Some Questions on the Course

Conformal Differential Geometry

Honours/PhD Special Topics Math3349/6209

Question 1 Show that inverse orthographic projection

$$\mathbb{R}^2 \ni (x,y) \mapsto (x,y,\sqrt{1-x^2-y^2}) \in \mathbb{R}^3$$

from the unit disc $\{(x,y) \text{ s.t. } x^2+y^2<1\}$ to the round 2-sphere $\{(x,y,z)\in\mathbb{R}^3 \text{ s.t. } x^2+y^2+z^2=1\}$ is not conformal except at $(0,0)$.

Question 2 Show that the mapping

$$\mathbb{R}^2 \ni \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x^3-3xy^2-x^2+4xy+y^2-x-2y+1 \\ 3x^2y-y^3-2x^2-2xy+2y^2+2x-y \end{pmatrix} \in \mathbb{R}^2$$

is conformal except at the points $(1,0)$ and $(0,1)$.

Question 3 Consider a smooth connected n-manifold equipped with a torsion-free affine connection ∇_a having curvature $R_{ab}{}^c{}_d$ defined by

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)X^c = R_{ab}{}^c{}_d X^d.$$

(i) Show that

$$\nabla_a X^b - \tfrac{1}{n}\delta_a{}^b\nabla_c X^c = 0 \iff \left\{ \begin{array}{lcl} \nabla_a X^b & = & \delta_a{}^b Y \\ \nabla_a Y & = & -\frac{1}{n-1}R_{ab}X^b \end{array} \right.$$

where $R_{bd} \equiv R_{ab}{}^a{}_d$.

(ii) Show that

$$\dim\{\sigma_{bc} \text{ s.t. } \sigma_{(ab)}=0 \text{ and } \nabla_{(a}\sigma_{b)c}=0\} \le \tfrac{n(n-1)(n+1)}{6}.$$

Question 4 Use the definition of the Levi-Civita connection to show that the Riemannian curvature of the surface in $\mathbb{R}^3$ defined by

$$z = \alpha x^2 + \beta y^2 + \text{higher order terms}$$

is given by

$$R_{abcd} = 4\alpha\beta(g_{bc}g_{ad} - g_{bc}g_{ad}) \quad \text{at the origin.}$$

Question 5 Solve the partial differential equations

$$\partial_a\Upsilon_b = \Upsilon_a\Upsilon_b \qquad \text{and} \qquad \partial_a\Upsilon_b = \Upsilon_a\Upsilon_b - \tfrac{1}{2}\Upsilon_c\Upsilon^c\delta_{ab}$$

in $\mathbb{R}^n$ for $n \ge 2$.

Answers due on 6 October at 9:30am