1) An example

trace-free part \( (\nabla a \sigma^b) = 0 \)

\[
\nabla_a \sigma_b{}^c - \delta_a^b \mu = \nabla_a (\delta_b{}^c - \delta_a{}^c) \mu
\]

\[
(1 - \eta) \nabla_a \mu = R_{abcd} \sigma^d
\]

\[
\nabla_a \mu + \frac{1}{n-1} R_{abcd} \sigma^d = 0
\]

So we conclude that

\[
\nabla_a \sigma_b{}^c - \delta_a^b \mu = 0
\]

\[
\nabla_a \mu + P_{abcd} \sigma^d = 0
\]

Compute:

\[
(\nabla_a \nabla_b - \nabla_b \nabla_a) (\sigma^c) = W_{abcd} \sigma^d (\nabla_a \delta^c_{bc} - \nabla_b \delta^c_{ac}) \sigma^c
\]

tractor curvature

\[
R_{abcd} = W_{abcd} + \text{trace pieces}
\]

we could have deduced this at \( \star \) (but we chose not to do).

(optimism begins to fade.)

For simplicity, let us suppose (w.l.g. in some sense) that the
Ricci curvature is symmetric

\[
R_{abcd} = W_{abcd} + \delta^c_a \delta^d_b - \delta^c_b \delta^d_a
\]

\[
P_{ab} = \text{Schouten tensor}
\]

projective
2) Another example

\[ \nabla_{(a} \sigma_{b)} = 0. \]

\[ \nabla_{a} \sigma_{b} = \mu_{ab} \] skew

\[ \Rightarrow \nabla_{a} \mu_{bc} = \nabla_{c} \mu_{ba} - \nabla_{b} \mu_{ca} \]

\[ = (\nabla_{c} \nabla_{b} - \nabla_{b} \nabla_{c}) \sigma_{a} \]

\[ = R_{bcd} a \sigma_{d} \]

\[ = (\delta_{d}^{c} \sigma_{a} + \delta_{b}^{d} P_{ca} - \delta_{c}^{d} P_{ba}) \sigma_{d} \]

\[ \Rightarrow \nabla_{a} (\sigma_{b}^{\mu_{bc}}) = (\nabla_{a} \sigma_{b} - \mu_{ab}^{d} \sigma_{d}) = 0 \]

The tractor connection

\[ 1^{2} (T) = \Lambda^{2} (\mathcal{O}) = \frac{\Lambda^{1}}{\Lambda^{2}} \]

a projectively invariant correction term

(Optimism is on the way out.)

Many more examples show that this is the norm rather than the exception (Dennis says that Robert says that Cartan does not construct a Cartan connection in his 5 variables paper).
General theory. \( \sigma(D) : \Lambda' \otimes E \to F \) (1st order for simplicity)

\[ \sigma(D) : \Lambda' \otimes E \to F \text{ surjective} \]

\[ K = \ker \sigma(D) \]

In our examples:

\[ K = \Lambda^2 \to \Lambda' \otimes \Lambda' \rightarrow \Lambda' \otimes E \]

\[ \Lambda' \to O^2 \Lambda' \]

\[ K = \Lambda^2 \to \Lambda' \otimes \Lambda' \to O^2 \Lambda' \]

Generally, \( 0 \to K \to K' \to E \to 0 \)

\[ 0 \to \Lambda' \otimes E \to J' \to E \to 0 \]

\[ 0 \to 0 \]

\[ F = F \]

\[ 0 \to 0 \]

Choose a splitting.

Induced splitting.

But symbol is no longer surjective:

\[ \Lambda' \otimes E \rightarrow \Lambda' \otimes E \]

\[ \Lambda' \otimes K \rightarrow \Lambda^2 \otimes E \]

\[ \Lambda' \otimes \Lambda' \otimes E \]

\[ (\sigma') \rightarrow (\nabla \sigma - \mu) \]

\[ (\mu) \rightarrow (\nabla \mu - \kappa \sigma) \]

\[ \text{in fact } D \text{ is canonically constructed from } D' \]

So force the issue by writing \( F' \) for \( \Lambda' \otimes E \) and \( \otimes \)

\[ \otimes \]

\[ \partial (\Lambda' \otimes K) \]

\[ \Lambda' \otimes E \]

and we've constructed \( D' : E' \to F' \)

Then \( K' = \ker \sigma(D') = (\Lambda' \otimes K) \otimes \otimes \Lambda' \otimes E \)

and now iterate.
In our examples:

1) \( K = \Lambda^0 \otimes \Lambda^0 T \)
\[ \Lambda^0 \otimes K \hookrightarrow \Lambda^0 \otimes \Lambda^0 \otimes E \rightarrow \Lambda^2 \otimes E \]
\[ \Lambda^2 \rightarrow \Lambda^0 \otimes \Lambda^0 \otimes E \rightarrow \Lambda^2 \otimes T \]
\( \mu a \mapsto \mu a \delta \zeta \rightarrow \mu [a \delta \zeta] \) injective so \( K' = 0 \)

and there's a canonical splitting

\[ \Lambda^2 \otimes T = \Lambda^2 \oplus \Lambda^2 \otimes T \]

so everything is true: prolongation gives a unique connection.

2) \( K = \Lambda^2 \hookrightarrow \Lambda^0 \otimes \Lambda^0 \)
\[ \Lambda^0 \otimes \Lambda^2 \rightarrow \Lambda^0 \otimes \Lambda^0 \otimes \Lambda^1 \rightarrow \Lambda^2 \otimes \Lambda^1 \]
\[ \oplus \]

is an isomorphism. No choice entailed in splitting, and get a connection.

More general theory

\[
\begin{align*}
K' & = \Lambda^0 \otimes E \\
K'' & = (\Lambda^0 \otimes K) \otimes \Lambda^2 \otimes E \\
K^{(l)} & = (\Lambda^0 \otimes K) \otimes \Lambda^2 \otimes E \\
K^{(l)} & = 0 \text{ for } l > 0 \Rightarrow \text{finite type and} \end{align*}
\]

They at least, prolongation terminates and we obtain a connection on

\[ E + K + K' + \cdots + K^{(l)} + \cdots \]

Hard algebra - last form than science. Philip Griffiths
Good news In some circumstances Kostant's BBW theorem computes \( K(2) \) (and much more because).

(\textbf{beware of the order we were playing the move})

**Example 1**

\[
\begin{align*}
000000 & = \sigma_6 + 100000 \\
\sigma_6 & \downarrow \\
-210001 & \downarrow \\
X & \downarrow
\end{align*}
\]

**Example 2**

\[
\begin{align*}
010000 & = X10000 + 101000 \\
X & \downarrow
\end{align*}
\]

Bad news Even in these parabolic-invariant cases, the follow-your-nose construction of the connection can fail

\textbf{EG conformal Killing equation}

\[ \nabla_\sigma \sigma_o = 0 \]

\[ \nabla_\sigma \sigma_o = \mu_{ab} + \delta_{ab} \]

Following your nose leads more down the projective path that the conformal

- seeExamples 0.0.1 \text{ awkward}
- 0.0.2 \text{ good}

This goodness is generalized by Hammerl, Samberg, Souček, Šilhan.
Here's the same example but done in two ways, one of which is better!

0.0.1. Example. Consider the partial differential equation

\( \nabla_\sigma \sigma_b = 0 \).

Even in this simple case, prolongation is already fairly involved. The details can be omitted on first reading and the main features are described in §0.0.3 below. We can rewrite (1) as

\( \nabla a_\sigma b = \mu_a + \nu g_{ab} \) where \( \mu_a \) is skew.

Then \( \nabla [a_\mu bc] = 0 \), so

\( \nabla a_\mu b = \nabla c_\mu a \sigma d - \nabla b_\mu c a = \nabla c(\nabla_b \sigma_a - \nu g_{ba}) - \nabla_b(\nabla_c \sigma_a - \nu g_{ca}) = R_{bc d} a_\sigma d - g_{ab} \nabla c_\nu + g_{ac} \nabla b_\nu. \)

Tracing over \( a \) and \( b \) gives

\( \nabla b_\mu b = -R_{d} a_\sigma d - (n - 1) \nabla c_\nu. \)

Let us introduce \( \rho_c = \frac{1}{n-1} \nabla b_\mu b \) and rearrange this last equation as

\( \nabla a_\nu = -\rho_a - \frac{1}{n-1} R_{a} b_\sigma b. \)

It may be used to eliminate \( \nabla c_\nu \) from (3) to obtain

\( \nabla a_\mu b = g_{ab} \rho_a - g_{ac} \rho_b + K_{abc}, \)

where

\( K_{abc} = R_{bc d} a_\sigma d + \frac{1}{n-1} g_{ab} R_{a} c_\sigma d - \frac{1}{n-1} g_{ac} R_{d} a_\sigma d. \)

Notice that \( K_{abc} \) is totally trace-free. Now apply \( \nabla d \) to (5) and skew over \( a \) and \( d \) to obtain

\( R_{da e} b_\mu c - R_{da e} c_\mu b = \nabla d K_{abc} - \nabla a K_{dcb} + g_{ab} \nabla d \rho_c - g_{db} \nabla a \rho_c - g_{ac} \nabla d \rho_b + g_{dc} \nabla a \rho_b. \)

Tracing over \( a \) and \( b \) gives

\( R_{d e} c_\mu c - R_{d e} c_\mu b = -\nabla b K_{dcb} + (n - 2) \nabla d \rho_c + g_{dc} \nabla b \rho_b \)

but tracing again, over \( c \) and \( d \), gives \( 0 = 2(n - 1) \nabla b \rho_b \). Therefore,

\( \nabla a_\rho b = \frac{1}{n-2} (R_{a} c_\mu b - R_{a} c d b_\mu c - \nabla c K_{abc}). \)

At this point it is clear that the system has closed: it comprises (2), (4), (5), and in (7) one has to expand \( \nabla c K_{abc} \) using (6) and (2).
0.0.2. Example. Consider the partial differential equation

\[ \nabla_{(a}\sigma_{b)} = 0. \]

Even in this simple case, prolongation is already fairly involved. The details can be omitted on first reading and the main features are described in §0.0.3 below. We can rewrite (8) as

\[ \nabla_{a}\sigma_{b} = \mu_{ab} + \nu g_{ab} \quad \text{where } \mu_{ab} \text{ is skew.} \]

Then \( \nabla_{a\mu_{bc}} = 0, \) so

\[ \nabla_{a}\mu_{bc} = \nabla_{c}\mu_{ba} - \nabla_{b}\mu_{ca} = \nabla_{c}(\nabla_{b}\sigma_{a} - \nu g_{ba}) - \nabla_{b}(\nabla_{a}\sigma_{c} - \nu g_{ca}) = R_{bc\ a\ d} - g_{ab}\nabla_{c}\nu + g_{ac}\nabla_{b}\nu. \]

If we introduce the totally trace-free Weyl curvature tensor \( C_{abcd} \) and symmetric Schouten tensor \( P_{ab} \) by

\[ R_{abcd} = C_{abcd} + P_{ac}g_{bd} - P_{bc}g_{ad} + P_{bd}g_{ac} - P_{ad}g_{bc}, \]

then (10) may be conveniently rewritten as

\[ \nabla_{a}\mu_{bc} = g_{ab}\rho_{c} - g_{ac}\rho_{b} - P_{ab}\sigma_{c} + P_{ac}\sigma_{b} + C_{bc\ d}\sigma_{d}, \]

where

\[ \nabla_{a}\nu = -\rho_{a} - P_{a\ b}\sigma_{b}. \]

Now apply \( \nabla_{d} \) to (12) and skew over \( a \) and \( d \) to obtain

\[ R_{da\ e\ b\ c} - R_{da\ c\ b\ e} = g_{ab}\nabla_{d}\rho_{c} - g_{db}\nabla_{a}\rho_{c} - g_{ac}\nabla_{d}\rho_{b} + g_{dc}\nabla_{a}\rho_{b} - P_{ab}\nabla_{d}\sigma_{c} + P_{db}\nabla_{a}\sigma_{c} + P_{ac}\nabla_{d}\sigma_{b} - P_{dc}\nabla_{a}\sigma_{b} - Y_{dab}\sigma_{c} + Y_{dac}\sigma_{b} - (\nabla_{d}C_{bc\ e})\sigma_{e} + C_{bc\ a\ d}\sigma_{e} - C_{bc\ d\ a}\sigma_{e}, \]

where

\[ Y_{abc} = \nabla_{a}P_{bc} - \nabla_{b}P_{ac}. \]

is the Cotton-York tensor. If we trace this equation over \( a \) and \( b \), substituting from (11) for \( R_{abcd} \) and using the Bianchi identity \( \nabla^{b}C_{bcde} = (n - 3)Y_{dec}, \) we find that

\[ (n - 2)\nabla_{d}\rho_{c} + g_{cd}\nabla^{a}\rho_{a} = (n - 2)P_{cd}\nu + g_{cd}P\nu - (n - 2)P_{d\ b}\mu_{bc} + (n - 2)Y_{d\ c\ b}, \]

where \( P = P_{a\ a}. \) If we trace this equation over \( a \) and \( b \) and substitute back in, then we discover that

\[ \nabla_{a}\rho_{b} = P_{a\ e}\mu_{b\ c} + P_{ab}\nu + Y_{ab\ c}\sigma_{c}, \]

and the system has closed: it comprises (9), (12), (13), and (14).
0.0.3. Discussion. The conclusion is that (1) is equivalent to \( \tilde{\nabla} \Sigma = 0 \) where
\[
\Sigma = \begin{pmatrix} \sigma_b & \mu_{bc} & \nu \\ \rho_b & & \end{pmatrix}
\]
is a section of the bundle \( V = \bigoplus \Lambda^l \Lambda^0 \) and \( \tilde{\nabla} : V \to \Lambda^1 \otimes V \) is an explicit connection of the form
\[
(15) \quad \tilde{\nabla} \begin{pmatrix} \sigma & \mu & \nu \\ \rho & & \end{pmatrix} = \begin{pmatrix} \nabla \sigma - \mu - \nu \\ \nabla \mu - \rho - R \nabla \sigma \\ \nabla \rho - \nabla \mu - R \nabla \nu - (\nabla R) \nabla \sigma \\ \end{pmatrix},
\]
where each \( \nabla \bullet \) indicates an appropriate linear combination of contractions of its ingredients.

The equation (1) is well-known. It says that the vector field \( \sigma^a \) is a conformal Killing field—its flow preserves the metric up to scale. From this geometric interpretation it follows easily that the space of solutions is bounded by \( \text{dim} \mathfrak{so}(n+1,1) \) since \( \mathfrak{so}(n+1,1) \) is the conformal algebra in the flat case. This bound is confirmed by the technique of prolongation:
\[
\text{rank} V = 2 \text{rank} \Lambda^1 + \text{rank} \Lambda^2 + \text{rank} \Lambda^0 = 2n + \frac{n(n-1)}{2} + 1 = \frac{(n+1)(n+2)}{2}.
\]
In [1], Semmelmann uses this technique to establish similar bounds on the dimension of spaces of conformal Killing forms. Specifically, he finds an explicit connection (also having the form (15)) on the bundle
\[
V = \bigoplus \Lambda^{p+1} \bigoplus \Lambda^p \bigoplus \Lambda^{p-1} \bigoplus \Lambda^0 \bigoplus \Lambda^1 \bigoplus \Lambda^2 \bigoplus \Lambda^0
\]
with rank \( \binom{n+2}{p+1} \), so that conformal Killing \( p \)-forms are equivalent to parallel sections of this bundle.

However! Example 0.0.2 is better since it may be written as \( \tilde{\nabla}_a \Sigma = 0 \) where \( \tilde{\nabla}_a \) is the connection on the bundle \( V \) given by
\[
\tilde{\nabla}_a \begin{pmatrix} \sigma_b & \mu_{bc} & \nu \\ \rho_b & & \end{pmatrix} = D_a \begin{pmatrix} \sigma_b & \mu_{bc} & \nu \\ \rho_b & & \end{pmatrix} - \begin{pmatrix} C_{bc} d^d \sigma_d \\ Y_{abc} \sigma_c \\ 0 \\ 0 \\ \end{pmatrix},
\]
where we recognise
\[
D_a \begin{pmatrix} \sigma_b & \mu_{bc} & \nu \\ \rho_b & & \end{pmatrix} = \begin{pmatrix} \nabla_a \sigma_b - \mu_{ab} - \nu g_{ab} \\ \nabla_a \rho_b - g_{ac} \rho_c + P_{ab} \sigma_c - P_{ac} \sigma_b \\ \nabla_a \nu + \rho_a + P_a b \sigma_b \\ \end{pmatrix}
\]
as the invariant tractor connection and the other curvature correction term as being conformally invariant to boot. So this one is gloriously orange whilst Example 0.0.1 is a sickly puce at best.

References