolation. The last statement follows in the same way. □

5. Poincaré inequalities and heat kernel bounds: the L^2 theory

The so-called De Giorgi property or Dirichlet property on the growth of the Dirichlet integral for harmonic functions was introduced by De Giorgi in [27], for L a second order divergence form differential operator with real coefficients on $\mathbb{R}^n$: there exists $\varepsilon \in (0,1)$ such that for all $r \leq R$, every pair of concentric balls B_r, B_R with radii r, R and all functions $u \in W^{1,2}(\mathbb{R}^n)$ harmonic in $2B_R$, i.e. $Lu = 0$ in $2B_R$, one has

$$\left(\fint_{B_r} |\nabla u|^2 d\mu\right)^{1/2} \lesssim \left(\frac{R}{r}\right)^{\varepsilon} \left(\fint_{B_R} |\nabla u|^2 d\mu\right)^{1/2}. \tag{5.1}$$

The De Giorgi property was subsequently used in many works and in various situations to prove Hölder regularity for solutions of inhomogeneous elliptic equations and systems (see for instance [40]).

The idea to look at the heat equation as a Laplace equation where the RHS is a time derivative, and to deduce parabolic regularity results from elliptic ones by using a non-homogeneous equivalent version of De Giorgi property was introduced in [1] for L a second order operator in divergence form on $\mathbb{R}^n$. In [2], the same ideas are applied in a discrete geometric setting, and the role of Poincaré inequalities clearly appears to ensure the elliptic regularity and the equivalence between the homogeneous and non-homogeneous versions of De Giorgi. This is the approach we will follow here, while taking full advantage of Theorem 3.5. We shall consider the following non-homogeneous version of De Giorgi property. With the help of Lemma 5.7 below, one shows that this formulation is a priori weaker than the one in [2, Proposition 4.4]. We shall see in the proof of Proposition 5.2 in Appendix B that under (UE) it is equivalent to (5.1).

Definition 5.1 (De Giorgi property)**.** *Let $(M, d, \mu, \mathcal{E})$ be a metric measure Dirichlet space with a "carré du champ" and L the associated operator. We say that $(DG_{2,\varepsilon})$ holds if the following is satisfied: for all $r \le R$, every pair of concentric balls B_r, B_R with respective radii r and R, and for every function $f \in \mathcal{D}$, one has*

$$(DG_{2,\varepsilon}) \qquad \left(\fint_{B_r} |\nabla f|^2 d\mu\right)^{1/2} \lesssim \left(\frac{R}{r}\right)^{\varepsilon} \left[\left(\fint_{B_R} |\nabla f|^2 d\mu\right)^{1/2} + R\|Lf\|_{L^\infty(B_R)}\right].$$

We sometimes omit the parameter ε, and write (DG_2) if $(DG_{2,\varepsilon})$ is satisfied for some $\varepsilon \in (0,1)$.

Let us now state the counterpart of a result of [2] in the discrete setting. For the convenience of the reader, we give a proof in Appendix B.

Proposition 5.2. *Let $(M, d, \mu, \mathcal{E})$ be a doubling metric measure Dirichlet space with a "carré du champ". Then (P_2) implies (DG_2).*

We are now in a position to give a simple proof of the main statement of [42], [62], and [70]. For simplicity let us denote in what follows, for B a ball and $f \in L^2_{loc}(M, \mu)$:

$$\mathrm{Osc}_B(f) := 2\text{-}\,\mathrm{Osc}_B(f) = \left(\fint_B |f - \fint_B f\,d\mu|^2\,d\mu\right)^{1/2}.$$

Theorem 5.3. *Let $(M, d, \mu, \mathcal{E})$ be a doubling metric measure Dirichlet space with a "carré du champ". Then (P_2) implies (LE).*

Proof. Applying (P_2) to $e^{-tL}f$ for $t > 0$ and $f \in L^2(M, \mu)$ on a ball B_r for $r > 0$ yields

$$(5.2) \qquad \mathrm{Osc}_{B_r}(e^{-tL}f) \lesssim r\left(\fint_{B_r} |\nabla e^{-tL}f|^2 d\mu\right)^{1/2}.$$

According to Proposition 5.2, $(DG_{2,\varepsilon})$ holds for some $\varepsilon > 0$, hence

$$\begin{aligned}(5.3)\qquad &\left(\fint_{B_r} |\nabla e^{-tL} f|^2 d\mu\right)^{1/2} \\ &\lesssim \left(\frac{\sqrt{t}}{r}\right)^{\varepsilon} \left[\left(\fint_{B_{\sqrt{t}}} |\nabla e^{-tL} f|^2 d\mu\right)^{1/2} + \sqrt{t}\operatorname*{ess\,sup}_{x\in B_{\sqrt{t}}} |Le^{-tL} f(x)|\right] \\ &\leq \left(\frac{\sqrt{t}}{r}\right)^{\varepsilon} \left(|B_{\sqrt{t}}|^{-1/2} \| |\nabla e^{-tL} f| \|_2 + \sqrt{t}\operatorname*{ess\,sup}_{x\in B_{\sqrt{t}}} |Le^{-tL} f(x)|\right)\end{aligned}$$

for some $\varepsilon \in (0,1)$, $0 < r \leq \sqrt{t}$ and $B_{\sqrt{t}}$ with radius $\sqrt{t}$ concentric to B_r. By (G_2),

$$(5.4)\qquad \| |\nabla e^{-tL} f| \|_2 \lesssim \frac{\|f\|_2}{\sqrt{t}}.$$

Now recall that under our assumptions, (UE) holds thanks to Proposition 2.1. By [43, Corollary 3.3] (one can also use the complex time bounds of [15, Proposition 4.1] and a Cauchy formula) the kernel of the operator tLe^{-tL} also satisfies pointwise Gaussian estimates. It follow that

$$(5.5)\qquad \operatorname*{ess\,sup}_{x\in 2B_{\sqrt{t}}} |tLe^{-tL} f(x)| \lesssim |B_{\sqrt{t}}|^{-1/2} \|f\|_2.$$

Putting together (5.2), (5.3), (5.4) and (5.5) yields

$$\mathrm{Osc}_{B_r}(e^{-tL} f) \lesssim \left(\frac{r}{\sqrt{t}}\right)^{1-\varepsilon} |B_{\sqrt{t}}|^{-1/2} \|f\|_2,$$

that is, $(H^{\eta}_{2,2})$ with $\eta = 1-\varepsilon > 0$. This implies (LE) according to Theorem 3.5. □

The original proofs of Theorem 5.3 went through the parabolic Harnack inequality; some [62–65,70] used a parabolic Moser iteration, another one [42] tricky geometric arguments. In [47, Section 4.2], a shorter proof was given, which went in three steps (with a fourth one, borrowed from [35], to deduce parabolic Harnack from (LE)). The first one is to derive an elliptic regularity estimate from (VD) and (P_2). We do not change this step, which relies on the elliptic Moser iteration; we give a proof for the sake of completeness in Proposition B.4 below. The second step is to obtain (UE). Our approach in Proposition 2.1 is particularly simple since $p = 2$. The third step is a lower bound on the Dirichlet heat kernel inside a ball whose radius is the square root of the time under consideration. This is not trivial (see [47, pp. 1457-1462]) and here lies our main simplification. We first push step one a little further by deducing (DG_2) from the elliptic regularity. We could then deduce the parabolic regularity as in [2, Section 4]). Instead, we use the self-improvement of Hölder regularity estimates on the semigroup from Proposition 3.1 and Theorem 3.5.

Introduce the scale-invariant local Sobolev inequality

$$(LS_q)\qquad \|f\|_q^2 \lesssim \frac{1}{V^{1-\frac{2}{q}}(x,r)} \left(\|f\|_2^2 + r^2\mathcal{E}(f)\right),$$

for every ball $B = B(x,r)$, every $f \in \mathcal{F}$ supported in $B(x,r)$, and for some $q > 2$. This inequality was introduced in [62] and was shown, under (VD), to be equivalent to (DUE) in the Riemannian setting. The equivalence was stated in our more general setting in [69]. See also [13] for many reformulations of (LS_q), an alternative proof of the equivalence with (DUE), and more references.

The main aim of [47] is to prove that the elliptic Harnack inequality, or an equivalent elliptic regularity estimate, together with (LS_q), or equivalently (UE), implies the parabolic Harnack inequality. It is enough in this respect to prove (LE), since as we already said the parabolic Harnack inequality follows from (UE) + (LE). This phenomenon falls in the circle of the ideas we are developing in the present work, and, using a transition trick from estimates for harmonic functions to estimates for all functions together with Theorem 3.5, we will now offer a simple proof of [47, Theorem 3.1]. Let us say that $u \in \mathcal{F}$ is harmonic on a ball B if $Lu = 0$ in the weak sense on B. Note that the following statement involves $\operatorname{diam}(M)$ as we want to treat by the same token the cases M bounded and unbounded. In a first reading one can certainly assume $\operatorname{diam}(M) = +\infty$.

Theorem 5.4. *Let $(M, d, \mu, \mathcal{E})$ be a doubling metric measure Dirichlet space with a "carré du champ" satisfying (LS_q) for some $q > 2$. Assume that the following elliptic regularity estimate holds: there exists $\alpha > 0$ and $\delta \in (0,1)$ such that for every $x_0 \in M$, $R > 0$ with $R < \delta \operatorname{diam}(M)$, $u \in \mathcal{F}$ harmonic in $B(x_0,R)$ and $x, y \in B(x_0, R/2)$, one has*

$$(ER) \qquad |u(x) - u(y)| \lesssim \left(\frac{d(x,y)}{R}\right)^{\alpha} \operatorname{Osc}_{B(x_0,R)}(u).$$

Then (LE) follows.

Remark 5.5. *It is known that (P_2) implies (LS_q) for some $q > 2$, see for instance* [62, Theorem 2.1], [26, Section 5]. *We shall also see in Proposition* B.4 *below that (P_2) implies (ER). Thus Theorem* 5.4 *gives back Theorem* 5.3.

Before we start the proof of Theorem 5.4, recall that (LS_q) for some $q > 2$ implies the following relative Faber-Krahn inequality

$$(FK) \qquad \left(\int_\Omega |f|^2 \, d\mu\right)^{1/2} \lesssim r \left(\frac{|\Omega|}{V(x,r)}\right)^{\beta} \left(\int_\Omega |\nabla f|^2 \, d\mu\right)^{1/2}$$

for some $\beta > 0$, all balls $B(x,r)$, $x \in M$, $r \in (0, \delta \operatorname{diam}(M))$ with some $\delta < 1$, and all $f \in \mathcal{F}$ supported in $\Omega \subset B(x,r)$. See for instance [47, Theorem 2.5], as well as [13, Section 3.3].

In particular, one has

$$(5.6) \qquad \left(\int_{B(x,r)} |f|^2 \, d\mu\right)^{1/2} \lesssim r \left(\int_{B(x,r)} |\nabla f|^2 \, d\mu\right)^{1/2},$$

for all balls $B(x,r)$, $x \in M$, $r \in (0, \delta \operatorname{diam}(M))$ with some $\delta < 1$, and all $f \in \mathcal{F}$ supported in $B(x,r)$.

We will need the following result inspired by [2, Lemma 4.2]. Note that the role classically played by ellipticity in such Lax-Milgram type arguments is played here by (5.6).

Lemma 5.6. *Let $(M, d, \mu, \mathcal{E})$ be a doubling metric measure Dirichlet space with a "carré du champ" satisfying* (5.6). *Let $f \in \mathcal{D}$ and consider an open ball $B \subset M$. Then, there exists $u \in \mathcal{F}$ such that $f - u \in \mathcal{F}$ is supported in the ball B and u is harmonic in B: for every $\phi \in \mathcal{F}$ supported on B then*

$$\int_M d\Gamma(u, \phi) = 0,$$

where we recall that $d\Gamma$ is the energy measure associated with the Dirichlet form $\mathcal{E}$.

Proof. Consider the space of functions

$$\mathcal{H} := \left\{\phi \in \mathcal{F} \subset L^2,\ \operatorname{supp}(\phi) \subset B\right\}.$$

Then, due to (5.6) the application

$$\phi \mapsto \|\phi\|_{\mathcal{H}} := \||\nabla \phi|\|_{L^2(B)}$$

defines a norm on $\mathcal{H}$. Consequently, $\mathcal{H}$ equipped with this norm is a Hilbert space, with the scalar product

$$\langle \phi_1, \phi_2 \rangle_{\mathcal{H}} := \int_B d\Gamma(\phi_1, \phi_2).$$

Since $f \in \mathcal{D} \subset \mathcal{F}$ then the linear form

$$\phi \mapsto \int_B d\Gamma(f, \phi)$$

is continuous on $\mathcal{H}$. By the representation theorem of Riesz, there exists $v \in \mathcal{H}$ such that for every $\phi \in \mathcal{H}$

$$\int_B d\Gamma(f, \phi) = \int_B d\Gamma(v, \phi).$$

We set $u := f - v$ so that $v = f - u$ being in $\mathcal{H}$ is supported in B. Moreover for every $\phi \in \mathcal{H}$, ϕ is supported in B so the previous equality yields

$$\int_M d\Gamma(u, \phi) = \int_M d\Gamma(f, \phi) - \int_M d\Gamma(v, \phi) = 0.$$

□

Proof of Theorem 5.4. Let $u \in \mathcal{F}$ be a function harmonic on a ball $B_R = B(x_0, R)$ and write $B_r = B(x_0, r)$ for $r \leq R$ with $R \leq \delta \operatorname{diam}(M)$ (where we have chosen for δ the minimum of the two parameters in (FK) and (ER)). From (ER), it follows that

$$\operatorname{Osc}_{B_r}(u) \lesssim \left(\frac{r}{R}\right)^{\alpha} \operatorname{Osc}_{B_R}(u). \tag{5.7}$$

Indeed, let $0 < r \leq R/4$. According to (ER), for every $x \in B_r$,

$$\begin{aligned}
\left| u(x) - \fint_{B_r} u(y)\, d\mu(y) \right| &\leq \fint_{B_r} |u(x) - u(y)|\, d\mu(y) \\
&\lesssim \left(\fint_{B_r} \left(\frac{d(x,y)}{R} \right)^{\alpha} d\mu(y) \right) \operatorname{Osc}_{B_R}(u) \\
&\lesssim \left(\frac{r}{R}\right)^{\alpha} \operatorname{Osc}_{B_R}(u).
\end{aligned}$$

Integrating over B_r then gives

$$\mathrm{Osc}_{B_r}(u) \lesssim \left(\frac{r}{R}\right)^{\alpha} \mathrm{Osc}_{B_R}(u).$$

The case $R/4 \le r \le R$ is trivial.

We will now extend this estimate to non-harmonic functions, namely prove that

$$\mathrm{Osc}_{B_r}(f) \lesssim \left(\frac{r}{R}\right)^{\alpha} \mathrm{Osc}_{B_R}(f) + \left(\frac{R}{r}\right)^{\nu/2} R^2 \left(\fint_{B_R} |Lf|^2 \, d\mu\right)^{1/2} \tag{5.8}$$

for all $f \in \mathcal{D}$ and concentric balls B_r, B_R with $0 < r \le R$. Let $f \in \mathcal{D}$. Since (FK) holds, one can invoke Lemma 5.6 below: there exists $u \in \mathcal{F}$ harmonic on B_R such that $f - u \in \mathcal{F}$ is supported in the ball B_R. One may write, using triangle inequality and (5.7),

$$\mathrm{Osc}_{B_r}(f) \le \mathrm{Osc}_{B_r}(u) + \mathrm{Osc}_{B_r}(f-u) \lesssim \left(\frac{r}{R}\right)^{\alpha} \mathrm{Osc}_{B_R}(u) + \mathrm{Osc}_{B_r}(f-u),$$

hence by triangle inequality again

$$\mathrm{Osc}_{B_r}(f) \le \left(\frac{r}{R}\right)^{\alpha} \left(\mathrm{Osc}_{B_R}(f) + \mathrm{Osc}_{B_R}(f-u)\right) + \mathrm{Osc}_{B_r}(f-u). \tag{5.9}$$

Let us start with estimating $\mathrm{Osc}_{B_R}(f-u)$:

$$\mathrm{Osc}_{B_R}(f-u) \lesssim \left(\fint_{B_R} |f-u|^2 \, d\mu\right)^{1/2},$$

and since $f-u$ is supported in B_R, by (FK) we have

$$\left(\fint_{B_R} |f-u|^2 \, d\mu\right)^{1/2} \lesssim R \left(\fint_{B_R} |\nabla(f-u)|^2 \, d\mu\right)^{1/2}.$$

Now, since $f-u$ is supported on B_R and u is harmonic on B_R

$$\begin{aligned}
\fint_{B_R} |\nabla(f-u)|^2 \, d\mu &= \frac{1}{|B_R|}\mathcal{E}(f-u, f-u) = \frac{1}{|B_R|} \int_M d\Gamma(f-u, f-u) \\
&= \frac{1}{|B_R|} \int_M d\Gamma(f, f-u) = \frac{1}{|B_R|} \int_{B_R} (f-u) Lf \, d\mu \\
&\le \frac{1}{|B_R|} \int_{B_R} |f-u||Lf| \, d\mu \le \left(\fint_{B_R} |f-u|^2\right)^{1/2} \left(\fint_{B_R} |Lf|^2 \, d\mu\right)^{1/2}.
\end{aligned}$$

From (FK), it follows that

$$\left(\fint_{B_R} |\nabla(f-u)|^2 \, d\mu\right)^{1/2} \lesssim R^{1/2} \left(\fint_{B_R} |\nabla(f-u)|^2\right)^{1/4} \left(\fint_{B_R} |Lf|^2 \, d\mu\right)^{1/4},$$

hence

$$\left(\fint_{B_R} |\nabla(f-u)|^2 \, d\mu\right)^{1/2} \lesssim R \left(\fint_{B_R} |Lf|^2 \, d\mu\right)^{1/2}. \tag{5.10}$$

Gathering the above inequalities yields

$$\mathrm{Osc}_{B_R}(f-u) \lesssim \left(\fint_{B_R} |f-u|^2 \, d\mu\right)^{1/2} \lesssim R^2 \left(\fint_{B_R} |Lf|^2 \, d\mu\right)^{1/2}. \tag{5.11}$$

Now for $\mathrm{Osc}_{B_r}(f-u)$: by doubling

$$\begin{aligned}\mathrm{Osc}_{B_r}(f-u) &\lesssim \left(\fint_{B_r} |f-u|^2\, d\mu\right)^{1/2} \\ &\lesssim \left(\frac{R}{r}\right)^{\nu/2} \left(\fint_{B_R} |f-u|^2\, d\mu\right)^{1/2},\end{aligned}$$

therefore by (5.11)

$$\mathrm{Osc}_{B_r}(f-u) \lesssim \left(\frac{R}{r}\right)^{\nu/2} R^2 \left(\fint_{B_R} |Lf|^2\, d\mu\right)^{1/2}. \tag{5.12}$$

Finally, putting together (5.9), (5.11), (5.12),

$$\mathrm{Osc}_{B_r}(f) \lesssim \left(\frac{r}{R}\right)^{\alpha} \mathrm{Osc}_{B_R}(f) + \left[\left(\frac{r}{R}\right)^{\alpha} + \left(\frac{R}{r}\right)^{\nu/2}\right] R^2 \left(\fint_{B_R} |Lf|^2\, d\mu\right)^{1/2},$$

which yields (5.8). A standard iteration argument, Lemma 5.7 below, with

$$A(s) := s^{-1}\,\mathrm{Osc}_{B_s}(f) \qquad \text{and} \qquad B(s) := \left(\fint_{B_s} |Lf|^2\, d\mu\right)^{1/2},$$

allows us to obtain for $\alpha' \in (0,\alpha)$

$$\mathrm{Osc}_{B_r}(f) \lesssim \left(\frac{r}{R}\right)^{\alpha'} \left(\mathrm{Osc}_{B_R}(f) + R^2 \|Lf\|_{L^\infty(B_R)}\right).$$

That holds for every $r \le R$ with $R \le \delta\,\mathrm{diam}(M)$.

Then if M is unbounded, we may choose $R = \sqrt{t}$ and replace f with $e^{-tL}f$, which yields

$$\begin{aligned}\mathrm{Osc}_{B_r}(e^{-tL}f) &\lesssim \left(\frac{r}{\sqrt{t}}\right)^{\alpha'} \left[\mathrm{Osc}_{B_{\sqrt{t}}}(e^{-tL}f) + \|tLe^{-tL}f\|_{L^\infty(B_{\sqrt{t}})}\right] \\ &\lesssim \left(\frac{r}{\sqrt{t}}\right)^{\alpha'} |B_{\sqrt{t}}|^{-1/2} \|f\|_2,\end{aligned} \tag{5.13}$$

where we used the Gaussian estimates for tLe^{-tL} (see the proof of Theorem 5.3). Property $(H^{\alpha'}_{2,2})$ follows and Theorem 3.5 yields (LE). If the ambient space M is bounded, as explained in Remark 3.7 to get (LE), it is sufficient to check (5.13) for the scales $\sqrt{t} \lesssim \mathrm{diam}(M)$, which is exactly what we just have proved. □

It remains to prove the next lemma which follows ideas of [1, Theorem 3.6] and [39, Lemma 2.1, Chapter III].

Lemma 5.7. *Let $0 < r < R$ and consider a function $A : [r,R] \to \mathbb{R}^+$ such that*

$$A(s) \lesssim \left(\frac{s'}{s}\right)^{\theta} A(s') \tag{5.14}$$

for all s, s' such that $r \le s \le s' \le R$ and for some $\theta > 0$. Let $B : [r,R] \to \mathbb{R}^+$, and assume that

$$A(s) \lesssim \left(\frac{s'}{s}\right)^{\varepsilon} A(s') + \left(\frac{s'}{s}\right)^{\gamma} s' B(s'), \tag{5.15}$$

for every $r \le s \le s' \le R$ *and for some* $\varepsilon \in (0,1)$ *and* $\gamma > 0$. *Then*

$$A(r) \lesssim \left(\frac{R}{r}\right)^{\varepsilon'} \left[A(R) + R \sup_{r\le u\le R} B(u)\right],$$

for every $\varepsilon' \in (\varepsilon, 1)$.

Proof. Applying (5.15) with s and $s' = Ks$ gives, for some numerical constant C,

$$A(s) \le CK^{\varepsilon} A(Ks) + CK^{\gamma} Ks B(Ks).$$

We choose $K > 1$ large enough such that $CK^{\varepsilon} \le K^{\varepsilon'}$ for some fixed $\varepsilon' \in (\varepsilon, 1)$. It follows that, for $r \le s < Ks \le R$,

$$\begin{aligned} A(s) &\le K^{\varepsilon'} A(Ks) + K^{\gamma+\varepsilon'-\varepsilon+1} s B(Ks) \\ &\le K^{\varepsilon'} A(Ks) + K^{\gamma+2} s \left(\sup_{r\le u\le R} B(u)\right). \end{aligned}$$

By iterating for $s = r, Kr, K^2 r, ..., K^{\lfloor\lambda\rfloor-1} r$, where λ is such that $K^{\lambda} r = R$, we deduce that

$$\begin{aligned} A(r) \le & K^{\lfloor\lambda\rfloor\varepsilon'} A(K^{\lfloor\lambda\rfloor} r) + \left(\sum_{\ell=0}^{\lfloor\lambda\rfloor-1} (K^{\ell} r) K^{\ell\varepsilon'}\right) K^{\gamma+2} \left(\sup_{r\le u\le R} B(u)\right) \\ = & K^{\lfloor\lambda\rfloor\varepsilon'} A(K^{\lfloor\lambda\rfloor} r) + \left(\sum_{\ell=0}^{\lfloor\lambda\rfloor-1} K^{\ell(1+\varepsilon')}\right) r K^{\gamma+2} \left(\sup_{r\le u\le R} B(u)\right) \\ \lesssim & K^{\lfloor\lambda\rfloor\varepsilon'} A(K^{\lfloor\lambda\rfloor} r) + K^{\lfloor\lambda\rfloor(1+\varepsilon')} r \left(\sup_{r\le u\le R} B(u)\right) \\ \le & K^{\lfloor\lambda\rfloor\varepsilon'} \left[A(K^{\lfloor\lambda\rfloor} r) + K^{\lfloor\lambda\rfloor} r \left(\sup_{r\le u\le R} B(u)\right)\right] \\ \le & K^{\lfloor\lambda\rfloor\varepsilon'} \left[A(K^{\lfloor\lambda\rfloor} r) + R \left(\sup_{r\le u\le R} B(u)\right)\right]. \end{aligned}$$

The claim follows by using (5.14). □

6. De Giorgi property and heat kernel bounds: the case $2 < p \le \nu$

We pursue the same ideas as in Section 5 but now for $p > 2$. To this end, we will rely on (G_p) and we will introduce L^p versions, for $p \in [1, +\infty)$, of the De Giorgi property.

Definition 6.1 (L^p De Giorgi property)**.** *Let* $(M, d, \mu, \mathcal{E})$ *be a metric measure Dirichlet space with a "carré du champ" and* L *the associated operator. For* $p \in [1, +\infty)$ *and* $\varepsilon \in (0,1)$, *we say that* $(DG_{p,\varepsilon})$ *holds if the following is satisfied: for all* $r \le R$, *every pair of concentric balls* B_r, B_R *with respective radii* r *and* R, *and for every function* $f \in \mathcal{D}$, *one has*

$$(DG_{p,\varepsilon}) \qquad \left(\fint_{B_r} |\nabla f|^p d\mu\right)^{1/p} \lesssim \left(\frac{R}{r}\right)^{\varepsilon} \left[\left(\fint_{B_R} |\nabla f|^p d\mu\right)^{1/p} + R\|Lf\|_{L^{\infty}(B_R)}\right].$$

We sometimes omit the parameter ε, and write (DG_p) if $(DG_{p,\varepsilon})$ is satisfied for some $\varepsilon \in (0,1)$.

Remark 6.2. *For $f \in \mathcal{D}$ and $0 < r < R$,*

$$\left(\fint_{B_r} |\nabla f|^p d\mu\right)^{1/p} \lesssim \left(\frac{|B_R|}{|B_r|}\right)^{1/p} \left(\fint_{B_R} |\nabla f|^p d\mu\right)^{1/p},$$

hence if (VD_ν) holds, then

$$\left(\fint_{B_r} |\nabla f|^p d\mu\right)^{1/p} \lesssim \left(\frac{R}{r}\right)^{\nu/p} \left(\fint_{B_R} |\nabla f|^p d\mu\right)^{1/p}.$$

Therefore if $p > \nu$, one always has $(DG_{p,\varepsilon})$ with $\varepsilon = \frac{\nu}{p} < 1$. This is why (DG_p) does not appear explicitly in Theorem 4.1. As a matter of fact, $(DG_{p,\frac{\nu}{p}})$ is implicit in its proof.

We have just seen that, for $\nu < p < +\infty$, (DG_p) is a trivial consequence of (VD_ν), and Proposition 5.2 states that (DG_2) follows from (VD) and (P_2). In the range $2 < p \leq \nu$, property (DG_p) is more mysterious. The main result of this section is that, together with (P_p) and (G_p), a weak version of (DG_p) implies (P_2). But as we already said, we do not know whether this is really an additional assumption, or whether it follows from $(P_p) + (G_p)$ in this range also.

Let us now introduce the weak version of (DG_p) that we shall use. An examination of the proof of Theorem 5.3 shows that we do not use the full strength of (DG_2): the integrals of the gradient can be taken on the whole space and not only on the larger ball (in the same flavour as in $(H^\eta_{p,p})$).

Definition 6.3 (non-local L^p De Giorgi property)**.** *For $p \in [1,+\infty)$ and $\varepsilon \in (0,1)$, we say that $(\overline{DG}_{p,\varepsilon})$ holds if the following is satisfied: for all $r \leq R$, every pair of concentric balls B_r, B_R with respective radii r and R, and for every function $f \in \mathcal{D}$, one has*

$$\left(\fint_{B_r} |\nabla f|^p d\mu\right)^{1/p} \lesssim \left(\frac{R}{r}\right)^{\varepsilon} \left[|B_R|^{-1/p} \left(\||\nabla f|\|_p + R\|Lf\|_p\right) + R\|Lf\|_{L^\infty(B_R)}\right].$$

We sometimes omit the parameter ε, and write $(\overline{DG}_p)$ if $(\overline{DG}_{p,\varepsilon})$ is satisfied for some $\varepsilon \in (0,1)$.

The fact that (DG_p) implies $(\overline{DG}_p)$ is obvious. We are now ready to give an extension to all $p \geq 2$ of Theorem 4.1. The case $p = 2$ already follows Theorem 5.3. Remember also that assumption $(\overline{DG}_p)$ is always fulfilled for $p > \nu$ in presence of (VD_ν) (see Remark 6.2).

Theorem 6.4. *Let $(M, d, \mu, \mathcal{E})$ be a doubling metric measure Dirichlet space with a "carré du champ". Assume (G_p), (P_p) and $(\overline{DG}_p)$ for some $p \in [2,+\infty)$. Then (LE) holds.*

Using the well-known implication from (LE) to (P_2) ([62], [70], see also [45, Theorem 2.31]), we can complement the first statement of Corollary 4.5 by saying that for $2 \leq p \leq \nu$, (G_p), (P_p) and $(\overline{DG}_p)$ imply (P_2).

Proof. Applying (P_p) to $e^{-tL}f$ for $t>0$ and $f\in L^p(M,\mu)$ on a ball B_r for $r>0$ yields

$$p\text{-}\operatorname{Osc}_{B_r}(e^{-tL}f)\lesssim r\left(\fint_{B_r}|\nabla e^{-tL}f|^p d\mu\right)^{1/p}.$$

By $(\overline{DG}_p)$, we can estimate

$$\left(\fint_{B_r}|\nabla e^{-tL}f|^p d\mu\right)^{1/p}$$

by

$$\left(\frac{\sqrt{t}}{r}\right)^{\varepsilon}\left[|B_{\sqrt{t}}|^{-1/p}\left(\||\nabla e^{-tL}f|\|_p+\|\sqrt{t}Le^{-tL}f\|_p\right)+\|\sqrt{t}Le^{-tL}f\|_{L^\infty(B_{\sqrt{t}})}\right]$$

for some $\varepsilon\in(0,1)$, if $0<r\le\sqrt{t}$ and $B_{\sqrt{t}}$ with radius $\sqrt{t}$ is concentric to B_r.

By (G_p) and the analyticity of $(e^{-tL})_{t>0}$ on $L^p(M,\mu)$,

$$\||\nabla e^{-tL}f|\|_p+\|\sqrt{t}Le^{-tL}f\|_p\lesssim\frac{\|f\|_p}{\sqrt{t}}.$$

Now under our assumptions, (UE) holds due to Proposition 2.1, hence tLe^{-tL} satisfies Gaussian pointwise estimates too (see the proof of Theorem 5.3). It follows easily that

$$\|\sqrt{t}Le^{-tL}f\|_{L^\infty(B_{\sqrt{t}})}\lesssim|B_{\sqrt{t}}|^{-1/p}\frac{\|f\|_p}{\sqrt{t}}.$$

Finally

$$p\text{-}\operatorname{Osc}_{B_r}(e^{-tL}f)\lesssim\left(\frac{r}{\sqrt{t}}\right)^{1-\varepsilon}|B_{\sqrt{t}}|^{-1/p}\|f\|_p,$$

that is, $(H^{\eta}_{p,p})$ with $\eta=1-\varepsilon>0$. This implies (LE) according to Theorem 3.5. □

Remark 6.5. *The fact that the combination (G_p), (P_p) and $(\overline{DG}_{p,\varepsilon})$ implies $(H^{1-\varepsilon}_{p,p})$ will be applied in* [10] *to the Sobolev algebra property.*

Let us now prove an extension of Corollary 4.5, whose proof will be more involved because of the presence of $(\overline{DG}_p)$.

Proposition 6.6. *Let $(M,d,\mu,\mathcal{E})$ be a doubling metric measure Dirichlet space with a "carré du champ". For $2\le q<p<+\infty$, we have*

$$(G_p)+(P_p)+(\overline{DG}_p)\Longrightarrow(G_q)+(P_q)+(DG_q).$$

If in addition $p>\nu$,

$$(G_p)+(P_p)\Longrightarrow(G_q)+(P_q)+(DG_q).$$

Moreover, if (G_p) holds for some $p\in(2,+\infty)$, then for every $q\in(2,\nu]$

$$(P_q)+(\overline{DG}_q)\Longleftrightarrow(P_2),$$

and for every $q\in(\nu,p]$

$$(P_q)\Longleftrightarrow(P_2).$$

Proof. Assume $(G_p)+(P_p)+(\overline{DG}_p)$. Then (LE) holds by Proposition 6.4, which in turn implies (P_2) according to [62], [70]. We thus get from Proposition 6.7 below that (DG_q) is true. Since (G_q) is a direct consequence of (G_p) and (P_q) of (P_2), we obtain the first and the third statements. The others follow from Remark 6.2. □

It remains to prove:

Proposition 6.7. *Let $(M,d,\mu,\mathcal{E})$ be a doubling metric measure Dirichlet space with a "carré du champ". Assume (P_2) and (G_p) for some $p\in(2,+\infty)$. Then (DG_q) holds for every $q\in[2,p]$.*

The proof of Proposition 6.7 will make use of an L^p version of the Caccioppoli inequality. For $p=2$, integration by parts yields easily a L^2-Caccioppoli inequality (see for instance Lemma B.1) which enables one to deduce (DG_2) from (P_2) (Proposition 5.2). But obtaining an L^p-Caccioppoli inequality for $p>2$ seems more difficult and cannot be handled using integration by parts directly. We use (G_p) and the finite propagation speed property instead, and obtain:

Proposition 6.8 (L^p Caccioppoli inequality)**.** *Let $(M,d,\mu,\mathcal{E})$ be a doubling metric measure Dirichlet space with a "carré du champ" satisfying (UE). Assume (G_p) for some $p\in[2,+\infty]$. Then for every $q\in(1,p]$,*

$$r\left(\fint_{B_r}|\nabla f|^q d\mu\right)^{1/q}\lesssim\left(\fint_{2B_r}|f|^q\,d\mu\right)^{1/q}+r^2\left(\fint_{2B_r}|Lf|^q\,d\mu\right)^{1/q} \tag{6.1}$$

for all $f\in\mathcal{D}$ and all balls B_r of radius r.

Note that is would be easier, and sufficient for our purposes in this section, to prove only a non-local version of the above inequality:

$$r\left(\fint_{B}|\nabla f|^q d\mu\right)^{1/q}\lesssim\sum_{\ell\geq 0}2^{-\ell N}\left[\left(\fint_{2^\ell B}|f|^q d\mu\right)^{1/q}+r^2\left(\fint_{2^\ell B}|Lf|^q d\mu\right)^{1/q}\right].$$

The above local version is used in the proof of Theorem 7.7. We feel anyway that (6.1) may be of independent interest, and that it is worth the extra effort, namely the use of the finite propagation speed property.

Since the heat semigroup satisfies Davies-Gaffney estimates (2.1), it is known (see e.g. [66,69] and [24, Section 3]) that $\sqrt{L}$ satisfies the finite speed propagation property (with a speed equal to 1 due to the normalization in (1.3)) for solutions of the corresponding wave equation. Consequently for every even function $\varphi\in\mathcal{S}(\mathbb{R})$ with $\operatorname{supp}\hat\varphi\subseteq[-1,1]$, every pair of Borel sets $E,F\subset M$ and every $r>0$, one has $\mathbb{1}_E\varphi(r\sqrt{L})\mathbb{1}_F=0$ if $\operatorname{dist}(E,F)>r$. This follows from the Fourier inversion formula and the bounded Borel functional calculus of $\sqrt{L}$, cf. [5, Lemma 4.4]. Moreover, since the Dirichlet form is strongly local, one also has $\mathbb{1}_E|\nabla\varphi(r\sqrt{L})\mathbb{1}_F|^2=0$ if $\operatorname{dist}(E,F)>r$.

Indeed, for every nonnegative, bounded and Lipschitz function χ_E supported in E and $g \in \mathcal{D}$ supported in F, it follows from (1.1), (1.2), and (1.6) that

$$\int_M \chi_E |\nabla \varphi(r\sqrt{L})g|^2 \, d\mu \le \int_M |\varphi(r\sqrt{L})g|^2 |\nabla \chi_E|^2 \, d\mu + \int_M \chi_E |\varphi(r\sqrt{L})g||L\varphi(r\sqrt{L})g| \, d\mu.$$

If $d(E,F) > \sqrt{r}$ then $\varphi(r\sqrt{L})g = 0$ in the support of χ_E, so

$$\int \chi_E |\nabla \varphi(r\sqrt{L})g|^2 \, d\mu = 0.$$

This holds for every nonnegative, bounded Lipschitz function χ_E supported in E, hence

$$\int_E |\nabla \varphi(r\sqrt{L})g|^2 \, d\mu = 0.$$

Proof of Proposition 6.8. Consider an even function $\varphi \in \mathcal{S}(\mathbb{R})$ with $\operatorname{supp} \hat{\varphi} \subseteq [-1,1]$ and $\varphi(0) = 1$. Consequently, $\varphi'(0) = 0$, and $x \mapsto x^{-1}\varphi'(x) \in \mathcal{S}(\mathbb{R})$ is even with Fourier support in $[-1,1]$, cf. [5, Lemma 6.1]. Fix a ball B_r of radius $r > 0$, an exponent $q \in (1,p]$ and split

$$f = \varphi(r\sqrt{L})f + (I - \varphi(r\sqrt{L}))f.$$

Since $\varphi(0) = 1$, one has

$$I - \varphi(r\sqrt{L}) = \int_0^r \sqrt{L}\varphi'(s\sqrt{L}) \, ds.$$

Using the finite propagation speed property applied to the functions φ and $x \mapsto x^{-1}\varphi'(x)$, we have that both $\varphi(r\sqrt{L})$ and $(r^2L)^{-1}(1-\varphi(r\sqrt{L}))$ satisfy the propagation property at speed 1 and so propagate at a distance at most r. As we have seen above, the same stills holds by composing with the gradient. Hence, for $f \in \mathcal{D}$,

$$\begin{aligned} \||\nabla f|\|_{L^q(B_r)} &\lesssim \||\nabla \varphi(r\sqrt{L})|\|_{q\to q} \|f\|_{L^q(2B_r)} \\ &\quad + r^2 \||\nabla (1 - \varphi(r\sqrt{L}))(r^2L)^{-1}|\|_{q\to q} \|Lf\|_{L^q(2B_r)}. \end{aligned} \tag{6.2}$$

Let us now estimate $\||\nabla \varphi(r\sqrt{L})|\|_{q\to q}$ and $\||\nabla(1-\varphi(r\sqrt{L}))(r^2L)^{-1}|\|_{q\to q}$. For $q > 2$, (G_q) holds by interpolation between (G_2) and (G_p), and for $q < 2$, (G_q) always holds as we already said. By writing the resolvent via the Laplace transform as $(1 + r^2L)^{-1} = \int_0^{+\infty} e^{-t(1+r^2L)} dt$, we deduce gradient bounds for the resolvent in L^q, that is

$$\||\nabla (1 + r^2L)^{-1}|\|_{q\to q} \lesssim \int_0^{+\infty} e^{-t} \||\nabla e^{-tr^2L}|\|_{q\to q} \, dt \lesssim \int_0^{+\infty} \frac{e^{-t}}{r\sqrt{t}} \, dt \lesssim r^{-1}.$$

Denote $\psi := \varphi$ or $\psi := x \mapsto (1 - \varphi(x))/x^2$, and consider $\lambda(x) = \psi(x)(1 + x^2)$. Hence

$$\psi(r\sqrt{L}) = (1 + r^2L)^{-1}\lambda(r\sqrt{L})$$

and therefore

$$\||\nabla \psi(r\sqrt{L})|\|_{q\to q} \le \||\nabla (1 + r^2L)^{-1}|\|_{q\to q} \|\lambda(r\sqrt{L})\|_{q\to q} \lesssim r^{-1} \|\lambda(r\sqrt{L})\|_{q\to q}.$$

Observe that $\lambda \in \mathcal{S}(\mathbb{R})$ since $\varphi(0) = 1$ and $\varphi'(0) = 0$. By a functional calculus result (see e.g. [30], Theorem 3.1) which relies on (UE), we then have

$$\sup_{r>0} \|\lambda(r\sqrt{L})\|_{q\to q} \lesssim 1,$$

and consequently,

$$\||\nabla\psi(r\sqrt{L})|\|_{q\to q} \lesssim r^{-1}.$$

Coming back to (6.2), we obtain

$$\||\nabla f|\|_{L^q(B_r)} \lesssim r^{-1}\|f\|_{L^q(2B_r)} + r\|Lf\|_{L^q(2B_r)},$$

which is the claim. □

We are now ready to prove Proposition 6.7. To pass from (DG_2) to (DG_q) for $2 < q \le p$ we shall use Hölder estimates as in [50, Lemma 2.3].

Proof of Proposition 6.7. Let B_R denote a ball of radius $R > 0$, and let $f \in \mathcal{F}$. Using Lemma 3.3 for $p = 2$ and (P_2), we can write

$$\|f\|_{C^\eta(B_R)} \lesssim \|f\|_{C^{\eta,2}(B_R)} \lesssim \sup_{\tilde{B}\subset 6B_R} r(\tilde{B})^{1-\eta} \left(\fint_{\tilde{B}} |\nabla f|^2\, d\mu\right)^{1/2}. \tag{6.3}$$

for any $\eta \in (0,1)$. Now according to Proposition 5.2, (P_2) yields $(DG_{2,\varepsilon})$ for some $\varepsilon \in (0,1)$. Choose $\eta = 1-\varepsilon$. Then it follows from (6.3) together with $(DG_{2,\varepsilon})$ that

$$\begin{aligned}\|f\|_{C^{1-\varepsilon,2}(B_R)} &\lesssim \sup_{\tilde{B}\subset 6B_R} r^{\varepsilon}(\tilde{B}) \left(\fint_{\tilde{B}} |\nabla f|^2\, d\mu\right)^{1/2} \\ &\lesssim R^{\varepsilon}\left[\left(\fint_{24B_R} |\nabla f|^2\, d\mu\right)^{1/2} + R \operatorname*{ess\,sup}_{x\in 24B_R} |Lf(x)|\right].\end{aligned}$$

We then apply Jensen's inequality and obtain for $q \ge 2$

$$\|f\|_{C^{1-\varepsilon}(B_R)} \lesssim R^{\varepsilon}\left[\left(\fint_{24B_R} |\nabla f|^q\, d\mu\right)^{\frac{1}{q}} + R \operatorname*{ess\,sup}_{x\in 24B_R} |Lf(x)|\right]. \tag{6.4}$$

We now deduce (DG_q) for $2 < q \le p$ from (6.4) and (G_p).

By (G_p) and Proposition 6.8 one has (6.1) for $2 < q \le p$. Replacing f with $f - \fint_{B_r} f d\mu$ in (6.1) yields

$$\left(\fint_{B_r} |\nabla f|^q d\mu\right)^{1/q} \lesssim r^{-1}\left(\fint_{2B_r} |f - \fint_{B_r} f d\mu|^q d\mu\right)^{1/q} + r\left(\fint_{2B_r} |Lf|^q d\mu\right)^{1/q}. \tag{6.5}$$

for every ball B_r with radius $r > 0$ and $B_r \subset B_R$.

Now we can write

$$\begin{aligned}\left(\fint_{2B_r} |f - \fint_{B_r} f d\mu|^q d\mu\right)^{1/q} &\le \operatorname*{ess\,sup}_{x,y\in 2B_r} |f(x) - f(y)| \\ &\le r^{1-\varepsilon}\|f\|_{C^{1-\varepsilon}(B_{2r})} \\ &\le r^{1-\varepsilon}\|f\|_{C^{1-\varepsilon}(B_{2R})}\end{aligned}$$

and (6.4) yields

$$\left(\fint_{2B_r} |f - \fint_{B_r} f d\mu|^q d\mu\right)^{1/q} \lesssim r^{1-\varepsilon} R^{\varepsilon} \left[\left(\fint_{48B_R} |\nabla f|^q d\mu\right)^{1/q} + R \operatorname*{ess\,sup}_{x\in 48B_R} |Lf(x)|\right].$$

Consequently

$$\begin{aligned}\left(\fint_{B_r} |\nabla f|^q d\mu\right)^{1/q} &\lesssim \left(\frac{R}{r}\right)^{\varepsilon} \left[\left(\fint_{48B_R} |\nabla f|^q d\mu\right)^{1/q} + R \operatorname*{ess\,sup}_{x\in 48B_R} |Lf(x)|\right] \\ &\quad + r\left(\fint_{2B_r} |Lf|^q d\mu\right)^{1/q} \\ &\lesssim \left(\frac{R}{r}\right)^{\varepsilon} \left[\left(\fint_{48B_R} |\nabla f|^q d\mu\right)^{1/q} + R \operatorname*{ess\,sup}_{x\in 48B_R} |Lf(x)|\right],\end{aligned}$$

which gives easily $(DG_{q,\varepsilon})$. □

Let us finish this section by noting a consequence of Theorem 7.7 below and Proposition 6.7 on the De Giorgi property.

Corollary 6.9. *Let $(M, d, \mu, \mathcal{E})$ be a doubling metric measure Dirichlet space with a "carré du champ" satisfying (P_2). Then there exists $\varepsilon > 0$ such that (DG_p) holds for every $p \in [2, 2+\varepsilon)$.*

7. Gradient estimates, Poincaré inequality and Riesz transform

Our aim in this section is to improve the understanding of the links between gradient estimates (G_p), boundedness of the Riesz transform (R_p) and Poincaré inequality (P_p), as started in [4] and [3].

Let us first state an improvement of one the main results in [4]: the point is that we are able to replace (P_2) by the weaker assumption (P_{p_0}) for $p_0 > 2$.

Theorem 7.1. *Let $(M, d, \mu, \mathcal{E})$ be a doubling metric measure Dirichlet space with a "carré du champ". If for some $p_0 \in (2, \infty)$ the combination (P_{p_0}) with (G_{p_0}) holds, then (R_p) is satisfied for every $p \in (1, p_0)$.*

Remark 7.2. *For $p_0 < \nu$, where ν is the exponent in (VD_ν), (P_{p_0}) is not necessary for (R_p) to hold for every $p \in (1, p_0)$, as the example of the connected sum of two copies of $\mathbb{R}^n$ shows (see [14]).*

We will use the following extrapolation result ([4], [6, Theorem 3.13]). Here, we denote by $\mathcal{M}$ the Hardy-Littlewood maximal operator, defined for $f \in L^1_{\mathrm{loc}}(M, \mu)$ and $x \in M$ by

$$\mathcal{M}f(x) := \sup_{B\ni x} \fint_B |f|\, d\mu, \tag{7.1}$$

where the supremum is taken over all balls $B \subset M$ with $x \in B$. For $p \in [1, +\infty)$, we abbreviate by $\mathcal{M}_p$ the operator defined by $\mathcal{M}_p(f) := [\mathcal{M}(|f|^p)]^{1/p}$, $f \in L^1_{\mathrm{loc}}(M, \mu)$. Note that $\mathcal{M}$ is bounded in $L^q(M, \mu)$ for all $q \in (1, +\infty]$, cf. [18, Chapitre III]. Consequently, $\mathcal{M}_p$ is bounded in $L^q(M, \mu)$ for all $q \in (p, +\infty]$.

Proposition 7.3. *Let $(M, d, \mu, \mathcal{E})$ be a doubling metric measure Dirichlet space. Let $(A_t)_{t>0}$ be a family of linear operators, uniformly bounded in $L^2(M, \mu)$. Let T be a sublinear operator which is bounded on $L^2(M, \mu)$. Assume that for some $q \in (2, +\infty)$, every ball B of radius $r > 0$ and every $f \in L^2(M, \mu)$, we have*

- *L^2-L^2 estimates of $T(I - A_{r^2})$:*

$$\left(\fint_B |T(I - A_{r^2})f|^2 d\mu\right)^{1/2} \lesssim \inf_{x \in B} \mathcal{M}_2(f)(x); \tag{7.2}$$

- *L^2-L^q estimates of $T(A_{r^2})$:*

$$\left(\fint_B |TA_{r^2} f|^q d\mu\right)^{1/q} \lesssim \inf_{x \in B} [\mathcal{M}_2(Tf)(x) + \mathcal{M}_2(f)(x)]. \tag{7.3}$$

Then, for every $p \in (2, q)$, T is bounded on $L^p(M, \mu)$.

In order to apply Proposition 7.3, we shall need the following ingredient, which relies on the self-improving property of reverse Hölder inequalities from Theorem C.1.

Proposition 7.4. *Let $(M, d, \mu, \mathcal{E})$ be a doubling metric measure Dirichlet space with a "carré du champ". Assume that for some $p_0 \in (2, \infty)$, (P_{p_0}) and (G_{p_0}) hold. Then for every function $f \in \mathcal{D}$ and every ball B of radius $r > 0$, we have*

$$\left(\fint_{B_r} |f - \fint_{B_r} f \, d\mu|^2 \, d\mu\right)^{1/2} \lesssim r \left(\fint_{2B_r} |\nabla f|^2 \, d\mu\right)^{1/2} + r^2 \|Lf\|_{L^\infty(4B_r)}.$$

Let us first recall the following folklore result (see for instance [46, Theorem 5.1, 1.], [36, Theorem 2.7] for similar statements).

Lemma 7.5. *Let $(M, d, \mu, \mathcal{E})$ be a doubling metric measure Dirichlet space with a "carré du champ". Assume that (P_p) holds for some $1 \leq p < +\infty$. Then if $q \in (p, +\infty)$ is such that $\nu \left(\frac{1}{p} - \frac{1}{q}\right) \leq 1$, the following Sobolev-Poincaré inequality holds:*

$$\left(\fint_{B_r} \left| f - \fint_{B_r} f \, d\mu \right|^q d\mu\right)^{1/q} \lesssim r \left(\fint_{B_r} |\nabla f|^p d\mu\right)^{1/p}, \tag{$P_{p,q}$}$$

for all $f \in \mathcal{F}$, $r > 0$, and all balls B_r with radius r.

Proof. This result is well-known if $1 < p < \nu$ (see [46, Theorem 5.1, 1.]). Note that the truncation property holds in our setting (see [59, Section 3,(o)]). If $p > \nu$, a stronger L^∞ inequality is true ([46, Theorem 5.1, 2.]), and one gets the above by integration. If $p = \nu$, one deduces the claim from [46, Theorem 5.1, 3.] by bounding from above the function $t \to t^q$ by an exponential. □

Proof of Proposition 7.4. Proposition 6.8 yields

$$\left(\fint_{B_r} |\nabla f|^{p_0} \, d\mu\right)^{1/p_0} \lesssim \frac{p_0\text{- Osc}_{2B_r}(f)}{r} + r \left(\fint_{2B_r} |Lf|^{p_0} \, d\mu\right)^{1/p_0}$$

for every ball B_r of radius r and every $f \in \mathcal{D}$. From [53] we know that (P_{p_0}) self-improves into $(P_{p_0-\varepsilon})$ for some $\varepsilon > 0$. Then , according to Lemma 7.5, if ε is small enough, $(P_{p_0-\varepsilon})$ self-improves again into the Sobolev-Poincaré inequality $(P_{p_0,p_0-\varepsilon})$ which yields

$$p_0\text{- Osc}_{2B_r}(f) \lesssim r \left(\fint_{2B_r} |\nabla f|^{p_0-\varepsilon} d\mu \right)^{1/(p_0-\varepsilon)}.$$

We therefore have

$$\left(\fint_{B_r} |\nabla f|^{p_0}\, d\mu \right)^{1/p_0} \lesssim \left(\fint_{2B_r} |\nabla f|^{p_0-\varepsilon}\, d\mu \right)^{1/(p_0-\varepsilon)} + r \left(\fint_{2B_r} |Lf|^{p_0}\, d\mu \right)^{1/p_0}. \tag{7.4}$$

Let us apply Theorem C.1 to the functional

$$a(B) := r \left(\fint_{2B} |Lf|^{p_0}\, d\mu \right)^{1/p_0},$$

which is regular, uniformly with respect to f. This gives

$$\left(\fint_{B_r} |\nabla f|^{p_0} d\mu \right)^{1/p_0} \lesssim \left(\fint_{2B_r} |\nabla f|^2\, d\mu \right)^{1/2} + r \left(\fint_{4B_r} |Lf|^{p_0}\, d\mu \right)^{1/p_0}.$$

Using Hölder's inequality and (P_{p_0}), we deduce that

$$\begin{aligned}
\left(\fint_{B_r} |f - \fint_{B_r} f\, d\mu|^2\, d\mu \right)^{1/2} &\leq \left(\fint_{B_r} |f - \fint_{B_r} f\, d\mu|^{p_0}\, d\mu \right)^{1/p_0} \\
&\lesssim r \left(\fint_{B_r} |\nabla f|^{p_0} d\mu \right)^{1/p_0} \\
&\lesssim r \left(\fint_{2B_r} |\nabla f|^2\, d\mu \right)^{1/2} + r^2 \left(\fint_{4B_r} |Lf|^{p_0}\, d\mu \right)^{1/p_0}.
\end{aligned}$$

Bounding the last term by $r^2 \|Lf\|_{L^\infty(4B_r)}$ then gives the desired estimate. □

Proof of Theorem 7.1. First, according to Proposition 2.1, we know that (P_{p_0}) with (G_{p_0}) implies (DUE) and consequently (UE). Thus, by [21], we know that the Riesz transform is bounded on $L^p(M,\mu)$ for every $p \in (1, 2]$. It remains to establish the L^p boundedness for $p \in (2, p_0)$. To do so, we apply Proposition 7.3 to the Riesz transform $\mathcal{R} := |\nabla L^{-1/2}|$. Let $q \in (2, p_0)$, and let $M \in \mathbb{N}$ with $M > \frac{\nu}{4}$. Moreover, let $f \in L^2(M,\mu)$ and B be a ball of radius $r > 0$. For $t > 0$, denote

$$P_t^{(M)} := I - (I - e^{-tL})^M \qquad \text{and} \qquad \tilde{P}_t^{(2M)} := \left[P_t^{(M)} \right]^2.$$

Then, following [4, Lemma 3.1], which only relies on the Davies-Gaffney estimates (2.1), we already know that (7.2) is satisfied for $T = \mathcal{R}$ and $A_{r^2} = P_{r^2}^{(M)}$. More precisely, it is proven that

$$\left(\fint_B |\nabla L^{-1/2}(I - P_{r^2}^{(M)}) f|^2 d\mu \right)^{1/2} \lesssim \sum_{j=0}^{\infty} 2^{j(\frac{\nu}{2} - 2M)} \left(\fint_{2^j B} |f|^2\, d\mu \right)^{1/2}. \tag{7.5}$$

To obtain (7.2) for $A_{r^2} = \tilde{P}_{r^2}^{(2M)}$, we first write

$$\begin{aligned}
I - \tilde{P}_{r^2}^{(2M)} = I - \left[P_{r^2}^{(M)} \right]^2 &= (I - P_{r^2}^{(M)})(I + P_{r^2}^{(M)}) \\
&= (I - P_{r^2}^{(M)})(2I - (I - e^{-r^2 L})^M).
\end{aligned}$$

By expanding $Q_{r^2} := (2I - (I - e^{-r^2L})^M)$ as a sum, one observes that $(Q_{r^2})_{r>0}$ satisfies Davies-Gaffney estimates as well. Using these, together with (7.5) and (VD), we obtain

$$\begin{aligned}\left(\fint_B |\nabla L^{-1/2}(I - \tilde{P}^{(2M)}_{r^2})f|^2 d\mu\right)^{1/2} &= \left(\fint_B |\nabla L^{-1/2}(I - P^{(M)}_{r^2})Q_{r^2} f|^2 d\mu\right)^{1/2} \\ &\lesssim \sum_{j=0}^{\infty} 2^{j(\frac{\nu}{2}-2M)} \left(\fint_{2^jB} |Q_{r^2} f|^2 \, d\mu\right)^{1/2} \\ &\lesssim \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} 2^{j(\frac{\nu}{2}-2M)} \left(\fint_{2^jB} |Q_{r^2} 1\!\!1_{S_k(2^jB)} f|^2 \, d\mu\right)^{1/2} \\ &\lesssim \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} 2^{j(\frac{\nu}{2}-2M)} e^{-(2^{j+k})^2} \left(\frac{|2^kB|}{|2^jB|}\right)^{1/2} \left(\fint_{2^kB} |f|^2 \, d\mu\right)^{1/2} \lesssim \inf_{x\in B} \mathcal{M}_2(f)(x).\end{aligned}$$

This establishes (7.2). Let us now check (7.3), again with $T = \mathcal{R}$ and $A_{r^2} = \tilde{P}^{(2M)}_{r^2}$. By expanding the operator $P^{(M)}_{r^2}$ as a sum, it suffices to check that

$$\left(\fint_B |\nabla L^{-1/2} e^{-kr^2L} P^{(M)}_{r^2} f|^q d\mu\right)^{\frac{1}{q}} \lesssim \inf_{x\in B} [\mathcal{M}_2(|\nabla L^{-1/2} f|)(x) + \mathcal{M}_2(f)(x)], \tag{7.6}$$

for every integer $k \in \{1, .., M\}$. Using the conservation property, we can write $\nabla L^{-1/2} e^{-kr^2L} P^{(M)}_{r^2} f = \nabla e^{-kr^2L} g$, with

$$g := L^{-1/2} P^{(M)}_{r^2} f - \left(\fint_B L^{-1/2} P^{(M)}_{r^2} f \, d\mu\right).$$

From [4, Lemma 3.2], which only relies on (G_q) and (UE), we know that

$$\left(\fint_B |\nabla e^{-kr^2L} g|^q \, d\mu\right)^{1/q} \lesssim \frac{1}{r} \sum_{\ell=0}^{\infty} e^{-2^\ell} \left(\fint_{2^\ell B} |g|^2 \, d\mu\right)^{1/2}. \tag{7.7}$$

We now aim to apply Proposition 7.4 to the right-hand side of (7.7). Observe that, denoting $h := L^{-1/2} P^{(M)}_{r^2} f$ and using Hölder's inequality and (VD), one has

$$\begin{aligned}\left(\fint_{2^\ell B} |g|^2 \, d\mu\right)^{1/2} &= \left(\fint_{2^\ell B} |h - \fint_B h \, d\mu|^2 \, d\mu\right)^{1/2} \\ &\leq \left(\fint_{2^\ell B} |h - \fint_{2^\ell B} h \, d\mu|^2 \, d\mu\right)^{1/2} + \sum_{j=1}^{\ell} \left(\fint_{2^jB} h \, d\mu - \fint_{2^{j-1}B} h \, d\mu\right) \\ &\lesssim \sum_{j=1}^{\ell} \left(\fint_{2^jB} |h - \fint_{2^jB} h \, d\mu|^2 \, d\mu\right)^{1/2}.\end{aligned}$$

Proposition 7.4 then yields

$$
\begin{aligned}
(7.8)\quad \left(\fint_{2^\ell B}|g|^2\,d\mu\right)^{1/2} &\lesssim \sum_{j=1}^{\ell}2^j r\left(\fint_{2^{j+1}B}|\nabla h|^2\,d\mu\right)^{1/2}+\sum_{j=1}^{\ell}(2^j r)^2\|Lh\|_{L^\infty(2^{j+2}B)}\\
&\lesssim \ell 2^{(\frac{\nu}{2}+1)\ell}r\left(\fint_{2^{\ell+1}B}|\nabla h|^2\,d\mu\right)^{1/2}+\ell 2^{2\ell}r^2\|Lh\|_{L^\infty(2^{\ell+2}B)}\\
&= \ell 2^{(\frac{\nu}{2}+1)\ell}r\left(\fint_{2^{\ell+1}B}|\nabla L^{-1/2}P^{(M)}_{r^2}f|^2\,d\mu\right)^{1/2}\\
&\qquad +\ell 2^{2\ell}r^2\|L^{1/2}P^{(M)}_{r^2}f\|_{L^\infty(2^{\ell+2}B)}.
\end{aligned}
$$

By covering $2^{\ell+1}B$ with approximately $2^{\nu\ell}$ balls of radius r, we get from (7.5)

$$
\begin{aligned}
&\left(\fint_{2^{\ell+1}B}|\nabla L^{-1/2}P^{(M)}_{r^2}f|^2\,d\mu\right)^{1/2}\\
&\quad\le \left(\fint_{2^{\ell+1}B}|\nabla L^{-1/2}f|^2\,d\mu\right)^{1/2}+\left(\fint_{2^{\ell+1}B}|\nabla L^{-1/2}(I-P^{(M)}_{r^2})f|^2\,d\mu\right)^{1/2}\\
&\quad\lesssim \left(\fint_{2^{\ell+1}B}|\nabla L^{-1/2}f|^2\,d\mu\right)^{1/2}+2^{\nu\ell}\sum_{j=0}^{\infty}2^{j(\frac{\nu}{2}-2M)}\left(\fint_{2^{\ell+j}B}|f|^2\,d\mu\right)^{1/2}\\
&\quad\lesssim \inf_{x\in B}[\mathcal{M}_2(|\nabla L^{-1/2}f|)(x)+2^{\nu\ell}\mathcal{M}_2(f)(x)].
\end{aligned}
$$

Moreover, by again covering $2^{\ell+2}B$ with approximately $2^{\nu\ell}$ balls of radius r, we obtain from Lemma 7.6

$$
r\|L^{1/2}P^{(M)}_{r^2}f\|_{L^\infty(2^{\ell+2}B)}\lesssim 2^{\nu\ell}\inf_{x\in B}\mathcal{M}_2(f)(x).
$$

Consequently, coming back to (7.8), we obtain that

$$
\begin{aligned}
&\left(\fint_{2^\ell B}|g|^2\,d\mu\right)^{1/2}\\
&\lesssim \ell 2^{(\frac{\nu}{2}+1)\ell}r\inf_{x\in B}[\mathcal{M}_2(|\nabla L^{-1/2}f|)(x)+2^{\nu\ell}\mathcal{M}_2(f)(x)]+\ell 2^{(\nu+2)\ell}r\inf_{x\in B}\mathcal{M}_2(f)(x)\\
&\lesssim \ell 2^{(\frac{3}{2}\nu+2)\ell}r\inf_{x\in B}[\mathcal{M}_2(|\nabla L^{-1/2}f|)(x)+\mathcal{M}_2(f)(x)].
\end{aligned}
$$

Plugging this estimate into (7.7) and using the definition of g, we deduce

$$
\left(\fint_B|\nabla L^{-1/2}e^{-kr^2L}P^{(M)}_{r^2}f|^q d\mu\right)^{1/q}\lesssim \inf_{x\in B}[\mathcal{M}_2(|\nabla L^{-1/2}f|)(x)+\mathcal{M}_2(f)(x)],
$$

which is (7.6). In this way, we obtain (7.3), and the proof is complete by Proposition 7.3. To be more precise, Proposition 7.3 implies that the Riesz transform is bounded on $L^p(M,\mu)$, for every $p\in(2,q)$ with arbitrary $q\in(2,p_0)$. □

Lemma 7.6. *Let $(M,d,\mu,\mathcal{E})$ be a doubling metric measure Dirichlet space. Assume (DUE). Then for every ball B of radius $r>0$ and every function $f\in L^2(M,\mu)$,* *we have*

$$r\|L^{1/2}e^{-r^2L}f\|_{L^\infty(B)} \lesssim \sup_{Q\supset B}\left(\fint_Q |f|^2\,d\mu\right)^{1/2},$$

where the supremum is taken over all balls Q containing B.

Proof. Let B be a ball of radius r and $f \in L^2(M,\mu)$. Abbreviate $g := (r^2L)^{1/2}e^{-r^2L}f$, and note that g is in the range of L, therefore $e^{-tL}g \to 0$ in $L^2(M,\mu)$, as $t \to +\infty$. This allows us to write $g = \int_0^\infty (sL)^{1/2}e^{-sL}g\,\frac{ds}{s}$, i.e.

$$(r^2L)^{1/2}e^{-r^2L}f = \int_0^\infty \left(\frac{r^2}{s}\right)^{1/2} sLe^{-(s+r^2)L}f\,\frac{ds}{s}. \tag{7.9}$$

Now recall that (UE) follows from (DUE). As stated in the proof of Theorem 5.3, this again implies by analyticity of the semigroup that for every $t > 0$, tLe^{-tL} has a kernel satisfying Gaussian pointwise estimates. This yields that $s \le r^2$, the operator $r^2Le^{-(s+r^2)L}$ has a kernel satisfying Gaussian pointwise estimates at scale r, hence

$$\|r^2Le^{-(s+r^2)L}f\|_{L^\infty(B)} \lesssim \sup_{Q\supset B}\left(\fint_Q |f|\,d\mu\right) \lesssim \sup_{Q\supset B}\left(\fint_Q |f|^2\,d\mu\right)^{1/2}.$$

Consequently,

$$\int_0^{r^2}\left(\frac{r^2}{s}\right)^{1/2}\|sLe^{-(s+r^2)L}f\|_{L^\infty(B)}\,\frac{ds}{s} \lesssim \sup_{Q\supset B}\left(\fint_Q |f|^2\,d\mu\right)^{1/2}.$$

For $s \ge r^2$ on the other hand, $sLe^{-(s+r^2)L}$ has a kernel satisfying Gaussian pointwise estimates at scale $\sqrt{s} \ge r$. Denoting by $\tilde{B} = \frac{\sqrt{s}}{r}B \supset B$ the dilated ball, we obtain in this case

$$\begin{aligned}\|sLe^{-(s+r^2)L}f\|_{L^\infty(B)} \le \|sLe^{-(s+r^2)L}f\|_{L^\infty(\tilde{B})} &\lesssim \sup_{Q\supset \tilde{B}}\left(\fint_Q |f|\,d\mu\right)\\ &\lesssim \sup_{Q\supset B}\left(\fint_Q |f|^2\,d\mu\right)^{1/2}.\end{aligned}$$

This gives

$$\int_{r^2}^{\infty}\left(\frac{r^2}{s}\right)^{1/2}\|sLe^{-(s+r^2)L}f\|_{L^\infty(B)}\,\frac{ds}{s} \lesssim \sup_{Q\supset B}\left(\fint_Q |f|^2\,d\mu\right)^{1/2}.$$

Putting the two parts together yields the conclusion. □

We are now going to give a more general version of the main result of [3]. More precisely, [3, Theorem 0.4] states that if M is a complete non-compact Riemannian manifold satisfying the doubling condition (VD) and L is its nonnegative Laplace-Beltrami operator, then under the Poincaré inequality (P_2) there exists $\varepsilon > 0$ such that (R_p) holds for $p \in [2, 2+\varepsilon)$. This result relies on the self-improvement of Poincaré inequalities from [53], but also on considerations on the Hodge projector that are specific to the Riemannian setting. We give here a proof that is valid in our more general Dirichlet space setting and also gives a L^{p_0}-version. Contrary to [3] where $p_0 = 2$ and part of the singular integral machinery of [4] had to be reworked, here we can directly use [4] through Theorem 7.1.

Theorem 7.7. *Let $(M, d, \mu, \mathcal{E})$ be a doubling metric measure Dirichlet space with a "carré du champ". If for some $p_0 \in [2, \infty)$, the combination (P_{p_0}) and (G_{p_0}) holds, then there exists $\varepsilon > 0$ such that (R_p) holds for $p \in [p_0, p_0 + \varepsilon)$.*

Proof. Since (P_{p_0}) implies (P_p) for $p > p_0$, by Theorem 7.1 it suffices to prove the statement for (G_p) instead of (R_p). Let $f \in L^{p_0}(M, \mu)$, $t > 0$ and set $g := e^{-tL} f$. Similarly as in Proposition 7.4, we have, for $\kappa > 0$ small enough,

$$\left(\fint_{B_r} |\nabla g|^{p_0} d\mu\right)^{1/p_0} \lesssim \left(\fint_{2B_r} |\nabla g|^{p_0-\kappa} d\mu\right)^{1/(p_0-\kappa)} + r\left(\fint_{2B_r} |Lg|^{p_0} d\mu\right)^{1/p_0}. \tag{7.10}$$

Now, for $B_{\sqrt{t}}$ a ball of radius $\sqrt{t}$ and $N > \nu$ a parameter to be chosen later, consider the function

$$h := |\nabla g| + c(f, B_{\sqrt{t}})$$

where

$$c(f, B_{\sqrt{t}}) := t^{-1/2} \sum_{\ell \geq 0} 2^{-\ell N} \left(\fint_{2^\ell B_{\sqrt{t}}} |f|\right)$$

is a constant. It follows from the pointwise Gaussian estimates for tLe^{-tL} (see the proof of Theorem 5.3) that for every ball $B_r \subset B_{\sqrt{t}}$ with radius r,

$$r\left(\fint_{2B_r} |Lg|^{p_0} d\mu\right)^{1/p_0} \leq \sqrt{t} \operatorname*{ess\,sup}_{B_{\sqrt{t}}} |Lg| \lesssim c(f, B_{\sqrt{t}}).$$

Therefore (7.10) yields

$$\left(\fint_{B_r} |\nabla g|^{p_0} d\mu\right)^{1/p_0} \lesssim \left(\fint_{2B_r} |\nabla g|^{p_0-\kappa} d\mu\right)^{1/(p_0-\kappa)} + c(f, B_{\sqrt{t}}). \tag{7.11}$$

It follows that

$$\left(\fint_{B_r} h^{p_0}\, d\mu\right)^{1/p_0} \lesssim \left(\fint_{2B_r} h^{p_0-\kappa}\, d\mu\right)^{1/(p_0-\kappa)}. \tag{7.12}$$

To see this, write

$$\left(\fint_{B_r} h^{p_0} d\mu\right)^{1/p_0} \leq \left(\fint_{B_r} |\nabla g|^{p_0} d\mu\right)^{1/p_0} + c(f, B_{\sqrt{t}}),$$

use (7.11), dominate the integrals in $|\nabla g|$ as well as $c(f, B_{\sqrt{t}})$ by integrals in h.

Now for $x \in B_{\sqrt{t}}$ consider the quantities

$$F_{p_0}(x) := \sup_{x \in B \subset 2B_{\sqrt{t}}} \left(\fint_B h^{p_0} d\mu\right)^{1/p_0}$$

and

$$F_{p_0-\kappa}(x) := \sup_{x \in B \subset 2B_{\sqrt{t}}} \left(\fint_B h^{p_0-\kappa} d\mu\right)^{1/(p_0-\kappa)},$$

where the supremum is taken over all the balls B containing x and included into $B_{\sqrt{t}}$. Let us first remark that, for $B_r \subset 2B_{\sqrt{t}}$ a ball of radius r with $\sqrt{t} \leq 4r \leq 8\sqrt{t}$, we have

$$\left(\fint_{B_r} |\nabla g|^{p_0-\kappa} d\mu\right)^{1/(p_0-\kappa)} \lesssim \left(\fint_{2B_{\sqrt{t}}} |\nabla e^{-tL} f|^{p_0} d\mu\right)^{1/p_0} \tag{7.13}$$
$$\lesssim t^{-1/2} \sum_{\ell \geq 0} e^{-c4^\ell} \left(\fint_{2^\ell B_{\sqrt{t}}} |f| d\mu\right)$$
$$\lesssim c(f, B_{\sqrt{t}}).$$

Since the constant $c(f, B_{\sqrt{t}})$ is bounded by every average of h, it follows that in the definition of F_{p_0} and $F_{p_0-\kappa}$, we can take only the supremum over balls $B_r \subset B_{\sqrt{t}}$ with $r \leq \sqrt{t}/4$.

Then, we have

$$F_{p_0}(x) \lesssim F_{p_0-\kappa}(x).$$

Indeed, for $B \subset 2B_{\sqrt{t}}$ a ball containing x, if $2B \subset 2B_{\sqrt{t}}$ then we may apply (7.12) and if $2B$ is not included in $2B_{\sqrt{t}}$ (since $x \in B \cap B_{\sqrt{t}}$) then $r(B) \simeq \sqrt{t}$ and so we directly apply the previous observation with (7.13).

We may apply Gehring's Lemma ([38, Lemma 2], [39,52] and also [57]), and we have for some $\varepsilon > 0$ that

$$\left(\fint_{B_{\sqrt{t}}} |h|^p d\mu\right)^{1/p} \lesssim \left(\fint_{2B_{\sqrt{t}}} |h|^{p_0} d\mu\right)^{1/p_0}$$

for every $p \in (p_0, p_0 + \varepsilon)$. Hence, with (7.12) applied for $r = \sqrt{t}$, it follows

$$\left(\fint_{B_{\sqrt{t}}} |h|^p d\mu\right)^{1/p} \lesssim \left(\fint_{4B_{\sqrt{t}}} |h|^{p_0-\kappa} d\mu\right)^{1/(p_0-\kappa)}.$$

Hence, we deduce that

$$\left(\fint_{B_{\sqrt{t}}} |\nabla e^{-tL} f|^p d\mu\right)^{1/p} \lesssim \left(\fint_{4B_{\sqrt{t}}} |\nabla e^{-tL} f|^{p_0-\kappa} d\mu\right)^{1/(p_0-\kappa)} + c(f, B_{\sqrt{t}}).$$

Interpolating the Davies-Gaffney estimates (2.1) with (G_{p_0}) yields $L^{p_0-\kappa} - L^{p_0-\kappa}$ off-diagonal estimates for $\sqrt{t}\nabla e^{-tL}$ and so for a large enough integer N

$$\left(\fint_{B_{\sqrt{t}}} |\sqrt{t}\nabla e^{-tL} f|^p d\mu\right)^{1/p} \lesssim \sum_{\ell \geq 0} 2^{-\ell N} \left(\fint_{2^\ell B_{\sqrt{t}}} |f|^p d\mu\right)^{1/p}.$$

By summing over a covering of M by balls with radius $\sqrt{t}$, using (VD) and taking N large enough, we then deduce that $\sqrt{t}|\nabla e^{-tL}|$ is bounded on L^p, uniformly with respect to $t > 0$, which is (G_p) as desired. □

By combining Theorems 7.1 and 7.7, the fact that (R_p) always implies (G_p) by L^p analyticity of the semigroup and the fact that the Poincaré inequality is weaker and weaker as p increases, we deduce the following statement, which encompasses Theorems 7.1 and 7.7 and extends both [4, Theorem 1.3] and [3, Theorem 0.4].

Theorem 7.8. *Assume that $(M,d,\mu,\mathcal{E})$ is a doubling metric measure Dirichlet space with a "carré du champ". If for some $p_0\in[2,\infty]$, the combination (P_{p_0}) with (G_{p_0}) holds, then there exists $p_1\in(p_0,+\infty]$ such that*

$$\{p\in(1,\infty),\ (R_p)\ holds\}=\{p\in(1,\infty),\ (G_p)\ holds\}=(1,p_1).$$

Appendix A. About the p-independence of $(H^\eta_{p,p})$

In this appendix, we study in more detail the p-independence of the property $(H^\eta_{p,p})$ for $p\in[1,+\infty]$ and $\eta\in(0,1]$. Recall that we denote by $\mathcal{M}$ the Hardy-Littlewood maximal operator as defined in (7.1), and by $\mathcal{M}_p$ the operator defined by $\mathcal{M}_p(f):=[\mathcal{M}(|f|^p)]^{1/p}$, $f\in L^1_{\text{loc}}(M,\mu)$, $p\in[1,+\infty)$. We set $\mathcal{M}_\infty(f):=\|f\|_\infty$, $f\in L^\infty(M,\mu)$.

In [32], gradient estimates for the heat semigroup are studied in the Riemannian setting, but the proofs rely only on the finite propagation speed property, therefore extend to the setting of a metric measure space with a "carré du champ". More precisely, it is proved that, under (VD) and (UE), the condition

$$\sup_{t>0}\sup_{x\in M}|B(x,\sqrt{t})|^{1-\frac{1}{q}}\|\sqrt{t}|\nabla p_t(x,\cdot)|\|_q<+\infty \tag{A.1}$$

is independent of $q\in[1,+\infty]$ and is in particular equivalent to Gaussian pointwise estimates for the gradient of the heat kernel. Since for $q=p'$

$$\sup_{x\in M}\|\sqrt{t}|\nabla p_t(x,\cdot)|\|_q=\|\sqrt{t}|\nabla e^{-tL}|\|_{p\to\infty},$$

this property can be thought of, at least in the polynomial volume growth situation $V(x,r)\simeq r^\nu$, as follows: the quantity $\|\sqrt{t}|\nabla e^{-tL}|\|_{p\to\infty}$ does not depend on the exponent $p\in[1,+\infty]$.

Even if the full version of this result in [32] is really non-trivial, it appears that a localised counterpart is indeed very easy: more precisely, the property

$$\sup_{t>0}\sqrt{t}|\nabla e^{-tL}f(x)|\lesssim\mathcal{M}_p(f)(x) \tag{A.2}$$

is p-independent. This fact directly follows by writing $\nabla e^{-tL}=\left(\nabla e^{-\frac{t}{2}L}\right)e^{-\frac{t}{2}L}$ with a semigroup $e^{-\frac{t}{2}L}$ satisfying all L^p-L^q off-diagonal estimates (since the heat kernel satisfies pointwise Gaussian estimates), so that for every $p,q\in[1,+\infty]$ with $p<q$, we have

$$\mathcal{M}_q(e^{-tL}f)(x)\lesssim\mathcal{M}_p(f)(x).$$

The estimate for $p\geq q$ follows from Hölder's inequality. In other words, the localised property (A.2) is much easier to prove than the full "global" version (A.1).

The inequality $(H^\eta_{p,p})$ is the Hölder counterpart of the L^p - L^∞ Lipschitz regularity property of the semigroup (A.1). Following the previous observation (and the results of [32], which can be extended to the situation of Hölder regularity instead of gradient estimates), it is natural to study the p-independence of $(H^\eta_{p,p})$ and to do so, we introduce localised versions of $(H^\eta_{p,p})$ as follows.

Definition A.1. *Let (M,d,μ,L) as at the beginning of Section 3 satisfy (VD) and (UE). Let $p,q \in [1,+\infty]$ and $\eta \in (0,1]$. We shall say that $(\overline{H}^{\eta}_{p,q})$ is satisfied, if for all $0 < r \le \sqrt{t}$, every ball B_r of radius, and every function $f \in L^p_{loc}(M,\mu)$,*

$$(\overline{H}^{\eta}_{p,q}) \qquad q\text{-}\operatorname{Osc}_{B_r}(e^{-tL}f) \lesssim \left(\frac{r}{\sqrt{t}}\right)^{\eta} \inf_{z\in B_{\sqrt{t}}} \mathcal{M}_p(f)(z).$$

Note that $(\overline{H}^{\eta}_{\infty,\infty}) = (H^{\eta}_{\infty,\infty})$.

With the help of this definition, we can prove the following p-independence of $(H^{\eta}_{p,p})$.

Theorem A.2. *Let (M,d,μ,L) as above satisfy (VD) and (UE). Let $\eta \in (0,1]$. The property $(\overline{H}^{\eta}_{p,p})$ is independent of $p \in [1,+\infty]$. The property "$(H^{\lambda}_{p,p})$ for every $\lambda < \eta$" is independent of $p \in [1,+\infty]$.*

The above theorem will be a direct consequence of self-improvement properties of $(H^{\eta}_{p,p})$ and $(\overline{H}^{\eta}_{p,p})$, which read as follows.

Proposition A.3. *Let (M,d,μ,L) as above satisfy (VD) and (UE). Let $p,q \in [1,+\infty]$ and $\eta \in (0,1]$. Then*

(i) $(\overline{H}^{\eta}_{p,p}) \Longrightarrow (\overline{H}^{\eta}_{1,\infty}) \Longrightarrow (\overline{H}^{\eta}_{q,q})$;

(ii) $(\overline{H}^{\eta}_{p,p}) \Longrightarrow (H^{\eta}_{p,p})$;

(iii) *For every $\lambda \in [0,\eta)$, $(H^{\eta}_{p,p}) \Longrightarrow (\overline{H}^{\lambda}_{p,p})$.*

Remark A.4. *As a consequence of Proposition A.3, the property: "there exists $\eta > 0$ such that $(H^{\eta}_{p,p})$ holds" is independent of $p \in [1,+\infty]$.*

Remark A.5. *All results of Appendix A remain true in the context of sub-Gaussian estimates, where we assume (UE_m) for some $m > 2$ instead of (UE). See Remark 3.8.*

Proof of Proposition A.3. Let us start with (i). First, we follow Proposition 3.1, and the same proof allows us to improve $(\overline{H}^{\eta}_{p,p})$ into $(\overline{H}^{\eta}_{p,\infty})$. Then, if $q \ge p$, we obtain from Jensen's inequality

$$\inf_{z\in B_{\sqrt{t}}} \mathcal{M}_p(f)(z) \le \inf_{z\in B_{\sqrt{t}}} \mathcal{M}_q(f)(z),$$

therefore

$$(\overline{H}^{\eta}_{p,\infty}) \Longrightarrow (\overline{H}^{\eta}_{q,\infty}) \Longrightarrow (\overline{H}^{\eta}_{q,q}).$$

Now let us focus on the case $q < p$. Consider $t > 0$ and set $s = \frac{t}{2}$. Let B_r be a ball of radius $r < \sqrt{t}$ and $B_{\sqrt{t}} = \frac{\sqrt{t}}{r} B_r$ the dilated ball of radius $\sqrt{t}$. If $r < \sqrt{s}$, we apply $(\overline{H}^{\eta}_{p,\infty})$ to $e^{-sL}f$, which yields

$$\operatorname*{ess\,sup}_{x,y\in B_r} \left|e^{-2sL}f(x) - e^{-2sL}f(y)\right| \lesssim \left(\frac{r}{\sqrt{s}}\right)^{\eta} \inf_{z\in B_{\sqrt{s}}} \mathcal{M}_p(e^{-sL}f)(z). \tag{A.3}$$

Using (UE) together with $t = 2s$, we then obtain

$$\operatorname*{ess\,sup}_{x,y\in B_r} \left|e^{-tL}f(x) - e^{-tL}f(y)\right| \lesssim \left(\frac{r}{\sqrt{t}}\right)^{\eta} \inf_{z\in B_{\sqrt{t}}} \mathcal{M}(f)(z),$$

which is $(\overline{H}^{\eta}_{1,\infty})$. The case $\sqrt{s} \leq r \leq \sqrt{t}$ is a direct consequence of (UE), since we have $r \simeq \sqrt{t}$ and so

$$\operatorname*{ess\,sup}_{x,y\in B_r} \left|e^{-tL}f(x) - e^{-tL}f(y)\right| \leq 2\|e^{-tL}f\|_{L^\infty(B_r)} \lesssim \|e^{-tL}f\|_{L^\infty(B_{\sqrt{t}})} \lesssim \inf_{z\in B_{\sqrt{t}}} \mathcal{M}(f)(z),$$

which yields $(\overline{H}^{\eta}_{1,\infty})$.

Now for (ii). Assume $(\overline{H}^{\eta}_{p,p})$ for some $p \in [1, +\infty]$. First, note that for $t = 2s$

$$\inf_{z\in B_{\sqrt{s}}} \mathcal{M}_p(e^{-sL}f)(z) \leq |B_{\sqrt{s}}|^{-1/p}\|e^{-sL}f\|_p + \sup_{x\in B_{\sqrt{s}}} |e^{-sL}f(x)| \lesssim |B_{\sqrt{t}}|^{-1/p}\|f\|_p,$$

where we used (UE). By applying the above estimate to (A.3), we can obtain $(H^{\eta}_{p,\infty})$ from $(\overline{H}^{\eta}_{p,p})$ with the same reasoning as in the proof of part (i). $(H^{\eta}_{p,p})$ then follows from Proposition 3.1.

Let us finally prove (iii). Assume $(H^{\eta}_{p,p})$ for some $\eta \in (0, 1]$ and $p \in [1, +\infty]$. Let B_r, $B_{\sqrt{t}}$ be a pair of concentric balls with respective radii r and $\sqrt{t}$, where $0 < r \leq \sqrt{t}$. Then we know that

$$p\text{-}\operatorname{Osc}_{B_r}(e^{-tL}f) \lesssim \left(\frac{r}{\sqrt{t}}\right)^{\eta} |B_{\sqrt{t}}|^{-1/p}\|f\|_p.$$

Let us split $f = \sum_{\ell\geq 0} f\mathbb{1}_{S_\ell(B_{\sqrt{t}})}$, and define for $\ell \geq 0$

$$I(\ell) := p\text{-}\operatorname{Osc}_{B_r}\left[e^{-tL}(f\mathbb{1}_{S_\ell(B_{\sqrt{t}})})\right].$$

We have $(H^{\lambda}_{p,p})$ for every $\lambda \in [0, \eta]$, therefore, for $\ell \leq 1$,

$$I(\ell) \lesssim \left(\frac{r}{\sqrt{t}}\right)^{\lambda} \left(\fint_{4B_{\sqrt{t}}} |f|^p d\mu\right)^{1/p} \lesssim \left(\frac{r}{\sqrt{t}}\right)^{\lambda} \inf_{z\in B_{\sqrt{t}}} \mathcal{M}_p(f)(z).$$

For $\ell \geq 2$, we similarly have

$$I(\ell) \lesssim \left(\frac{r}{\sqrt{t}}\right)^{\eta} 2^{\ell\frac{\nu}{p}} \left(\fint_{2^\ell B_{\sqrt{t}}} |f|^p d\mu\right)^{1/p}. \tag{A.4}$$

Moreover, using again (UE), we have

$$I(\ell) \leq 2\left(\fint_{B_r} \left|e^{-tL}(f\mathbb{1}_{S_\ell(B_{\sqrt{t}})})d\mu\right|^p d\mu\right)^{1/p} \lesssim e^{-c4^\ell} \left(\fint_{2^\ell B_{\sqrt{t}}} |f|^p d\mu\right)^{1/p}, \tag{A.5}$$

which yields

$$\left(\fint_{B_r} \left|e^{-tL}(f\mathbb{1}_{S_\ell(B_{\sqrt{t}})})d\mu\right|^p d\mu\right)^{1/p} \leq \|e^{-tL}(f\mathbb{1}_{S_\ell(B_{\sqrt{t}})})\|_{L^\infty(B_r)}$$

$$\leq \|e^{-tL}(f\mathbb{1}_{S_\ell(B_{\sqrt{t}})})\|_{L^\infty(B_{\sqrt{t}})} \lesssim e^{-c4^\ell} \left(\fint_{2^\ell B_{\sqrt{t}}} |f|^p d\mu\right)^{1/p}.$$

By interpolating between (A.4) and (A.5), we get for every $\lambda \in [0,\eta)$, with c_λ a constant depending on λ,

$$I(\ell) \lesssim \left(\frac{r}{\sqrt{t}}\right)^{\lambda} e^{-c_\lambda 4^\ell} \left(\fint_{2^\ell B_{\sqrt{t}}} |f|^p d\mu\right)^{1/p}.$$

By summing over $\ell \geq 0$, we obtain

$$\left(\fint_{B_r} \left| e^{-tL} f - \fint_{B_r} e^{-tL} f \, d\mu \right|^p d\mu\right)^{1/p} \leq \sum_{\ell \geq 0} I(\ell) \lesssim \left(\frac{r}{\sqrt{t}}\right)^{\lambda} \inf_{z \in B_{\sqrt{t}}} \mathcal{M}_p(f)(z),$$

which is $(\overline{H}^{\lambda}_{p,p})$. □

In [10], we will introduce and use another notion of L^p Hölder regularity of the heat semigroup, which is p-dependent and can be seen as the Hölder version of (G_p) instead of (A.1).

Appendix B. From Poincaré to De Giorgi

In this appendix, we give a self-contained proof for the fact that, under (VD), (P_2) implies the De Giorgi property (DG_2), as stated in Proposition 5.2. One method is to use the elliptic Moser iteration process from [60], see for instance [2, Sections 5 and 6]. The proof is given there in a discrete time and space setting, but adapts to our current setting. Another proof in [2] by-passes the difficult part of the Moser iteration process, namely the John-Nirenberg lemma, and uses instead the self-improvement property of Poincaré inequalities. This is the one we will present here.

In this appendix, we will assume for simplicity that $\mathrm{diam}(M) = +\infty$. If on the contrary $\mathrm{diam}(M) < +\infty$, it is enough to assume that $R \leq \delta \,\mathrm{diam}(M)$, where δ is the parameter that has to appear in (FK) in that case, and to use doubling at the end of the argument.

We first need to state a version of the Caccioppoli inequality (6.1) in L^2, but for subharmonic functions.

Lemma B.1. *Let $(M,d,\mu,\mathcal{E})$ be a doubling metric measure Dirichlet space with a "carré du champ". For every $x \in M$, $0 < r < R$ and every $u \in \mathcal{D}$ with $uLu \leq 0$ on $B(x,R)$, one has*

$$\int_{B(x,r)} |\nabla u|^2 \, d\mu \lesssim \frac{1}{(R-r)^2} \int_{B(x,R)} |u|^2 \, d\mu. \tag{B.1}$$

Proof. Consider a function χ belonging to $\mathcal{F}$, supported on $B(x,R)$, with values in $[0,1]$, and such that $\chi \equiv 1$ on $B(x,r)$ and $\||\nabla\chi|\|_\infty \lesssim (R-r)^{-1}$. Such a function can easily be built in our setting from the metric d (see for instance [45, Section 2.2.6] for details).

Since $uLu \leq 0$ on $B(x,R)$ one may write

$$\begin{aligned} 0 &\leq -\int_M \chi^2 u L u \, d\mu = -\mathcal{E}(\chi^2 u, u) \\ &= -\frac{1}{2}\mathcal{E}(\chi^2, u^2) - \int_M \chi^2 |\nabla u|^2 d\mu, \end{aligned}$$

where one uses (1.1). Consequently, by (1.2) and (1.6),

$$I := \int_M \chi^2 |\nabla u|^2 d\mu \leq \frac{1}{2} \left| \mathcal{E}(\chi^2, u^2) \right| \leq 2 \int_M \chi |u| |\nabla \chi| |\nabla u| \, d\mu$$

and we deduce by Cauchy-Schwarz that

$$I \lesssim \frac{1}{R-r} \int_M |u| |\nabla u| \chi \, d\mu \lesssim \frac{1}{R-r} \sqrt{I} \left[\int_{B(x,R)} |u|^2 \, d\mu \right]^{1/2},$$

which yields (B.1). ☐

First, the relative Faber-Krahn inequality (FK) (see Section 5 for a definition) implies an L^2 mean value property for harmonic functions.

Proposition B.2. *Let $(M, d, \mu, \mathcal{E})$ be a doubling metric measure Dirichlet space with a "carré du champ" satisfying (UE). Assume (FK). Then one has, for all $R > 0$, $x_0 \in M$, and $u \in \mathcal{F}$ harmonic in $B(x_0, R)$,*

$$\max_{x \in B(x_0, R/2)} u(x) \lesssim \left(\fint_{B(x_0,R)} u_+^2 \, d\mu \right)^{1/2}.$$

Proof. Let $x_0 \in M$, $R > 0$ and $u \in \mathcal{F}$ such that $Lu = 0$ in $B(x_0, R)$. For $h, r > 0$ define

$$M(r) = \max_{x \in B(x_0,r)} u(x), \qquad m(r) = \min_{x \in B(x_0,r)} u(x),$$
$$A(h,r) = \{x \in B(x_0, r);\ u(x) \geq h\}, \qquad a(h,r) = \mu(A(h,r)).$$

Consider $\rho < r \leq 2\rho$ with $r \leq R$. Let χ be a Lipschitz function supported on $B(x_0, r)$, that equals 1 on $B(x_0, \rho)$ and such that $\|\nabla \chi\|_\infty \lesssim (r-\rho)^{-1}$. Applying (FK) to $\chi(u-h)_+$ (which is supported on $B(x_0, r)$) for $h \in \mathbb{R}$ gives

$$\begin{aligned}
&\int_{B(x_0,\rho)} (u-h)_+^2 \, d\mu \leq \int \chi^2 (u-h)_+^2 \, d\mu \\
&\lesssim r^2 \left(\frac{a(h,r)}{V(x_0,r)} \right)^{2\beta} \left[\int_{B(x_0,r)} |\nabla (u-h)_+|^2 \, d\mu + \frac{1}{(r-\rho)^2} \int_{B(x_0,r)} (u-h)_+^2 \, d\mu \right],
\end{aligned}$$

where we use the Leibniz rule for the gradient. Then, since $u-h$ is harmonic, that is $L(u-h) = 0$ in $B(x_0, R)$, by [68, Lemma 2] we deduce that $(u-h)_+$ is nonnegative and subharmonic, that is $L(u-h)_+ \leq 0$, in $B(x_0, R)$. So $(u-h)_+ L(u-h)_+ \leq 0$ in $B(x_0, R)$. We then deduce from (B.1) that

$$\begin{aligned}
\int_{B(x_0,\rho)} (u-h)_+^2 \, d\mu &\lesssim \frac{r^2}{(r-\rho)^2} V(x_0,r)^{-2\beta} a(h,r)^{2\beta} \int_{B(x_0,r)} (u-h)_+^2 \, d\mu \\
&\lesssim \frac{\rho^2}{(r-\rho)^2} V(x_0,\rho)^{-2\beta} a(h,r)^{2\beta} \int_{B(x_0,r)} (u-h)_+^2 \, d\mu,
\end{aligned}$$

where we used the doubling property and $\rho \simeq r$. Set

$$u(h,\rho) = \int_{B(x_0,\rho)} (u-h)_+^2 \, d\mu = \int_{A(h,\rho)} (u-h)^2 \, d\mu.$$

One has

$$u(h,\rho)\lesssim \frac{\rho^2}{(r-\rho)^2}V(x_0,\rho)^{-2\beta}a(h,r)^{2\beta}u(h,r). \tag{B.2}$$

Moreover, for $h>k$,

$$(h-k)^2a(h,r)\le \int_{A(h,r)}(u-k)^2\,d\mu\le \int_{A(k,r)}(u-k)^2\,d\mu,$$

that is

$$a(h,r)\le \frac{1}{(h-k)^2}u(k,r). \tag{B.3}$$

Then (B.2) and (B.3) yield for $\rho<r\le 2\rho$

$$u(h,\rho)\le C\frac{\rho^2}{(r-\rho)^2}V(x_0,\rho)^{-2\beta}\frac{1}{(h-k)^{4\beta}}u(k,r)^{1+2\beta}. \tag{B.4}$$

Set $k_n=(1-\frac{1}{2^n})d$ and $\rho_n=\frac{R}{2}(1+\frac{1}{2^n})$, $n\in\mathbb{N}$, where

$$d=C^{\frac{1}{4\beta}}2^{\frac{|\nu\beta-1|}{2\beta}}2^{\frac{1}{4\beta^2}+\frac{2}{\beta}+1}V(x_0,R)^{-1/2}u(0,R)^{\frac12}.$$

Inequality (B.4) (which can be applied since $\rho_{n+1}<\rho_n\le 2\rho_{n+1}$) yields

$$u(k_{n+1},\rho_{n+1})\le C\frac{2^{2(n+2)+4\beta(n+1)}\rho_{n+1}^2}{R^2d^{4\beta}}V(x_0,\rho_{n+1})^{-2\beta}u(k_n,\rho_n)^{1+2\beta},$$

therefore by (VD_ν),

$$u(k_{n+1},\rho_{n+1})\le C2^{2|\nu\beta-1|}\frac{2^{2(n+2)+4\beta(n+1)}}{d^{4\beta}}V(x_0,R)^{-2\beta}u(k_n,\rho_n)^{1+2\beta}.$$

Due to the definition of d, this yields

$$u(k_{n+1},\rho_{n+1})\le 2^{2(n-2)+4\beta n-\frac{1}{\beta}}u(0,R)^{-2\beta}u(k_n,\rho_n)^{1+2\beta}. \tag{B.5}$$

From (B.5) one proves by induction that

$$u(k_n,\rho_n)\le \frac{u(0,R)}{2^{(2+\beta^{-1})n}},\qquad \forall n\in\mathbb{N}.$$

By letting n go to infinity, one concludes that $u(d,R/2)=0$. This means that for all $x\in B(x_0,R/2)$,

$$u(x)\le d=C\left(\frac{1}{V(x_0,R)}\int_{A(0,R)}u^2d\mu\right)^{1/2}.$$

□

To go further, we will need to use scaled Poincaré inequalities and their consequences. Assuming (P_2), we know that there exists $\varepsilon>0$ such that $(P_{2-\varepsilon})$ holds [53]. We will be working with the modified version

$$\left(\fint_B |f|^{2-\varepsilon}\,d\mu\right)^{\frac{1}{2-\varepsilon}}\lesssim r\frac{|B|}{|\{y\in B;\ f(y)=0\}|}\left(\fint_B|\nabla f|^{2-\varepsilon}d\mu\right)^{\frac{1}{2-\varepsilon}}, \tag{$\widetilde{P}_{2-\varepsilon}$}$$

where f ranges in $\mathcal{D}$ and B in balls in M of radius r.
The inequality $(\widetilde{P}_{2-\varepsilon})$ is a consequence of $(P_{2-\varepsilon})$, as can be seen by checking the inequality

$$\left(\int_B |f|^{2-\varepsilon}\,d\mu\right)^{\frac{1}{2-\varepsilon}} \leq C_\varepsilon \frac{|B|}{|\{y\in B;\ f(y)=0\}|}\left(\int_B |f-\fint_B f\,d\mu|^{2-\varepsilon}\,d\mu\right)^{\frac{1}{2-\varepsilon}}.$$

To prove the latter, abbreviate $B_{N(f)} := \{y\in B;\ f(y)=0\}$ and write

$$\begin{aligned}\left(\int_B |f|^{2-\varepsilon}\,d\mu\right)^{\frac{1}{2-\varepsilon}} &= \left(\int_{B\setminus B_{N(f)}} |f|^{2-\varepsilon}\,d\mu\right)^{\frac{1}{2-\varepsilon}}\\
&\leq \left(\int_{B\setminus B_{N(f)}} |f-\fint_B f\,d\mu|^{2-\varepsilon}\,d\mu\right)^{\frac{1}{2-\varepsilon}} + \left(\int_{B\setminus B_{N(f)}} \left|\fint_B f\,d\mu\right|^{2-\varepsilon}\,d\mu\right)^{\frac{1}{2-\varepsilon}}\\
&= \left(\int_{B\setminus B_{N(f)}} |f-\fint_B f\,d\mu|^{2-\varepsilon}\,d\mu\right)^{\frac{1}{2-\varepsilon}} + |B\setminus B_{N(f)}|^{\frac{1}{2-\varepsilon}}\ \left|\fint_B f\,d\mu\right|\\
&\leq \left(\int_B |f-\fint_B f\,d\mu|^{2-\varepsilon}\,d\mu\right)^{\frac{1}{2-\varepsilon}} + \left(\frac{|B\setminus B_{N(f)}|}{|B|}\right)^{\frac{1}{2-\varepsilon}}\left(\int_B |f|^{2-\varepsilon}\,d\mu\right)^{\frac{1}{2-\varepsilon}}.\end{aligned}$$

From this, we deduce

$$\begin{aligned}\left(\int_B |f|^{2-\varepsilon}\,d\mu\right)^{\frac{1}{2-\varepsilon}} &\leq \left(1-\left(\frac{|B\setminus B_{N(f)}|}{|B|}\right)^{\frac{1}{2-\varepsilon}}\right)^{-1}\left(\int_B |f-\fint_B f\,d\mu|^{2-\varepsilon}\,d\mu\right)^{\frac{1}{2-\varepsilon}}\\
&\lesssim \frac{|B|}{|B_{N(f)}|}\left(\int_B |f-\fint_B f\,d\mu|^{2-\varepsilon}\,d\mu\right)^{\frac{1}{2-\varepsilon}},\end{aligned}$$

where in the last step we have used the elementary inequality $1-(1-x)^{1/p}\geq \frac{x}{p}$ for $x\in[0,1]$ and $p\in[1,+\infty)$.

Lemma B.3. *Let $(M,d,\mu,\mathcal{E})$ be a doubling metric measure Dirichlet space with a "carré du champ" satisfying (UE) and (FK). Let $\varepsilon\in(0,1]$ and assume $(\widetilde{P}_{2-\varepsilon})$. Let $x_0\in M$, $R>0$ and $u\in\mathcal{F}$ harmonic in $B(x_0,R)$. Set*

$$k_i = M(R) - \left(\frac{M(R)-m(R)}{2^{i+1}}\right),\quad i\in\mathbb{N}.$$

Assume that $a(k_0,R/2)\leq \frac{1}{2}V(x_0,R/2)$. Then for all integer $i\in\mathbb{N}^*$

$$\frac{a(k_i,R/2)}{V(x_0,R/2)}\leq C i^{-\varepsilon/2},$$

where C does not depend on x_0, R, or u.

Proof. For $h>k>k_0$, set $v=(u-k)_+\wedge(h-k)$. By assumption,

$$\begin{aligned}|\{x\in B(x_0,R/2);\ v(x)=0\}| &= |B(x_0,R/2)\setminus A(k,R/2)|\\
&\geq |B(x_0,R/2)\setminus A(k_0,R/2)| \geq 1/2V(x_0,R/2).\end{aligned}$$

The Poincaré inequality $(\widetilde{P}_{2-\varepsilon})$ therefore yields

$$\int_{B(x_0,R/2)} |v|^{2-\varepsilon}\, d\mu \lesssim R^{2-\varepsilon} \int_{B(x_0,R/2)} |\nabla v|^{2-\varepsilon}\, d\mu.$$

Hence

$$(h-k)^{2-\varepsilon} a(h,R/2) \lesssim R^{2-\varepsilon} \int_{B(x_0,R/2)} |\nabla v|^{2-\varepsilon}\, d\mu.$$

Now

$$(h-k)^{2-\varepsilon} a(h,R/2) \lesssim R^{2-\varepsilon} \int_{A(k,R/2)\setminus A(h,R/2)} |\nabla u|^{2-\varepsilon}\, d\mu.$$

By Hölder,

$$(h-k)^{2-\varepsilon} a(h,R/2) \lesssim R^{2-\varepsilon} (a(k,R/2)-a(h,R/2))^{\varepsilon/2} \left(\int_{A(k,R/2)\setminus A(h,R/2)} |\nabla u|^2\, d\mu \right)^{\frac{2-\varepsilon}{2}}.$$

Now

$$\int_{A(k,R/2)\setminus A(h,R/2)} |\nabla u|^2\, d\mu \le \int_{A(k,R/2)} |\nabla u|^2\, d\mu = \int_{B(x_0,R/2)} |\nabla (u-k)_+|^2\, d\mu.$$

As we already observed at the beginning of the proof of Proposition B.2 in a similar situation, $(u-k)_+$ is subharmonic, therefore (B.1) yields

$$\int_{B(x_0,R/2)} |\nabla (u-k)_+|^2\, d\mu \lesssim R^{-2} \int_{B(x_0,R)} (u-k)_+^2\, d\mu \lesssim R^{-2} V(x_0,R)(M(R)-k)^2.$$

Thus

$$(h-k)^{2-\varepsilon} a(h,R/2) \lesssim V(x_0,R)^{\frac{2-\varepsilon}{2}} (M(R)-k)^{2-\varepsilon} (a(k,R/2)-a(h,R/2))^{\frac{\varepsilon}{2}}.$$

Since $k_i - k_{i-1} = \frac{M(R)-k_0}{2^i}$ and $M(R)-k_{i-1} = \frac{M(R)-k_0}{2^{i-1}}$, the above inequality yields

$$a(k_i,R/2)^{\frac{2}{\varepsilon}} \lesssim V(x_0,R)^{\frac{2-\varepsilon}{\varepsilon}} (a(k_{i-1},R/2)-a(k_i,R/2)).$$

Using the fact that $a(k_i,R/2)$ is non-increasing in i, one obtains

$$\begin{aligned} i a(k_i,R/2)^{\frac{2}{\varepsilon}} &\le \sum_{j=1}^{i} a(k_j,R/2)^{\frac{2}{\varepsilon}} \lesssim V(x_0,R)^{\frac{2-\varepsilon}{\varepsilon}} (a(k_0,R/2)-a(k_i,R/2)) \\ &\le V(x_0,R)^{\frac{2-\varepsilon}{\varepsilon}} a(k_0,R/2). \end{aligned}$$

Hence

$$\frac{a(k_i,R/2)}{V(x_0,R)} \lesssim \left(i^{-1} \frac{a(k_0,R/2)}{V(x_0,R)} \right)^{\frac{\varepsilon}{2}} \le \left(\frac{i^{-1}}{2} \right)^{\frac{\varepsilon}{2}}.$$

□

We are now in a position to deduce the elliptic regularity estimate (ER) introduced in Theorem 5.4 from (P_2).

Proposition B.4. *Let $(M,d,\mu,\mathcal{E})$ be a doubling metric measure Dirichlet space with a "carré du champ" satisfying (P_2). Then (ER) holds.*

Proof. Recall first that (FK) holds according to Remark 5.5. Fix $x_0 \in M$, $R > 0$, and let $u \in \mathcal{F}$ be harmonic in $B(x_0, R)$. Let $r \in (0, R/2]$. By applying Proposition B.2 to $u - K$ for any $K \le M(2R)$, one obtains

$$M(R/2) - K \lesssim (M(2R) - K)\left(\frac{a(K,R)}{V(x_0,R)}\right)^{1/2}. \tag{B.6}$$

According to (B.6) applied in $B(x_0, r)$ with $K = k_i := k_i(2r) = M(2r) - \frac{M(2r)-m(2r)}{2^{i+1}}$, there exists a constant C, independent of the main parameters, such that

$$M(r/2) \le k_i(2r) + C(M(2r) - k_i(2r))\left(\frac{a(k_i,r)}{V(x_0,r)}\right)^{1/2}.$$

Assume that $a(k_0, r) \le \frac{1}{2}V(x_0, r)$, otherwise work with $-u$. Since (P_2) implies $(\widetilde{P}_{2-\varepsilon})$ as we already pointed out, we can apply Lemma B.3. This yields

$$\frac{a(k_i,r)}{V(x_0,r)} \le C i^{-\varepsilon/2},$$

therefore one can choose i large enough so that

$$C\left(\frac{a(k_i,r)}{V(x_0,r)}\right)^{1/2} \le \frac{1}{2}.$$

One obtains

$$M(r/2) \le M(2r) - \frac{1}{2^{i+2}}(M(2r) - m(2r)),$$

hence

$$M(r/2) - m(r/2) \le (M(2r) - m(2r))\left(1 - \frac{1}{2^{i+2}}\right).$$

Set $\omega(r) = M(r) - m(r)$. One has

$$\omega(r/2) \le \eta\omega(2r), \qquad \forall r \in (0, R/2],$$

where $\eta = 1 - \frac{1}{2^{i+2}} \in (0,1)$. It follows that there exist $C, \alpha > 0$ such that

$$\omega(\rho) \le C\left(\frac{\rho}{R}\right)^{\alpha}\omega(R/2), \qquad \forall \rho \in (0, R/2].$$

In particular,

$$|u(x) - u(y)| \le C'\left(\frac{d(x,y)}{R}\right)^{\alpha}\max_{B(x_0,R/2)}|u|, \qquad \forall x, y \in B(x_0, R/2).$$

Now, it follows easily from Proposition B.2 that

$$\max_{B(x_0,R/2)}|u| \lesssim 2\text{-}\operatorname{Osc}_{B(x_0,R)}(u),$$

hence the claim. □

We can now prove Proposition 5.2. This will be done in two steps: first (ER) yields a De Giorgi property for harmonic functions similar to (5.1). Second one derives the full (DG_2) by classical L^2 techniques.

Proof of Proposition 5.2. Consider a function $u \in \mathcal{F}$ harmonic in $B_R = B(x_0, R)$. By Proposition B.4, we have (ER), and we have seen at the beginning of the proof of Theorem 5.4 that this implies

$$\mathrm{Osc}_{B_r}(u) \lesssim \left(\frac{r}{R}\right)^{\alpha} \mathrm{Osc}_{B_R}(u),$$

for $B_r = B(x_0, r)$ and $0 < r \le R$. Using the Caccioppoli inequality (B.1) and (P_2), we obtain

$$\left(\fint_{B(x_0,r)} |\nabla u|^2 \, d\mu\right)^{1/2} \lesssim \left(\frac{R}{r}\right)^{1-\alpha} \left(\fint_{B(x_0,R)} |\nabla u|^2 \, d\mu\right)^{1/2} \tag{B.7}$$

for $0 < r \le R/2$. If $R/2 \le r \le R$ then the inequality still holds by (VD).

We can now deduce (DG_2) as follows (cf. [39, Theorem 1.1] or [1, Theorem 3.6]). Let $f \in \mathcal{D}$, $x_0 \in M$, and $R > 0$. As in the proof of Theorem 5.4, from Lemma 5.6 (since (FK) follows from (P_2) let consider $u \in \mathcal{F}$ be harmonic on $B(x_0, R)$ such that $f - u \in \mathcal{F}$ is supported in the ball $B(x_0, R)$. From (1.1), (1.2), and (1.6), we deduce

$$\| |\nabla u| \|_{L^2(B(x_0,R))} \lesssim \| |\nabla f| \|_{L^2(B(x_0,R))} + R \|Lf\|_{L^2(B(x_0,R))}. \tag{B.8}$$

Using triangle inequality, then (VD) and (B.7), write, for $0 < r \le R$,

$$\begin{aligned}
\left(\fint_{B(x_0,r)} |\nabla f|^2 d\mu\right)^{1/2} &\le \left(\fint_{B(x_0,r)} |\nabla u|^2 d\mu\right)^{1/2} + \left(\fint_{B(x_0,r)} |\nabla (f-u)|^2 d\mu\right)^{1/2} \\
&\le \left(\fint_{B(x_0,r)} |\nabla u|^2 d\mu\right)^{1/2} + \left(\frac{R}{r}\right)^{\frac{\nu}{2}} \left(\fint_{B(x_0,R)} |\nabla (f-u)|^2 d\mu\right)^{1/2} \\
&\le \left(\frac{R}{r}\right)^{1-\alpha} \left(\fint_{B(x_0,R)} |\nabla u|^2 \, d\mu\right)^{1/2} + \left(\frac{R}{r}\right)^{\frac{\nu}{2}} \left(\fint_{B(x_0,R)} |\nabla (f-u)|^2 d\mu\right)^{1/2}.
\end{aligned}$$

Since (FK) follows from our assumptions, we can use (5.10) and together with (B.8) it follows that

$$\begin{aligned}
\left(\fint_{B(x_0,r)} |\nabla f|^2 d\mu\right)^{1/2} &\lesssim \left(\frac{R}{r}\right)^{1-\alpha} \left[\left(\fint_{B(x_0,R)} |\nabla f|^2 d\mu\right)^{1/2} + R \left(\fint_{B(x_0,R)} |Lf|^2 d\mu\right)^{1/2}\right] \\
&\quad + \left(\frac{R}{r}\right)^{\nu/2} R \left(\fint_{B(x_0,R)} |Lf|^2 d\mu\right)^{1/2},
\end{aligned}$$

hence

$$\left(\fint_{B(x_0,r)} |\nabla f|^2 d\mu\right)^{1/2} \lesssim \left(\frac{R}{r}\right)^{1-\alpha} \left(\fint_{B(x_0,R)} |\nabla f|^2 d\mu\right)^{1/2} + \left(\frac{R}{r}\right)^{\gamma} R \left(\fint_{B(x_0,R)} |Lf|^2 d\mu\right)^{1/2},$$

where $\gamma = \max\{1-\alpha, \frac{\nu}{2}\}$. Applying Lemma 5.7 and since

$$\sup_{r \le R} \left(\fint_{B(x_0,r)} |Lf|^2 d\mu\right)^{1/2} \lesssim \|Lf\|_{L^\infty(B_R)},$$

one obtains $(DG_{2,\varepsilon})$ for every $\varepsilon \in (1-\alpha, 1)$. □

Appendix C. A self-improving property for reverse Hölder inequalities

In this appendix, we shall describe a general self-improving property of reverse Hölder inequalities which is used in Proposition 7.4. These results are not new and already appeared in the literature (see e.g. [51, Theorem 2] and [49, Subsection 3.38]). We give a proof for the sake of completeness.

Consider (M,d,μ) a doubling metric measure space. Let $\mathcal{Q}$ be the collection of all balls of the ambient space M, and consider a functional $a:\mathcal{Q}\to[0,\infty)$. We say that a is regular if there exists a constant $c>0$ such that for every pair of balls $B,\tilde{B}$ with $\tilde{B}\subset B\subset 2\tilde{B}$

$$ca(\tilde{B})\leq a(B)\leq c^{-1}a(2\tilde{B}).$$

Theorem C.1. *Let $1<p<q\leq+\infty$. Consider a regular functional a. Let $\omega\in L^1_{\mathrm{loc}}(M,\mu)$ be a non-negative function such that for every ball $B\subset M$*

$$\left(\fint_B \omega^q\,d\mu\right)^{1/q}\lesssim\left(\fint_{2B}\omega^p\,d\mu\right)^{1/p}+a(B). \tag{C.1}$$

Then, for every $\eta\in(0,1)$ and every ball $B\subset M$,

$$\left(\fint_B \omega^q\,d\mu\right)^{1/q}\lesssim\left(\fint_{2B}\omega^{\eta p}\,d\mu\right)^{1/(\eta p)}+a(2B).$$

In other words, the right-hand side exponent of a reverse Hölder inequality always self-improves.

Proof. Fix $\eta\in(0,1)$. For every $\varepsilon\in(0,1)$, consider

$$K(\varepsilon,\eta):=\sup_{B\in\mathcal{Q}}\frac{\left(\fint_B\omega^p\,d\mu\right)^{1/p}}{\left(\fint_{2B}\omega^{\eta p}\,d\mu\right)^{1/(\eta p)}+a(2B)+\varepsilon\left(\fint_{2B}\omega^p\,d\mu\right)^{1/p}}.$$

It is easy to observe that $K(\varepsilon,\eta)$ is finite and bounded by ε^{-1}. We claim that $K(\varepsilon,\eta)$ is uniformly bounded, with respect to ε.

Indeed, assume that $K(\varepsilon,\eta)\geq 1$ (else there is nothing to prove) and take a ball $B\in\mathcal{Q}$. Consider $(B_i)_i$ a finite collection of balls which covers B with $\ell(B_i)\simeq\ell(B)$ and $4B_i\subset 2B$. Then

$$\left(\fint_B\omega^p\,d\mu\right)^{1/p}\lesssim\sum_i\left(\fint_{B_i}\omega^p\,d\mu\right)^{1/p}=\sum_i\left(\fint_{B_i}(\omega^\delta\omega^{1-\delta})^p\,d\mu\right)^{1/p},$$

for any $\delta\in[0,1]$. Using Hölder's inequality with the particular choice $\delta=\frac{\eta(q-p)}{q-\eta p}$ gives us

$$\left(\fint_{B_i}(\omega^\delta\omega^{1-\delta})^p\,d\mu\right)^{1/p}\leq\left(\fint_{B_i}\omega^{\eta p}\,d\mu\right)^{\frac{\delta}{\eta p}}\left(\fint_{B_i}\omega^q\,d\mu\right)^{(1-\delta)/q},$$

since then

$$\frac{1}{p}=\frac{\delta}{\eta p}+\frac{1-\delta}{q}.$$

Using (C.1), the assumption $K(\varepsilon,\eta)\geq 1$ and the fact that $4B_i\subset 2B$, one obtains

$$\left(\fint_{B_i}(\omega^\delta\omega^{1-\delta})^p\,d\mu\right)^{1/p}\lesssim\left(\fint_{B_i}\omega^{\eta p}\,d\mu\right)^{\frac{\delta}{\eta p}}\left[\left(\fint_{2B_i}\omega^p\,d\mu\right)^{1/p}+a(B_i)\right]^{1-\delta}$$

$$\lesssim\left(\fint_{B_i}\omega^{\eta p}\,d\mu\right)^{\frac{\delta}{\eta p}}K(\varepsilon,\eta)^{1-\delta}\left[\left(\fint_{4B_i}\omega^{\eta p}\,d\mu\right)^{1/(\eta p)}+\varepsilon\left(\fint_{4B_i}\omega^p\,d\mu\right)^{1/p}+a(2B)\right]^{1-\delta}.$$

By summing over the finite collection of balls (B_i), we deduce that

$$\left(\fint_B\omega^p\,d\mu\right)^{1/p}$$

$$\lesssim\left(\fint_B\omega^{\eta p}\,d\mu\right)^{\frac{\delta}{\eta p}}K(\varepsilon,\eta)^{1-\delta}\left[\left(\fint_{2B}\omega^{\eta p}\,d\mu\right)^{1/(\eta p)}+\varepsilon\left(\fint_{2B}\omega^p\,d\mu\right)^{1/p}+a(2B)\right]^{1-\delta}$$

$$\lesssim K(\varepsilon,\eta)^{1-\delta}\left[\left(\fint_{2B}\omega^{\eta p}\,d\mu\right)^{1/(\eta p)}+\varepsilon\left(\fint_{2B}\omega^p\,d\mu\right)^{1/p}+a(2B)\right].$$

Taking the supremum over all balls B then yields

$$K(\varepsilon,\eta)\lesssim K(\varepsilon,\eta)^{1-\delta},$$

which in turn yields, since $K(\varepsilon,\eta)$ is finite and $\delta>0$ due to $p<q$,

$$K(\varepsilon,\eta)\lesssim 1.$$

This last estimate is uniform with respect to ε. Hence, by letting $\varepsilon\to 0$, we obtain the desired conclusion. □

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