Higher Symmetries of the Laplacian

Michael Eastwood

Australian National University
References


Thanks: Erik van den Ban, David Calderbank, Andreas Čap, Rod Gover, Robin Graham, Keith Hannabuss, Bertram Kostant, Toshio Oshima, Paul Tod, Misha Vasiliev, Nolan Wallach, Edward Witten, and Joseph Wolf.
A simple question on $\mathbb{R}^n$, $n \geq 3$

**Question:** Which linear differential operators preserve harmonic functions? **Answer on $\mathbb{R}^3$:**

**Zeroth order** \( f \mapsto \text{constant} \times f \)

**First order**

\[
\begin{align*}
\nabla_1 &= \frac{\partial}{\partial x_1} \\
\nabla_2 &= \frac{\partial}{\partial x_2} \\
\nabla_3 &= \frac{\partial}{\partial x_3}
\end{align*}
\]

\[
\begin{align*}
x_1 \nabla_2 - x_2 \nabla_1 & \quad & \text{&c.} \\
x_1 \nabla_1 + x_2 \nabla_2 + x_3 \nabla_3 + 1/2 & \quad & 1 \\
(x_1^2 - x_2^2 - x_3^2) \nabla_1 + 2x_1x_2 \nabla_2 + 2x_1x_3 \nabla_3 + x_1 & \quad & 3
\end{align*}
\]

**Dimensions**

\[
[D_1, D_2] \equiv D_1D_2 - D_2D_1
\]

**Lie Algebra** = $\mathfrak{so}(4, 1) =$ **conformal algebra** ← NB!
Second order

Boyer-Kalnins-Miller (1976)

Extras: $\propto$ Laplacian ($f \mapsto h\Delta f$ for any smooth $h$)
plus a 35-dim family of new ones!

\[ \{\mathcal{D}_1, \mathcal{D}_2\} \equiv \mathcal{D}_1\mathcal{D}_2 + \mathcal{D}_2\mathcal{D}_1 \]

$\otimes^2 \mathfrak{so}(4,1) = ?$  $\dim = 10 \times 11/2 = 55$

55 = 35 + 14 + 1 + 5

Separation of variables (Bôcher, Bateman, . . .).
Third order . . .?
Conformal geometry

Action of $\text{SO}(n + 1, 1)$ on $S^n$ by conformal transformations

stereographic projection

‘flat’ model

{generators}
Conformal Laplacian  Dirac 1935

\[ r \equiv x_1^2 + \cdots + x_n^2 + x_{n+1}^2 - x_{n+2}^2 \]

\[ \tilde{\Delta} \equiv \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial x_{n+1}^2} - \frac{\partial^2}{\partial x_{n+2}^2} \]

\( f \) on null cone \( \subset \mathbb{R}^{n+2} \) homogeneous of degree \( w \) \( \rightsquigarrow \)

- ambiently extend to \( \tilde{f} \) of degree \( w \)
- freedom \( \tilde{f} \mapsto \tilde{f} + rg \) for \( g \) of degree \( w - 2 \)
- calculate: \( \tilde{\Delta}(rg) = r\tilde{\Delta}g + 2(n + 2w - 2)g \)

\( w = 1 - n/2 \Rightarrow f \mapsto (\tilde{\Delta}\tilde{f})|_{r=0} \) is invariantly defined.

On \( \mathbb{R}^n \) it’s \( \Delta \)  

On \( S^n \) it’s \( \Delta - \frac{n-2}{4(n-1)} R \)

AdS/CFT  Fefferman-Graham ‘ambient’ metric
Symmetries of $\Delta$

$\mathcal{D}$ a symmetry $\iff \Delta\mathcal{D} = \delta\Delta$ for some $\delta$.

trivial example: $\mathcal{D} = \mathcal{P}\Delta$ for any $\mathcal{P}$

equivalence: $\mathcal{D}_1 \equiv \mathcal{D}_2 \iff \mathcal{D}_1 - \mathcal{D}_2 = \mathcal{P}\Delta$

$\mathbb{R}^n \leadsto \mathcal{A}_n \equiv$ algebra of symmetries under composition up to equivalence

Write $\mathcal{D} = V^{bc\cdots d}_{\nabla b \nabla c \cdots \nabla d} + \text{lower order terms}$

symbol normalise w.l.g. to be trace-free
Theorems

\[ \mathcal{D} \text{ a symmetry } \Rightarrow \text{ trace-free part of } \nabla^{(a} V^{bc\cdots d)} = 0 \]

Easy

On \( \mathbb{R}^n \), such a \underline{conformal Killing tensor} \( V^{bc\cdots d} \) \( \leadsto \mathcal{D}_V \)

Not So Easy

\( \mathcal{D}_V \) is a canonically associated symmetry of the form

\[ \mathcal{D}_V = V^{bc\cdots d} \nabla_b \nabla_c \cdots \nabla_d + \text{ lower order terms} \]

• E.g. First order

\[ \mathcal{D}_V f = V^a \nabla_a f + \frac{n-2}{2n} (\nabla_a V^a) f \]

• E.g. Second order

\[ \mathcal{D}_V f = V^{ab} \nabla_a \nabla_b f + \frac{n}{n+2} (\nabla_a V^{ab}) \nabla_b f + \frac{n(n-2)}{4(n+2)(n+1)} (\nabla_a \nabla_b V^{ab}) f \]
Ingredients of proof

- We can solve the conformal Killing tensor equation

\[ \nabla^{(a} V^{bc \cdots d)} = g^{(ab} \chi^{c \cdots d)} \]

on \( \mathbb{R}^n \) by prolongation and/or BGG machinery:

\[ \begin{array}{c}
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
\end{array}
\end{array} \]

\[ \circ \]

w.r.t. \( \mathfrak{so}(n + 1, 1) \).

# of columns = # of indices on \( V^{bc \cdots d} \)

E.g. \( V^b = s^b + m^{bc} x_c + \lambda x^b + r^c x_c x^b - \frac{1}{2} x_c x_c r^b \)

= translation + rotation + dilation + inversion.

- Use ‘ambient’ methods to construct \( \mathcal{D}_V \).
Corollary

As a vector space

\[ A_n = \bigoplus_{s=0}^{\infty} \left( \bigotimes_s g \right) \]

Question: What about the algebra structure?

Cf.: let \( \mathfrak{g} \) be a Lie algebra. As a vector space

\[ \mathcal{U}(\mathfrak{g}) = \bigoplus_{s=0}^{\infty} \left( \bigotimes_s \mathfrak{g} \right) \]

but the algebra structure is opaque viewed this way.
The algebra structure

\[ \mathcal{U}(\mathfrak{g}) = \frac{\bigotimes \mathfrak{g}}{(X \otimes Y - Y \otimes X - [X, Y])} \]

Theorem

\[ \mathcal{A}_n = \frac{\bigotimes \mathfrak{so}(n + 1, 1)}{(X \otimes Y - X \circ Y - \frac{1}{2}[X, Y] + \frac{n-2}{4n(n+1)} \langle X, Y \rangle)} \]

Cartan \hspace{1cm} Lie \hspace{1cm} Killing

Equivalently,

\[ \mathcal{A}_n = \mathcal{U}(\mathfrak{so}(n + 1, 1))/\text{Joseph Ideal}. \]
Proof of algebra structure

Calculate by ambient means that

\[ \mathcal{D}_X \mathcal{D}_Y = \mathcal{D}_X \circ Y + \frac{1}{2} \mathcal{D}[X,Y] - \frac{n-2}{4n(n+1)} \mathcal{D}\langle X,Y \rangle \]

and use properties of Cartan product (due to Kostant).

**Remark:** simple Lie algebra \( g \neq sl(2, \mathbb{C}) \) \( \Rightarrow \)

\[ \dim \left( \bigotimes g \right) \]

\[ \left( X \otimes Y - X \circ Y - \frac{1}{2} [X, Y] - \lambda \langle X, Y \rangle \right) \]

for precisely one value of \( \lambda \) (Braverman and Joseph)

\[ \leadsto \text{graded algebra} \bigoplus_{s=0}^{\infty} \bigcirc^s g. \]
Curved analogues

For any vector field $V^a$,

$$f \mapsto V^a \nabla_a f + \frac{n-2}{2n} (\nabla_a V^a) f$$

is conformally invariant.

For any trace-free symmetric tensor field $V^{ab}$,

$$f \mapsto V^{ab} \nabla_a \nabla_b f + \frac{n}{n+2} (\nabla_a V^{ab}) \nabla_b f + \frac{n(n-2)}{4(n+2)(n+1)} (\nabla_a \nabla_b V^{ab}) f$$

$$- \frac{n+2}{4(n+1)} R_{ab} V^{ab} f = \text{curvature correction terms}$$

is conformally invariant & c. & c.
Curved symmetries?

- $V^a$ is a conformal Killing vector $\Rightarrow$

$$\mathcal{D}_V f \equiv V^a \nabla_a f + \frac{n-2}{2n} (\nabla_a V^a) f$$

is symmetry of the conformal Laplacian.

- $V^{ab}$ is a conformal Killing tensor $\Rightarrow$

$$\mathcal{D}_V f \equiv V^{ab} \nabla_a \nabla_b f + \frac{n}{n+2} (\nabla_a V^{ab}) \nabla_b f$$

$$+ \frac{n(n-2)}{4(n+2)(n+1)} (\nabla_a \nabla_b V^{ab}) f - \frac{n+2}{4(n+1)} R_{ab} V^{ab} f$$

is a symmetry of the conformal Laplacian. **Unknown!**
Another operator

In even dimensions, there is the **Dirac operator**

\[ D : \mathbb{S}^+ \rightarrow \mathbb{S}^- . \]

A **symmetry** of \( D \) is an operator \( \mathcal{D} : \mathbb{S}^+ \rightarrow \mathbb{S}^+ \) s.t.

\[
\begin{array}{ccc}
\mathbb{S}^+ & \xrightarrow{D} & \mathbb{S}^- \\
\mathcal{D} \downarrow & & \downarrow \delta \\
\mathbb{S}^+ & \xrightarrow{D} & \mathbb{S}^- 
\end{array}
\]

commutes for some differential operator \( \delta : \mathbb{S}^- \rightarrow \mathbb{S}^- \).

- The symbol of \( \mathcal{D} \) satisfies a conformally invariant overdetermined system of equations.
- First order symmetries: Benn and Kress.
- Higher order symmetries in the flat case: E, Somberg, and Souček.
Yet another operator

For the square of the Laplacian (E and Leistner)
symmetry algebra =

\[ \bigotimes \mathfrak{so}(n + 1, 1) \]

\[
X \otimes Y - X \odot Y - X \bullet Y - \frac{1}{2} [X, Y] + \frac{(n-4)(n+4)}{4n(n+1)(n+2)} \langle X, Y \rangle
\]

and some fourth order elements

with graded counterpart

new

different
THANK YOU

THE END