Representation theory and the X-ray transform

Projective differential geometry

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Summary of Lecture 1

Curvature on $\mathbb{RP}_n$

$$R_{abcd} = g_{ac}g_{bd} - g_{bc}g_{ad}$$

This Lecture!

$$\nabla_{[a}\nabla_{b]}\mu_c = g_{c[a}\mu_{b]}$$

Curvature on $\mathbb{CP}_n$

$$R_{abcd} = g_{ac}g_{bd} - g_{bc}g_{ad} + J_{ac}J_{bd} - J_{bc}J_{ad} + 2J_{ab}J_{cd}$$

Model embeddings

$\mu : \mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$ totally geodesic

2-form lemma

$$\mu^*\psi_{ab} = 0\ \forall\mu \quad\Leftrightarrow\quad \psi^{\perp}_{ab} = 0$$

Curvature lemma

$$\mu^*\psi_{abcd} = 0\ \forall\mu \quad\Leftrightarrow\quad \psi^{\perp}_{abcd} = 0$$

Topics

- Tractor bundle and connection on $\mathbb{RP}_n$
- Projective differential geometry
- Projective tractor bundles and connections
- Bernstein-Gelfand-Gelfand resolutions on $\mathbb{RP}_n$
- Lie algebra cohomology
- Range of the Killing operator
- Higher Killing operators

Tractors on real projective space

Consider the connection on $\mathbb{T} \equiv \Lambda^0 \oplus \Lambda^1$ given by

$$\begin{bmatrix} \sigma \\ \mu_b \end{bmatrix} \stackrel{\nabla_a}{\longmapsto} \begin{bmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + g_{ab} \sigma \end{bmatrix}$$

Calculate

$$\nabla_a \nabla_b \begin{bmatrix} \sigma \\ \mu_c \end{bmatrix} = \nabla_a \begin{bmatrix} \nabla_b \sigma - \mu_b \\ \nabla_b \mu_c + g_{bc} \sigma \end{bmatrix}$$

$$= \begin{bmatrix} \nabla_a(\nabla_b \sigma - \mu_b) - \nabla_b \mu_a - g_{ba} \sigma \\ \nabla_a(\nabla_b \mu_c + g_{bc} \sigma) + g_{ac}(\nabla_b \sigma - \mu_b) \end{bmatrix}$$

$$\Rightarrow \nabla_{[a} \nabla_{b]} \begin{bmatrix} \sigma \\ \mu_c \end{bmatrix} = \begin{bmatrix} 0 \\ \nabla_{[a} \nabla_{b]} \mu_c - g_{c[a} \mu_{b]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Flat!

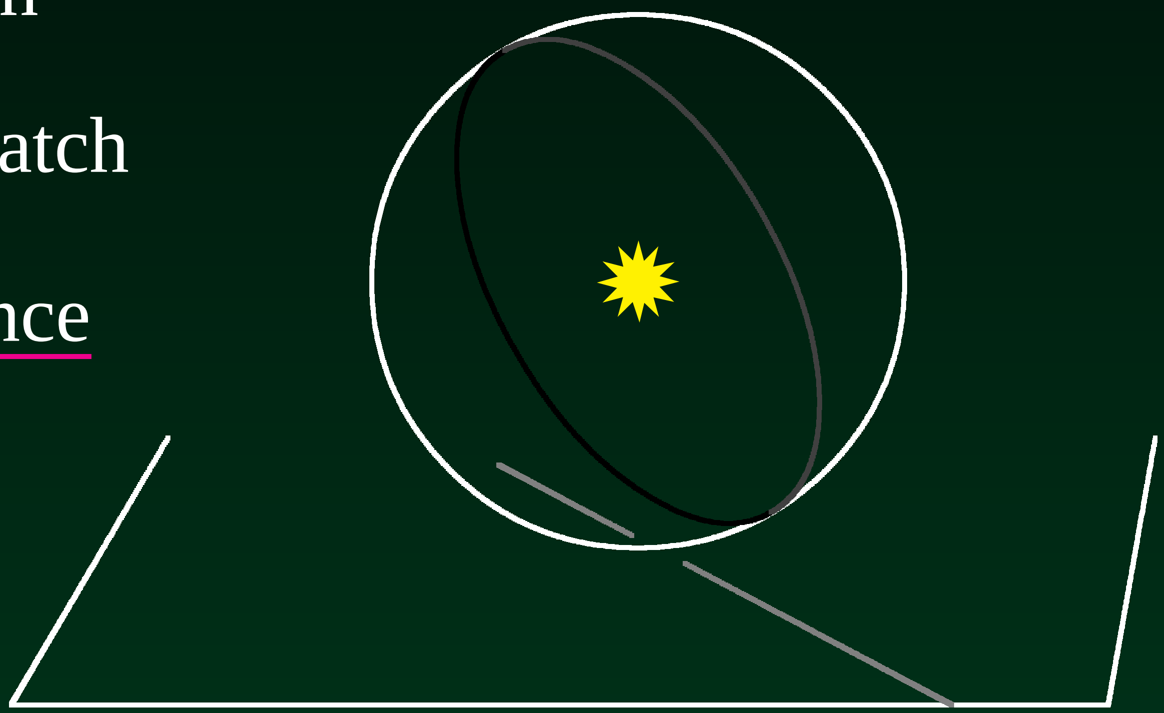
Projective differential geometry

Suppose ∇_a and $\hat{\nabla}_a$ are torsion-free affine connections.

Def$^{\text{n}}$ ∇_a and $\hat{\nabla}_a$ are projectively equivalent iff they have the same geodesics (as unparameterised curves).

EG The round sphere is projectively flat by central projection

Affine coördinate patch $\mathbb{R}^n \hookrightarrow \mathbb{RP}_n$ is a projective equivalence



Projective tractors

- Can always choose ∇_a s.t. R_{ab} is symmetric
- Then define a connection on $\mathbb{T} \equiv \Lambda^0 \oplus \Lambda^1$ by

$$\left[\begin{array}{c}\sigma\\ \mu_b\end{array}\right] \stackrel{\nabla_a}{\longmapsto} \left[\begin{array}{c}\nabla_a\sigma - \mu_a\\ \nabla_a\mu_b + \mathrm{P}_{ab}\sigma\end{array}\right] \qquad \mathrm{P}_{ab} \equiv \tfrac{1}{n-1}R_{ab}$$

- It is projectively invariant (T.Y. Thomas)
- Equivalent to a Cartan connection (É. Cartan)
- It is flat iff ∇_a is projectively flat
- Hence the flat tractor connection on $\mathbb{RP}_n$
- NB: $\mathbb{RP}_n$ is a homogeneous space $\mathrm{SL}(n+1,\mathbb{R})/P$

Back to real projective space

Recall $\mathbb{T} \equiv \Lambda^0 \oplus \Lambda^1$ with its flat connection.

Thus, $\mathbb{V} \equiv \Lambda^2\mathbb{T} = \Lambda^1 \oplus \Lambda^2$ also has a flat connection

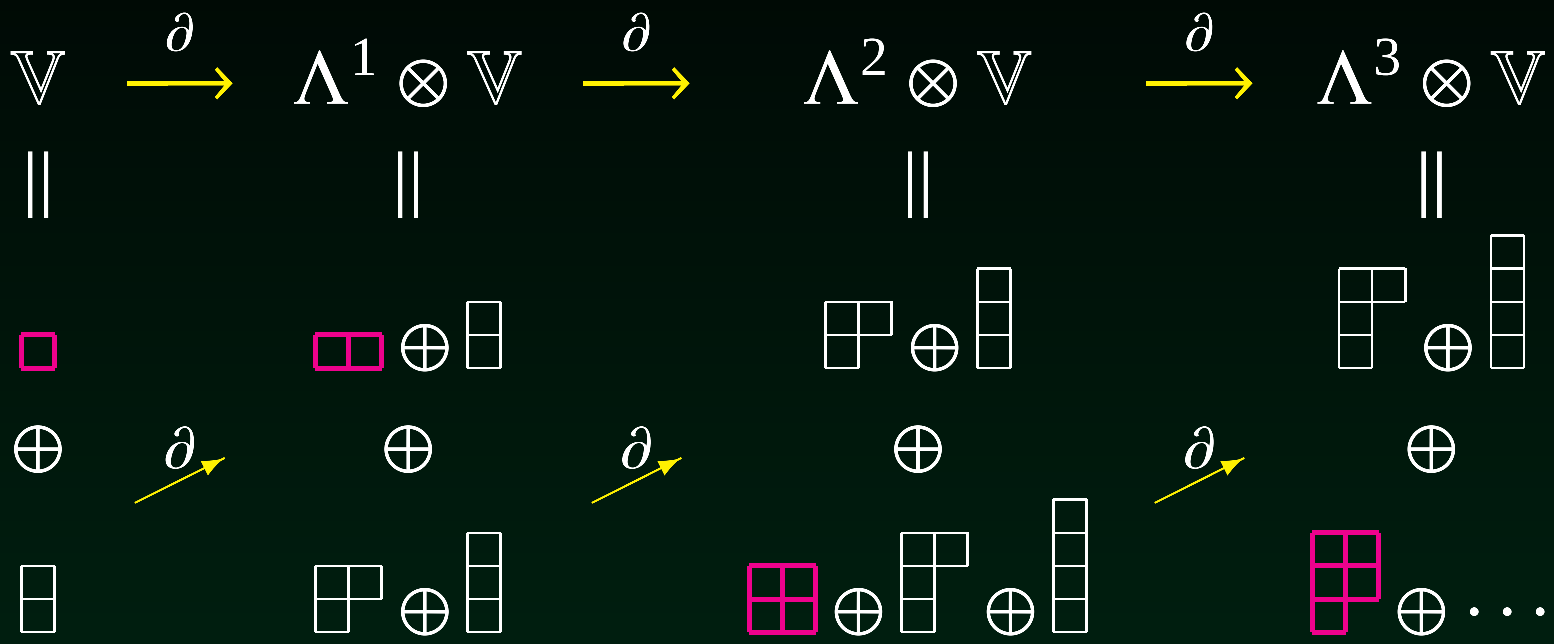
$$\left[\begin{array}{c}\sigma_b\\ \mu_{bc}\end{array}\right] \stackrel{\nabla_a}{\longmapsto} \left[\begin{array}{c}\nabla_a\sigma_b - \mu_{ab}\\ \nabla_a\mu_{bc} + g_{ab}\sigma_c - g_{ac}\sigma_b\end{array}\right].$$

$$\begin{array}{ccccccccc}
\mathbb{V} & \xrightarrow{\nabla} & \Lambda^1\otimes\mathbb{V} & \xrightarrow{\nabla} & \Lambda^2\otimes\mathbb{V} & \xrightarrow{\nabla} & \Lambda^3\otimes\mathbb{V} & \xrightarrow{\nabla} & \cdots\\
\| & & \| & & \| & & \| & & \\
\Lambda^1 & & \Lambda^1\otimes\Lambda^1 & & \Lambda^2\otimes\Lambda^1 & & \Lambda^3\otimes\Lambda^1 & & \cdots\\
\oplus & & \oplus & & \oplus & & \oplus & & \\
\Lambda^2 & & \Lambda^1\otimes\Lambda^2 & & \Lambda^2\otimes\Lambda^2 & & \Lambda^3\otimes\Lambda^2 & & \cdots
\end{array}$$

NB

$$\Lambda^p\otimes\Lambda^2 \ni \mu_{a\cdots bcd} \stackrel{\partial}{\longmapsto} \mu_{[a\cdots bc]d} \in \Lambda^{p+1}\otimes\Lambda^1$$

Decompose into irreducibles



Lie algebra cohomology! (Kostant 1961)

Suspend disbelief

EG: $\ker : \Lambda^2 \otimes \Lambda^2 \xrightarrow{\partial} \Lambda^3 \otimes \Lambda^1$

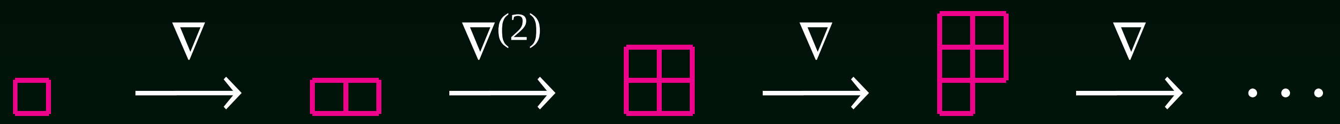
$= \{\psi_{abcd} = \psi_{[ab][cd]} \text{ s.t. } \psi_{[abc]d} = 0\}$

$=$ {Riemann curvature tensors}!

BGG resolutions

Diagram chasing $\rightsquigarrow$
locally exact complexes

Riemannian deformation



$$\Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \Lambda^3 \xrightarrow{d} \cdots$$

Bernstein-Gelfand-Gelfand resolutions.

de Rham

$$\mathbb{RP}_n = \mathrm{SL}(n+1,\mathbb{R}) / \left\{ \begin{bmatrix} * & * \cdots * \\ 0 & \\ \vdots & * \\ 0 & \end{bmatrix} \right\} = G/P,$$

where G is semisimple and P is parabolic.

Lie algebra cohomology

$\mathfrak{u}$ = Lie algebra $\mathcal{V}$ = $\mathfrak{u}$-module

$$0 \to \mathcal{V} \xrightarrow{\partial} \mathrm{Hom}(\mathfrak{u}, \mathcal{V}) \xrightarrow{\partial} \mathrm{Hom}(\Lambda^2\mathfrak{u}, \mathcal{V}) \xrightarrow{\partial} \cdots$$

$$\partial v(X) = Xv \qquad \partial\phi(X \wedge Y) = \phi([X,Y]) - X\phi(Y) + Y\phi(X) \qquad \cdots$$

$$\rightsquigarrow \boxed{H^p(\mathfrak{u}, \mathcal{V})}$$

$$\mathfrak{sl}(n+1, \mathbb{R}) \ni \left[\begin{array}{c|ccc} * & * & \cdots & * \\ \hline * & & & \\ \vdots & & * & \\ * & & & \end{array}\right]$$

$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$

Let $\mathfrak{u} = \mathfrak{g}_{-1}$ ($\Rightarrow \mathfrak{u}^* = \mathfrak{g}_1$)

Let $\mathcal{V} = \Lambda^2\mathbb{R}^{n+1}|_{\mathfrak{g}_{-1}}$

$$0 \ \to \ \mathcal{V} \ \xrightarrow{\partial} \ \mathfrak{g}_1 \otimes \mathcal{V} \ \xrightarrow{\partial} \ \Lambda^2\mathfrak{g}_1 \otimes \mathcal{V} \ \xrightarrow{\partial}$$

Geometric import

Kostant's Bott-Borel-Weil Theorem $\Longrightarrow$

$H^p(\mathfrak{g}_{-1}, \mathcal{V}) = \square, \square\!\square, \begin{smallmatrix}\square\!\square\\ \square\!\square\end{smallmatrix}, \cdots$ as $\mathrm{SL}(n,\mathbb{R})$-modules

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{V} & \xrightarrow{\partial} & \mathfrak{g}_1 \otimes \mathcal{V} & \xrightarrow{\partial} & \Lambda^2\mathfrak{g}_1 \otimes \mathcal{V} & \xrightarrow{\partial} & \Lambda^3\mathfrak{g}_1 \otimes \mathcal{V} \\
 & & \wr & & \wr & & \wr & & \wr \\
0 & \to & \mathbb{V} & \xrightarrow{\partial} & \Lambda^1 \otimes \mathbb{V} & \xrightarrow{\partial} & \Lambda^2 \otimes \mathbb{V} & \xrightarrow{\partial} & \Lambda^3 \otimes \mathbb{V} \\
 & & \| & & \| & & \| & & \| \\
 & & \Lambda^1 & & \Lambda^1 \otimes \Lambda^1 & & \Lambda^2 \otimes \Lambda^1 & & \Lambda^3 \otimes \Lambda^1 \\
 & & \oplus & \nearrow & \oplus & \nearrow & \oplus & \nearrow & \oplus \\
 & & \Lambda^2 & & \Lambda^1 \otimes \Lambda^2 & & \Lambda^2 \otimes \Lambda^2 & & \Lambda^3 \otimes \Lambda^2
\end{array}$$

Previously suspected tensor identities are justified !!

Back to BGG

de Rham
$$\Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2$$
$$\phi \mapsto \nabla_a \phi$$
$$\omega_a \mapsto \nabla_{[a}\omega_{b]}$$

On $\mathbb{RP}_n$ for $n \geq 2$, $\boxed{\omega_a = \nabla_a \phi \Leftrightarrow \nabla_{[a}\omega_{b]} = 0}$

Riemannian deformation $\phi_a \mapsto \nabla_{(a}\phi_{b)} =$ Killing

$$\square \xrightarrow{\nabla} \square\!\square \xrightarrow{\nabla^{(2)}} \boxplus$$
$$\phi_a \mapsto \nabla_{(a}\phi_{b)}$$
$$\omega_{ab} \mapsto \pi(\nabla_{(a}\nabla_{c)}\omega_{bd} + g_{ac}\omega_{bd})$$

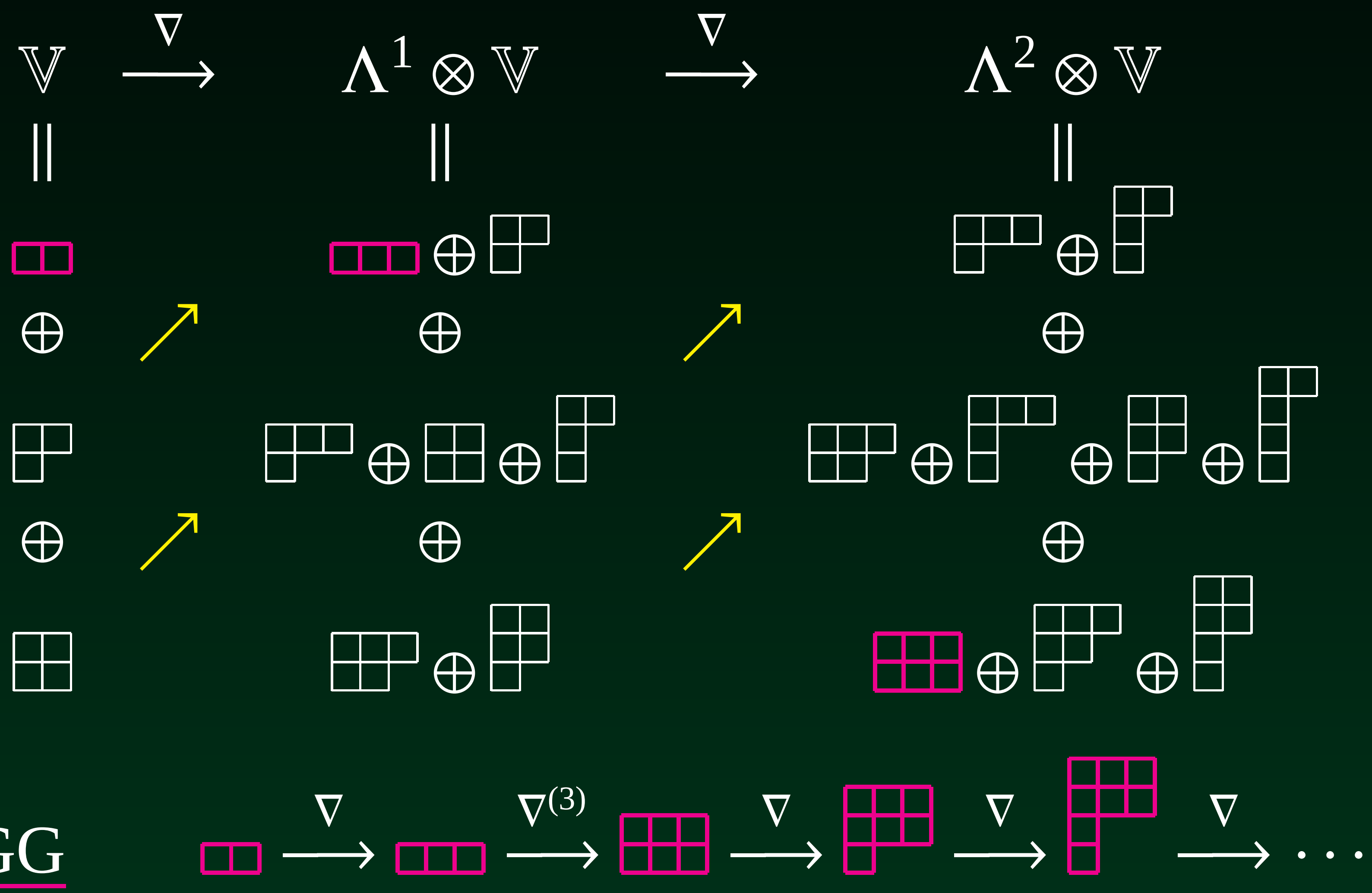
$$\boxed{\omega_{ab} = \nabla_{(a}\phi_{b)} \Leftrightarrow \pi(\nabla_{(a}\nabla_{c)}\omega_{bd} + g_{ac}\omega_{bd}) = 0,}$$

where $\pi(\Phi_{acbd} = \Phi_{[ab][cd]})$ $\quad \square\!\square \otimes \square\!\square \longrightarrow \boxplus$

Higher Killing operators

$$\phi_{bc} = \phi_{(bc)} \mapsto \nabla_{(a}\phi_{bc)}$$

prolong using Lie algebra cohomology



Summary

Range of Killing operators on $\mathbb{RP}_n$ for $n \geq 2$

$$\omega_a = \nabla_a \phi \Leftrightarrow \pi(\nabla_a \omega_b) = \nabla_{[a}\omega_{b]} = 0$$

$$\omega_{ab} = \nabla_{(a}\phi_{b)} \Leftrightarrow \pi(\nabla_{(a}\nabla_{c)}\omega_{bd} + g_{ac}\omega_{bd}) = 0$$

$$\omega_{abc} = \nabla_{(a}\phi_{bc)} \Leftrightarrow \pi(\nabla_{(a}\nabla_c\nabla_{e)}\omega_{bdf} + 4g_{(ac}\nabla_{e)}\omega_{bdf}) = 0$$

Combinatorics

1					$\ddots$ $\ddots$
1	1				$\ddots$
1	4				
1	10	9			
1	20	64			
1	35	259	225		
1	56	784	2304		
1	84	1974	12916	11025	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

THANK YOU

END Of PART TWO