Local geometry of integral curves in parabolic geometries

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(joint work with Igor Zelenko)

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Canberra, 19/09/2011
Outline

1. The problem, examples and motivations
   - Definitions
   - Basic example
   - Motivations

2. Construction of the canonical moving frame
   - Distinguished curves in parabolic homogeneous spaces
   - Normalization conditions
   - Existence of the normal moving frame

3. Examples and discussion
   - Classical examples
   - Ruled surfaces
   - $G_2$ examples
   - Generalizations
Let $M = G/P$ be an arbitrary parabolic homogeneous space:

$\mathfrak{g} = \sum_{i \in \mathbb{Z}} \mathfrak{g}_i$ is a graded semisimple Lie algebra of the Lie group $G$ and $p = \sum_{i \geq 0} \mathfrak{g}$ is a parabolic subalgebra of $\mathfrak{g}$.
Integral curves in parabolic homogeneous spaces

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- $M$ is naturally equipped with an invariant bracket-generating vector distribution $D$: $D \subset TM$ is defined as a $G$-invariant distribution equal to $\mathfrak{g}_{-1} \mod \mathfrak{p}$ at $o = eP$.
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- We consider unparametrized curves in $M$, integral to the distribution $D$. 

Main questions: the natural moving frame, the number of fundamental differential invariants, the existence of the natural projective parameter on such curves.
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- Main questions: the natural moving frame, the number of fundamental differential invariants, the existence of the natural projective parameter on such curves.
Basic example

Curves in Grassmann varieties $\gamma \subset \text{Gr}(r, V)$. 

Define osculating flag:

$0 \subset \gamma \subset \gamma' \subset \ldots \subset \gamma^{(k)} \subset V$,

where $\gamma^{(t)} = \langle v_1^{(t)}, \ldots, v_r^{(t)} \rangle$,

$\gamma'^{(t)} = \langle v_1^{(t)}, \ldots, v_r^{(t)}, v_1'^{(t)}, \ldots, v_r'^{(t)} \rangle$.

This definition does not depend on the basis $\{v_i^{(t)}\}$ and the choice of the parameter $t$ on $\gamma$.

We get the natural embedding of $\gamma$ into the flag variety $F_{r_0, \ldots, r_k}(V)$, $r_i = \text{dim} \gamma^{(i)}$, and $\gamma$ becomes an integral curve in the flag variety.

More generally, the curve $0 \subset W_0^{(t)} \subset W_1^{(t)} \subset \ldots \subset W_k^{(t)} \subset V$ in $F_{r_0, \ldots, r_k}(V)$ is integral if and only if $W'_i^{(t)} \subset W_{i+1}^{(t)}$. 

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Prototype


There is a canonical moving frame $E \subset \text{SL}(3, \mathbb{R})$ such that for each section $\sigma: \gamma \to E$ the pull-back $\sigma^* \omega$ of the Maurer-Cartan form $\omega$ on $\text{SL}(3, \mathbb{R})$ is:

$$
\begin{pmatrix}
\omega_{00} & \omega_{01} & \omega_{02} \\
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-\omega_{00} & \omega_{01} & 0
\end{pmatrix}
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\end{pmatrix}
+ 
\begin{pmatrix}
0 & 0 & \omega_{02} \\
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\end{pmatrix}
$$

The first summand is a natural projective connection on the curve. It is necessarily flat and defines the natural projective parameter on the curve. The second summand defines the relative projective invariant $k = \omega_{02}/\omega_{01}$. The plane curve is locally a conic if and only if $k = 0$.

Similar results for non-degenerate curves in $\mathbb{P}^n$ are known (Wilczynski, Griffits, Green, Se-ashi).

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- Similar results for non-degenerate curves in $P^n$ are known (Wilczynski, Griffits, Green, Se-ashi).
Motivations

- The linearization of any non-linear ODE is:

\[ y^{(n+1)}(t) + p_1(t)y^{(n)}(t) + \cdots + p_{n+1}(t)y(t) = 0. \]

Geometry of such ODEs is equivalent to the geometry of projective curves in \( P^n \) (Wilczynski).
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- A \textit{pseudo-product structure on a smooth manifold} is a pair of complementary completely integrable distributions \( E, F \) such that their sum \( E \oplus F \) is bracket generating. If \( \dim E = 1, \dim F = r \) (often the case), then the linearization of \( F \) along fibers of \( \pi: M \to M/E \) is a family of (unparametrized) curves in Grassmann varieties:
  \[ \gamma \to \text{Gr}_r(T_\gamma(M/E)), \quad t \mapsto d_t\pi(F_t). \]
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\[ \gamma \to \text{Gr}_r(T \gamma(M/E)), \quad t \mapsto d_t \pi(F_t). \]

- Let \( N \subset T^*M \) be an odd-dimensional submanifold, \( \pi: T^*M \to M \), and let \( \Omega \) be the symplectic structure on \( T^*M \). Define a pair \( E \oplus F \) as \( E = \ker \Omega|_N, \ F = \ker d\pi \). Suppose \( \dim E = 1 \). Then the linearization of \( F \) along \( E \) is a curve in the isotropic Grassmannian of the symplectic group.
Type of integral curves

- The first natural invariant of such curve $\gamma$ is a type of its tangent line ($=1$-st jet) viewed as an element of $PD$, projectivization of the vector bundle $D \subset TM$. 

Types of integral curves are in one-to-one correspondence with orbits of $G$ on $P(g-1)$. Due to E.Vinberg, there is only a finite number of such orbits ("The Weyl group of a graded Lie algebra", English transl. Math. USSR, Izvestija 10 (1976)).

Type of the curve is constant in generic point, but may degenerate at singular points. We consider only integral curves of constant type.
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- Fix an element $x \in \mathfrak{g}$ of degree $-1$. We say that an integral curve $\gamma$ is of type $x$, if the lift of $\gamma$ to $PD$ lies in the orbit of the line $\mathbb{R}x$ under the action of $G$ on $PD$. Curve $\gamma$ is of type $x$ if for each point $p \in \gamma$ there exists an element $g \in G$ such that $g.o = p$ and $g_*(x) \subset T_p\gamma$.

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- Element $x$ can always be completed by elements $h \in g_0$, $y \in g_1$ to the basis of $\mathfrak{sl}_2$ subalgebra (Morozov, Vinberg).
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- The corresponding subgroup acts transitively on the distinguished curve. Thus, any distinguished curve is always rational curve or its cover. The projective parameter on it does not depend on the choice of $h$ and $y$. 
Let $S$ be the symmetry group of $\gamma_0$. The corresponding subalgebra $\mathfrak{s} \subset \mathfrak{g}$ is a graded subalgebra in $\mathfrak{g}$ of the form:

- $s_i = 0$ for $i \leq -2$;
- $s_{-1} = \langle x \rangle$;
- $s_i = (\text{ad } x)^{-1}(s_{i-1}) = \{ u \in \mathfrak{g}_i \mid [x, u] \in s_{i-1} \}$ for all $i \geq 0$. 

It is characterized as a largest graded subalgebra in $\mathfrak{g}$ such that its negative part is $R_x$. $\mathfrak{s} = \langle x, h, y \rangle \oplus n$, where $n$ is the largest ideal of $\mathfrak{s}$ concentrated in the non-negative degree. $\mathfrak{s}$ is often reductive: rational normal curves in projective spaces, conformal circles, etc. In this case $n = Z_{\mathfrak{g}}(x) \subset \mathfrak{g}_0$. 

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Symmetry algebras of distinguished curves

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Notion of a normal moving frame

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- $\pi: G \to G/P$ is the standard principle $P$-bundle

Moving frame is (any) subbundle of this bundle:

We construct a normal moving frame by imposing conditions on $\omega(T_pE) \subset g$ for $p \in E$.

Example (Cartan) of such conditions for projective curves in $P^2$:

$$\omega|_E = \begin{bmatrix}
\omega_{00} & \omega_{01} & \omega_{02} \\
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Normalization conditions

- Define normalization conditions for any curve of type $x$ as a graded subspace $W \subset g$ such that:
  
  (1) $W_i = 0$ for $i < 0$;
  
  (2) $W_i$ is complementary to $s_i + [x, g_{i+1}]$;
  
  (3) $W$ is invariant with respect to $s^{(0)} = \sum_{i \geq 0} s_i$. 

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  3. $W$ is invariant with respect to $s^{(0)} = \sum_{i \geq 0} s_i$.

- Such $W$ may or may not exist. It always exists if $s$ is reductive:

  $$W = \{ u \in s_{\geq 0}^\perp \mid [u, y] = 0 \},$$

  where $s^\perp$ is the orthogonal complement to $s$ w.r.t. to Killing form of $\mathfrak{g}$, and $s_{\geq 0}^\perp$ is its part of non-negative degree.
Main result

Theorem

Fix normalization conditions for a given integral curve type \( x \in g_{-1} \). Then there exists a unique moving frame \( E \rightarrow \gamma \) for any curve \( \gamma \) of type \( x \) such that:

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\omega(T_pE) \subset s \oplus W \quad \text{for all } p \in E.
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Decompose \( \omega|_E \) as \( \omega_s + \omega_W \) correspondingly.
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Decompose $\omega|_E$ as $\omega_\mathfrak{s} + \omega_W$ correspondingly.

- The form $\omega_\mathfrak{s}$ is a flat projective connection on $\gamma$. It defines the canonical projective parameter on $\gamma$. 

Main result

Theorem

Fix normalization conditions for a given integral curve type $x \in g_{-1}$. Then there exists a unique moving frame $E \to \gamma$ for any curve $\gamma$ of type $x$ such that:

$$\omega(T_pE) \subset s \oplus W \quad \text{for all } p \in E.$$  

Decompose $\omega|_E$ as $\omega_s + \omega_W$ correspondingly.

- The form $\omega_s$ is a flat projective connection on $\gamma$. It defines the canonical projective parameter on $\gamma$.

- The form $\omega_W$ is a vertical equivariant form defining the complete system of fundamental invariants of $\gamma$. In particular, $\gamma$ is locally equivalent to the distinguished curve of type $x$ if and only if $\omega_W = 0$. 

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Classical examples

- Non-degenerate curves in $P^n$ can be naturally lifted to integral curves in the flag variety $F_{1,2,...,n}(\mathbb{R}^{n+1})$. The corresponding curve type is given by:

$$x = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix}$$

Distinguished curves are osculating flags of rational normal curves. Symmetry algebra is $s = \langle x, h, y \rangle = sl_2$. Normalization condition $W = \langle y^2, \ldots, y^n \rangle$. 
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- Curves of generic type in $\text{Gr}_n(\mathbb{R}^{rn})$ for any $r \geq 2$. They correspond to linear systems of $r$ ODEs of order $n$. (Se-ashi)
No invariant normalization conditions

- What if invariant normalization conditions do not exist?

Take any graded subspace $W$ complementary to $s + [x, g]$. We can still define the normal moving frame bundle, but it is no longer a principal fiber bundle. Thus, Cartan connection does not survive, and projective parametrization is not defined uniformly for all curves of type $x$.

Smallest example without invariant space $W$ is curves in $\text{Gr}_2(\mathbb{R}^5) = \text{ruled surfaces in } \mathbb{P}^4$. More generally, the normalization condition does exist for ruled surfaces in $\mathbb{P}^{2k+1}$ and fails to exist for ruled surfaces in $\mathbb{P}^{2k}$ for generic curve type.
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- $g_{-1} \equiv S^3(\mathbb{R}^2)$ under the irreducible action of $G_0 \equiv GL(2, \mathbb{R})$;

Three curve types depending on the multiplicity of roots of cubic polynomial. Normalization conditions always exist:

- Triple root: curve is naturally lifted to $G/B$; after the lift (in the non-degenerate case) we get the symmetry algebra $s = sl_2$ and 1 fundamental invariant.
- Double root: symmetry algebra $s$ is 5-dimensional and is non-reductive. Yet, normalization conditions do exist, and there are 2 fundamental invariants.
- No multiple roots: symmetry algebra $s = sl_2$, there are 3 fundamental invariants.
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Generalizations

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- **Integral submanifolds of higher dimension in parabolic homogeneous spaces.** Needs modification of normalization conditions: \( W \) is an \( s^{(0)} \)-invariant splitting of the exact sequence:

\[
0 \to B_+^1(s_{-1}, g/s) \to Z_+^1(s_{-1}, g/s) \to H_+^1(s_{-1}, g/s) \to 0.
\]

The rest works similarly. In particular, if \( H_+^1(s_{-1}, g/s) = 0 \), all submanifolds of type \( s_{-1} \subset g_{-1} \) are flat (= locally equivalent to the orbit of \( \exp(s_{-1}) \)).