Representation theory and the X-ray transform

Differential geometry on real and complex projective space

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Topics

- Connections
- Affine connections
- Levi-Civita connection
- Round sphere and real projective space
- Complex projective space
- Fubini-Study curvature
- Model embeddings
- Kähler form and model pullback
- Representation theory
- Tensors under model pullback
- Pullback of curvature et cetera

Connections

E = smooth vector bundle (real or complex)

Connection

$$\nabla : E \to \Lambda^1 \otimes E \quad \text{s.t.} \quad \nabla(f\sigma) = f\nabla\sigma + df \otimes \sigma$$

Coupled de Rham

$$E \xrightarrow{\nabla} \underbrace{\Lambda^1 \otimes E \xrightarrow{\nabla} \Lambda^2 \otimes E}_{\omega \otimes \sigma \mapsto d\omega \otimes \sigma - \omega \wedge \nabla\sigma} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Lambda^n \otimes E \to 0$$

Curvature

$$\begin{array}{l} \nabla^2 : E \longrightarrow \Lambda^2 \otimes E \\ \nabla^2(f\sigma) = f\nabla^2\sigma \end{array} \quad \rightsquigarrow \quad \kappa \in \Gamma(\Lambda^2 \otimes \operatorname{End}(E))$$

Bianchi identity: $\nabla^3 = 0$ or $\nabla\kappa = 0$

Induced connections

If E and F are equipped with connections, then there are induced connections on the following bundles.

- E^*
- $E \otimes E$
- $E \wedge E$
- $E \odot E$
- $\mathrm{End}(E) = E^* \otimes E$
- $E \otimes F$
- $\mathrm{Hom}(E, F) = E^* \otimes F$
- and so on...

Leibniz rule, e.g. $\nabla(\sigma \otimes \tau) = (\nabla\sigma) \otimes \tau + \sigma \otimes (\nabla\tau)$

Existence and freedom

$\nabla, \hat{\nabla}$ connections on $E \Rightarrow$

$$h\nabla + (1-h)\hat{\nabla} \quad \text{also a connection.}$$

Therefore, partition of unity $\Rightarrow$ existence.

Freedom: $\nabla, \hat{\nabla}$ connections on $E \Rightarrow$

$$\hat{\nabla} = \nabla + \Gamma$$

for $\Gamma : E \to \Lambda^1 \otimes E$ a homomorphism.

Affine connections

Affine $\equiv$ connection on Λ^1 (or the tangent bundle)

Problem:

$$\nabla : \Lambda^1 \to \Lambda^1 \otimes \Lambda^1 \xrightarrow{\wedge} \Lambda^2$$

may not agree with the exterior derivative!

Solution:

$$T \equiv \wedge \circ \nabla - d : \Lambda^1 \to \Lambda^2 \hookrightarrow \Lambda^1 \otimes \Lambda^1$$

is a homomorphism $\equiv$ torsion $\in \Gamma(\Lambda^1 \otimes \mathrm{End}(\Lambda^1))$.

$\hat{\nabla} \equiv \nabla - T$ is a torsion free connection on Λ^1.

Consequence: $\Lambda^1 \otimes \Lambda^1 \xrightarrow{\hat{\nabla}} \Lambda^2 \otimes \Lambda^1$ is unambiguous.

Indices

Covariant tensors $\phi_a, \psi_{ab}, \ldots$. (Anti)-symmetrisation:

$$\phi_{[ab]} \equiv \tfrac{1}{2}(\phi_{ab} - \phi_{ba}) \qquad \phi_{(ab)c} \equiv \tfrac{1}{2}(\phi_{abc} + \phi_{bac}) \qquad \cdots$$

Contravariant tensors, e.g. X^a = vector field.

E.g. torsion tensor

$$T_{ab}{}^c \quad \text{s.t.} \quad T_{ab}{}^c = T_{[ab]}{}^c \quad \text{or} \quad T_{(ab)}{}^c = 0$$

Einstein summation convention: $X \lrcorner\, \omega \equiv X^a \omega_a$

Curvature of a torsion-free affine connection

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) X^c = R_{ab}{}^c{}_d X^d$$

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) \omega_d = -R_{ab}{}^c{}_d \omega_c$$

Levi-Civita connection

Given g_{ab} a metric, $\exists!$ affine ∇_a characterised by

- ∇_a is torsion-free
- $\nabla_a g_{bc} = 0$.

Proof Choose any $\hat{\nabla}_a$ torsion-free. Consider

$$\nabla_a \phi_b = \hat{\nabla}_a \phi_b - \Gamma_{ab}{}^c \phi_c.$$

Want: $\boxed{\Gamma_{abc} = \Gamma_{(ab)c}}$ and $\boxed{\Gamma_{a(bc)} = \frac{1}{2}\hat{\nabla}_a g_{bc}}$ but

$$\begin{array}{ccc} \Lambda^1 \otimes \Lambda^2 & \xrightarrow{\cong} & \Lambda^2 \otimes \Lambda^1 \\ K_{abc} & \longmapsto & K_{[ab]c} \\ \parallel & & \\ K_{a[bc]} & & \end{array}$$

NB

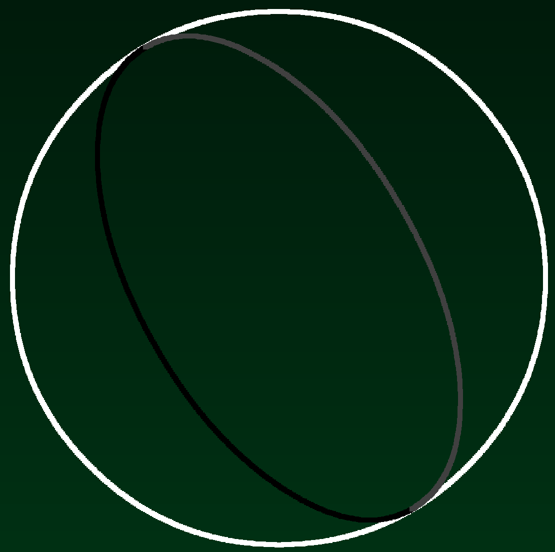
QED!

Round sphere

Riemannian curvature tensor

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\omega_d = -R_{ab}{}^c{}_d\omega_c$$

$$R_{abcd} = R_{[ab][cd]} \quad R_{[abc]d} = 0 \quad (\Rightarrow R_{abcd} = R_{cdab})$$



$= \mathrm{SO}(n+1)/\mathrm{SO}(n)$

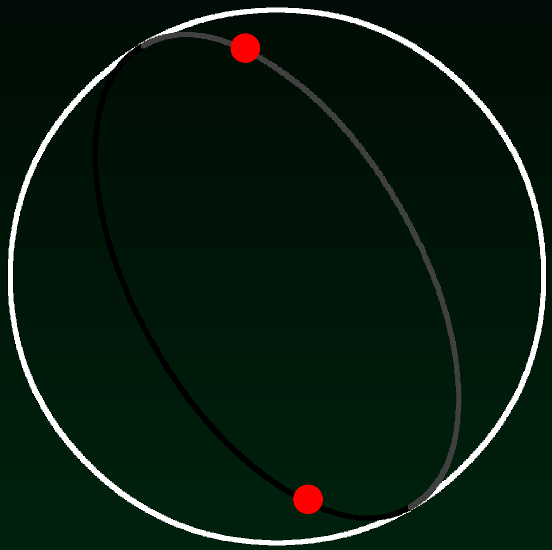
(J.A. Wolf, 'Spaces of . . . ')

$$\boxed{R_{abcd} = g_{ac}g_{bd} - g_{bc}g_{ad}}$$

constant curvature

$$R_{ab} \equiv R_{ca}{}^c{}_b = (n-1)g_{ab} \quad R \equiv R_a{}^a = n(n-1)$$

Real projective space



antipodal identification

$$= \mathrm{SO}(n+1)/\mathrm{S}(\mathrm{O}(1)\times\mathrm{O}(n))$$

$$\boxed{R_{abcd} = g_{ac}g_{bd} - g_{bc}g_{ad}}$$

constant curvature

i.e. $(\nabla_b\nabla_c - \nabla_c\nabla_b)\omega_a = g_{ab}\omega_c - g_{ac}\omega_b$

$\mathbb{RP}_n = \{\text{linear } L \subset \mathbb{R}^{n+1} \text{ s.t. } \dim L = 1\}$

Complex projective space

$$\mathbb{CP}_n = \{\text{linear } L \subset \mathbb{C}^{n+1} \text{ s.t. } \dim L = 1\}$$
$$= SU(n+1)/S(U(1) \times U(n))$$

Fubini-Study metric g_{ab}

$$\mathbb{RP}_n \xhookrightarrow{\text{complex span}} \mathbb{CP}_n$$ totally geodesic embedding

$J_a{}^b$ s.t. $J_a{}^b J_b{}^c = -\delta_a{}^c$ complex structure (orthogonal)

J_{ab} $(\equiv J_a{}^c g_{bc})$ Kähler form (skew)

$$\boxed{\nabla_a J_{bc} = 0}$$

Fubini-Study curvature

$$R_{abcd} = g_{ac}g_{bd} - g_{bc}g_{ad} + J_{ac}J_{bd} - J_{bc}J_{ad} + 2J_{ab}J_{cd}$$

NB: $R_{ab}{}^c{}_d J_{ce} = 2J_{a(d}g_{e)b} - 2J_{b(d}g_{e)a} + 2J_{ab}g_{de}$ ✓ □

Model embeddings

$$\left.\begin{array}{l}\text{Recall totally geodesic embedding } \mathbb{RP}_n \overset{\iota}{\hookrightarrow} \mathbb{CP}_n \\ \text{Recall } \mathrm{SU}(n+1) \text{ acting on } \mathbb{CP}_n \text{ by isometries}\end{array}\right\} \rightsquigarrow$$

Model embeddings: $\mathbb{RP}_n \overset{\mu}{\hookrightarrow} \mathbb{CP}_n$.

J is type $(1,1) \Rightarrow \iota^* J = 0$. Therefore $\mu^* J = 0$,

i.e. model embeddings are Lagrangian. Conversely,

$$\text{Linear Algebra} \Rightarrow \begin{array}{c}\text{Model embeddings though } p \in \mathbb{CP}_n \\ \updownarrow \\ \text{Lagrangian subspaces of } T_p\mathbb{CP}_n\end{array}$$

(Lagrangian Grassmannian $\cong \mathrm{U}(n)/\mathrm{O}(n)$
E.g. Helgason ‘Geometric Analysis . . . ,’ Exercise I.A.4(ii))

Kähler form

$\nabla_a J_{bc} = 0 \Rightarrow$ dJ=0
J_{ab} is non-degenerate

$\leadsto$ $\mathbb{CP}_n$ is a symplectic manifold

Let ψ be a 2-form, i.e. with indices $\psi_{ab} = \psi_{[ab]}$. Then

$$\psi_{ab} = \underbrace{\psi_{ab} - \frac{1}{2n} J^{cd}\psi_{cd}J_{ab}}_{J\text{-trace-free}} + \frac{1}{2n} J^{cd}\psi_{cd}J_{ab}$$

$$\begin{aligned} \psi_{ab} &= \psi^{\perp}_{ab} + \theta J_{ab} \\ \psi &= \psi_{\perp} + \theta J \\ \Lambda^2 &= \Lambda^2_{\perp} \oplus \Lambda^0 J \end{aligned}$$

Symplectic decomposition

Representation theory

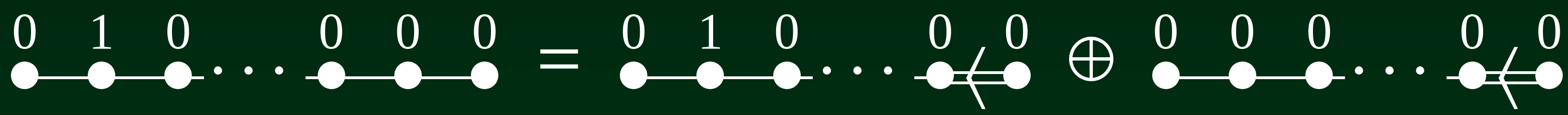
$\Lambda^2 = \Lambda^2_\perp \oplus \Lambda^0 J$ (on any symplectic manifold)

Let J be a non-degenerate skew $2n \times 2n$ real matrix.

$\mathrm{Sp}(2n, \mathbb{R}) \equiv \{A = 2n \times 2n$ matrix s.t. $AJA^t = J\}$

Defining representation : $\mathbb{R}^{2n}$

$$\Lambda^2 \mathbb{R}^{2n} = \Lambda^2_\perp \mathbb{R}^{2n} \oplus \mathbb{R}$$



$2n-1$ nodes $\quad$ n nodes $\quad$ n nodes

Branching for $\mathrm{SL}(2n, \mathbb{R}) \supset \mathrm{Sp}(2n, \mathbb{R})$.

(Induced vector bundles from co-frame bundle.)

Model pullback

Recall model embeddings $\mu : \mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$.
Suppose ψ is a two-form on $\mathbb{CP}_n$.

Lemma: $\mu^*\psi = 0 \, \forall \mu \quad \Leftrightarrow \quad \psi = \theta J$

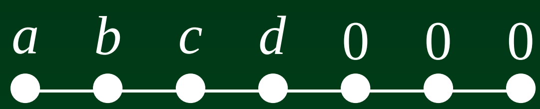
$$\Leftrightarrow \quad \psi_\perp = 0.$$

Proof: Fix attention on $p \in \mathbb{CP}_n$.

$$\psi(p) \in \Lambda^2_p \text{ s.t. } \psi(p)|_L = 0 \, \forall \text{ Lagrangian } L \subset T_p\mathbb{CP}_n$$

NB: invariant under $\mathrm{Sp}(2n, \mathbb{R})$ acting on $T_p\mathbb{CP}_n$.
Recall $\mu^* J = 0$. $\exists \psi$ s.t. $\mu^*\psi \neq 0$. Schur! $\square$

Generalises to suitable tensors, using



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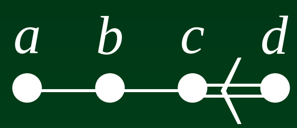

$\oplus \cdots$ (case $n = 4$).

Curvature pullback

$$\left.\begin{array}{l}\psi_{abcd} = \psi_{[ab][cd]} \\ \psi_{[abc]d} = 0\end{array}\right\} \equiv \text{Riemann tensor symmetries}$$

$$\psi_{abcd} \in \Gamma(\overset{0}{\bullet}\!-\!\overset{2}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}) \quad (\text{case } n = 4).$$

Branch to $\mathrm{SL}(8,\mathbb{R}) \supset \mathrm{Sp}(8,\mathbb{R})$

$$\overset{0}{\bullet}\!-\!\overset{2}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet} = \overset{0}{\bullet}\!-\!\overset{2}{\bullet}\!-\!\overset{0}{\bullet}\!\Leftarrow\!\overset{0}{\bullet} \oplus \overset{0}{\bullet}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\!\Leftarrow\!\overset{0}{\bullet} \oplus \overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!\Leftarrow\!\overset{0}{\bullet}$$

$$\psi_{abcd} = \psi^{\perp}_{abcd} + \phi_{ab} \bowtie J_{cd} + \theta J_{ab} \bowtie J_{cd}$$

cf. Weyl   cf. Ricci   cf. Scalar

Lemma: $\mu^*\psi_{abcd} = 0 \; \forall$ models $\mu \quad \Leftrightarrow \quad \psi^{\perp}_{abcd} = 0$

Summary

Curvature on $\mathbb{RP}_n$

$$R_{abcd} = g_{ac}g_{bd} - g_{bc}g_{ad}$$

Curvature on $\mathbb{CP}_n$

$$R_{abcd} = g_{ac}g_{bd} - g_{bc}g_{ad} + J_{ac}J_{bd} - J_{bc}J_{ad} + 2J_{ab}J_{cd}$$

Model embeddings

$\mu : \mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$ totally geodesic

2-form lemma

$$\mu^*\psi_{ab} = 0\,\forall\mu \quad \Leftrightarrow \quad \psi^{\perp}_{ab} = 0$$

Curvature lemma

$$\mu^*\psi_{abcd} = 0\,\forall\mu \quad \Leftrightarrow \quad \psi^{\perp}_{abcd} = 0$$

THANK YOU

END OF PART ONE