How to Draw a Cube

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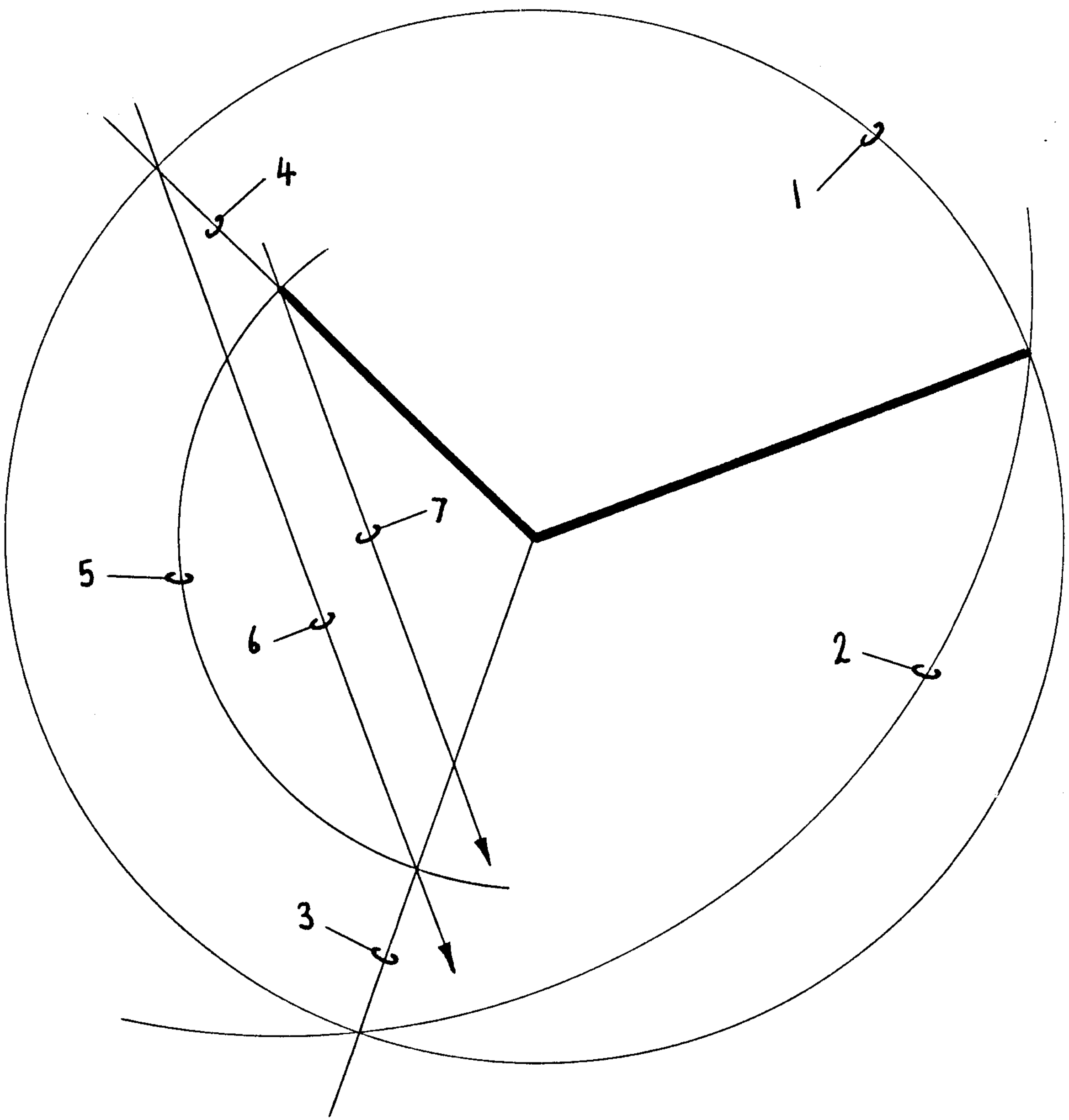
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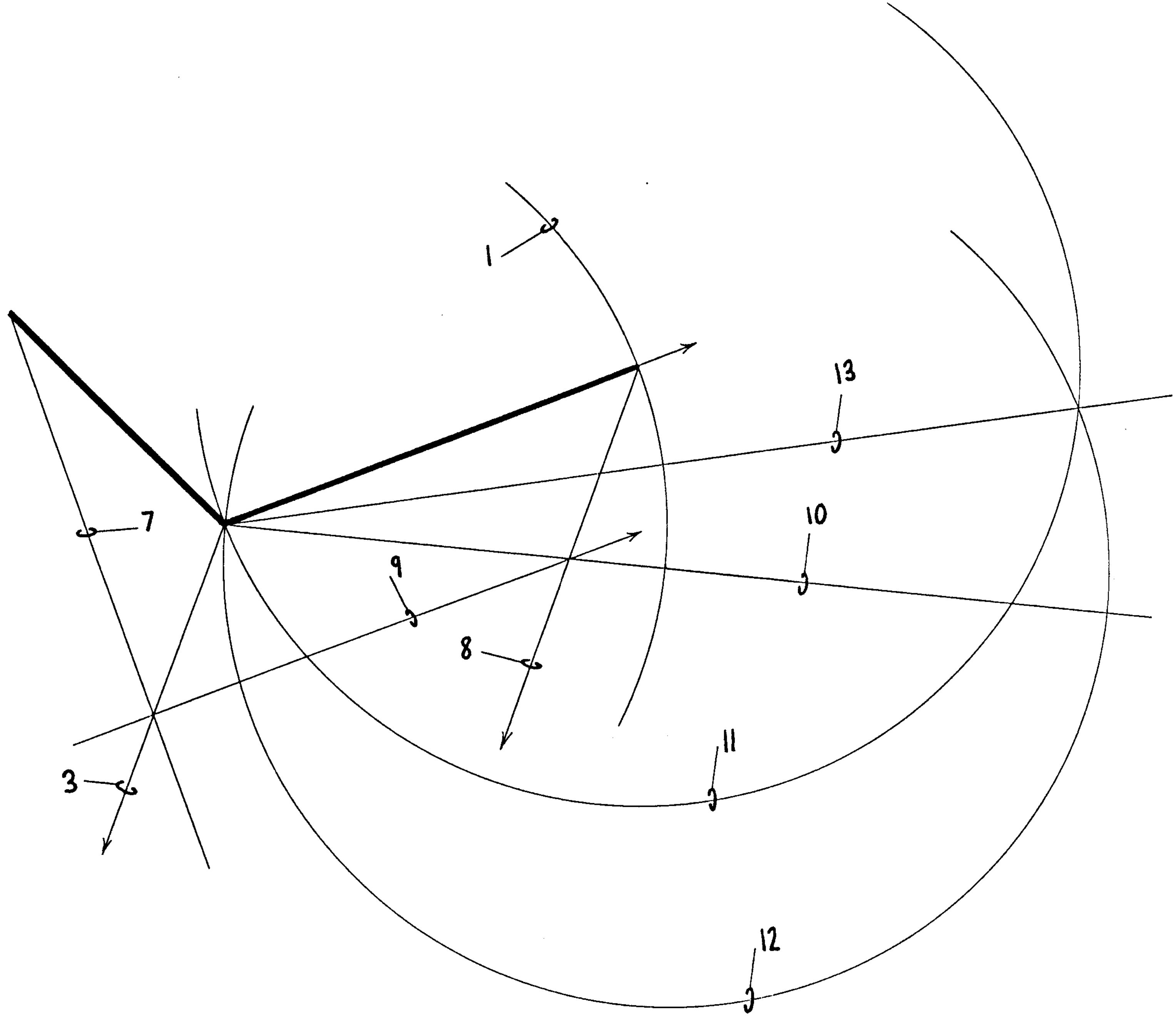
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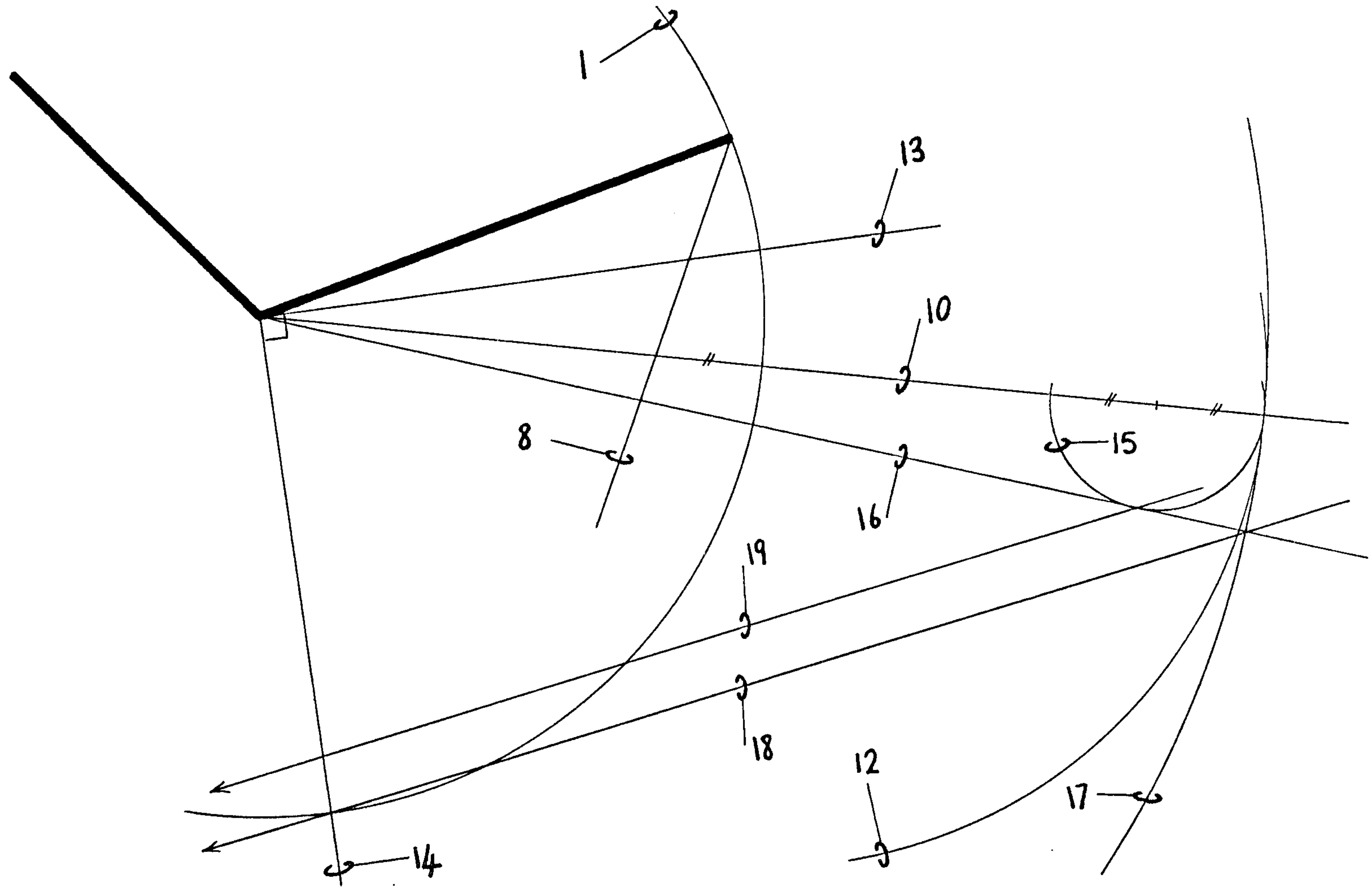

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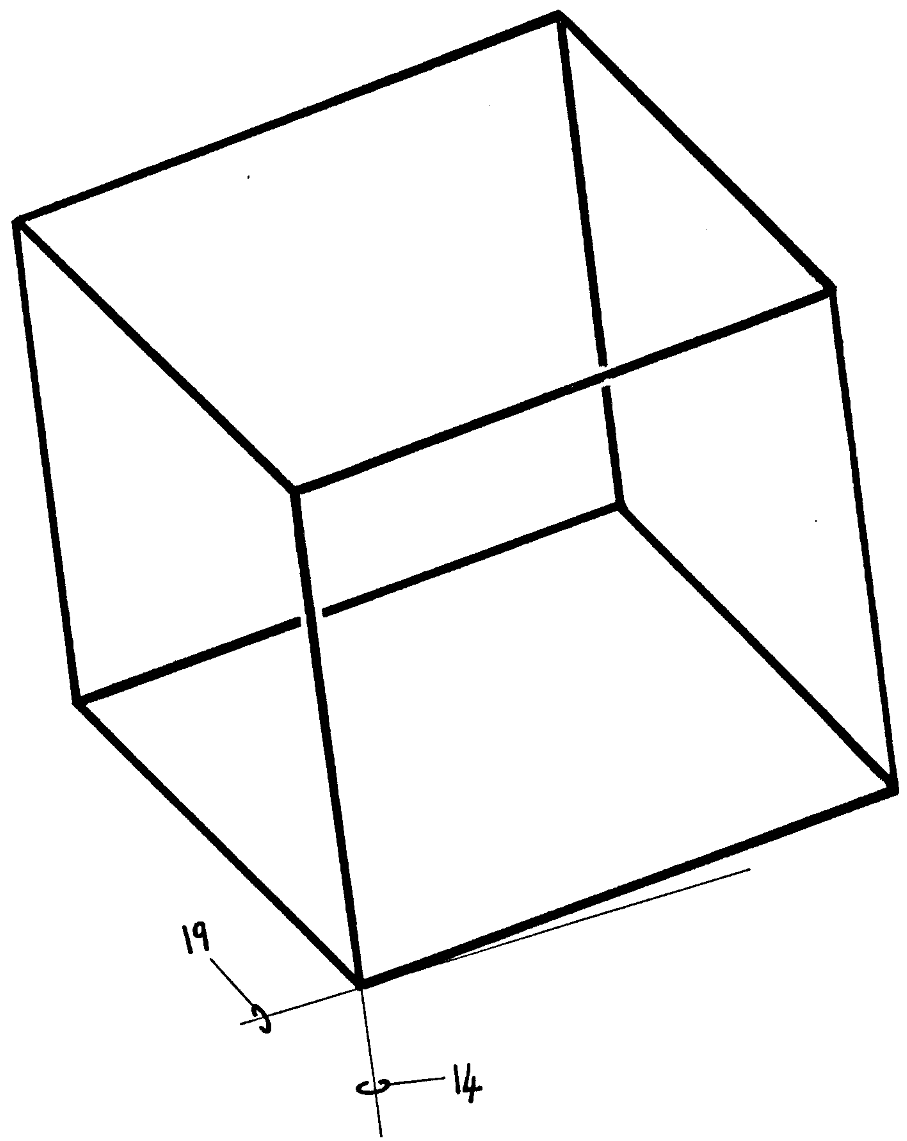
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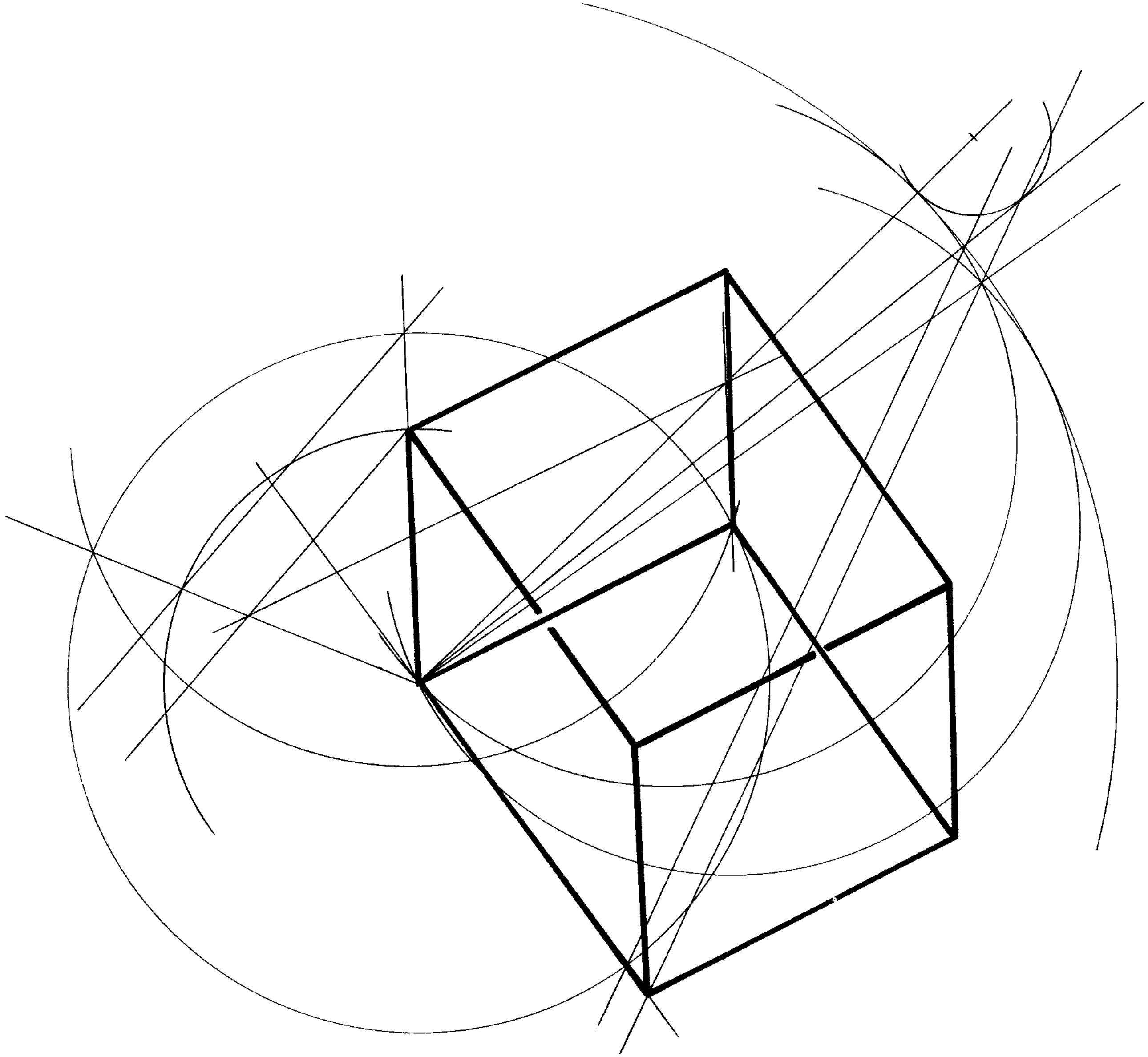
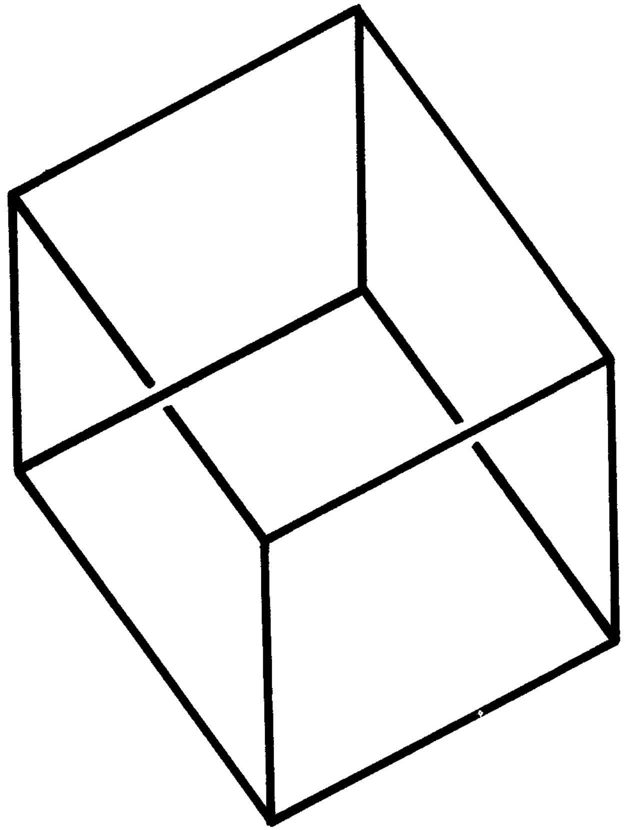
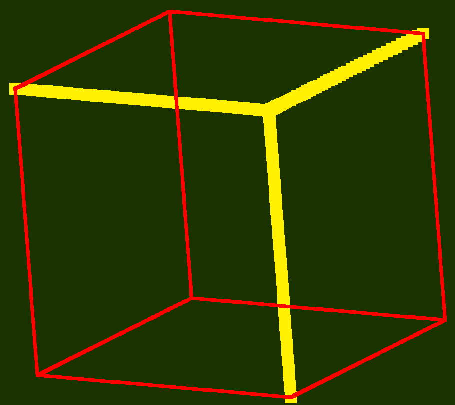




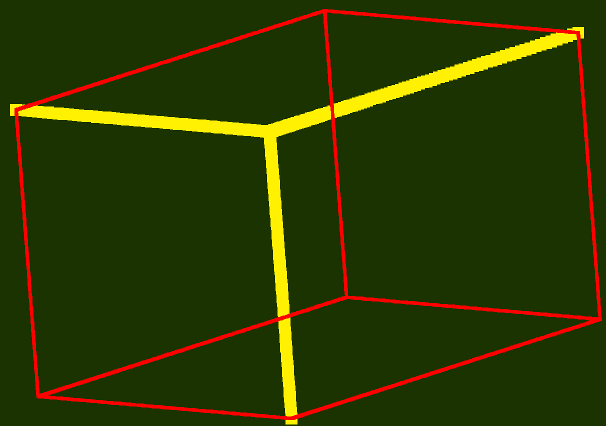
Orthographic projection

$$\begin{array}{rcl} \mathbb{R}^3 & \longrightarrow & \mathbb{R}^2 \\ (x, y, z) & \longmapsto & (x, y) \end{array}$$

mutually orthogonal and the same length $\left\{ \begin{array}{ccc} a & & \alpha \\ b & \longmapsto & \beta \\ c & & \gamma \end{array} \right.$



Right

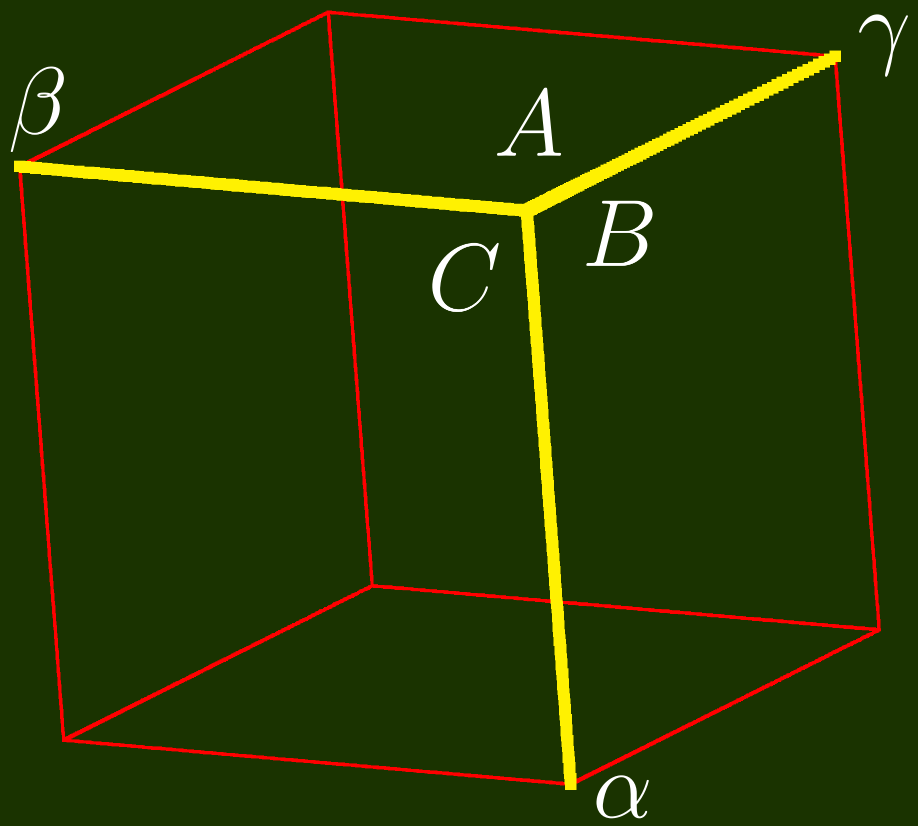


Wrong

in ~~So Many~~ Ways

Two

Weisbach 1844

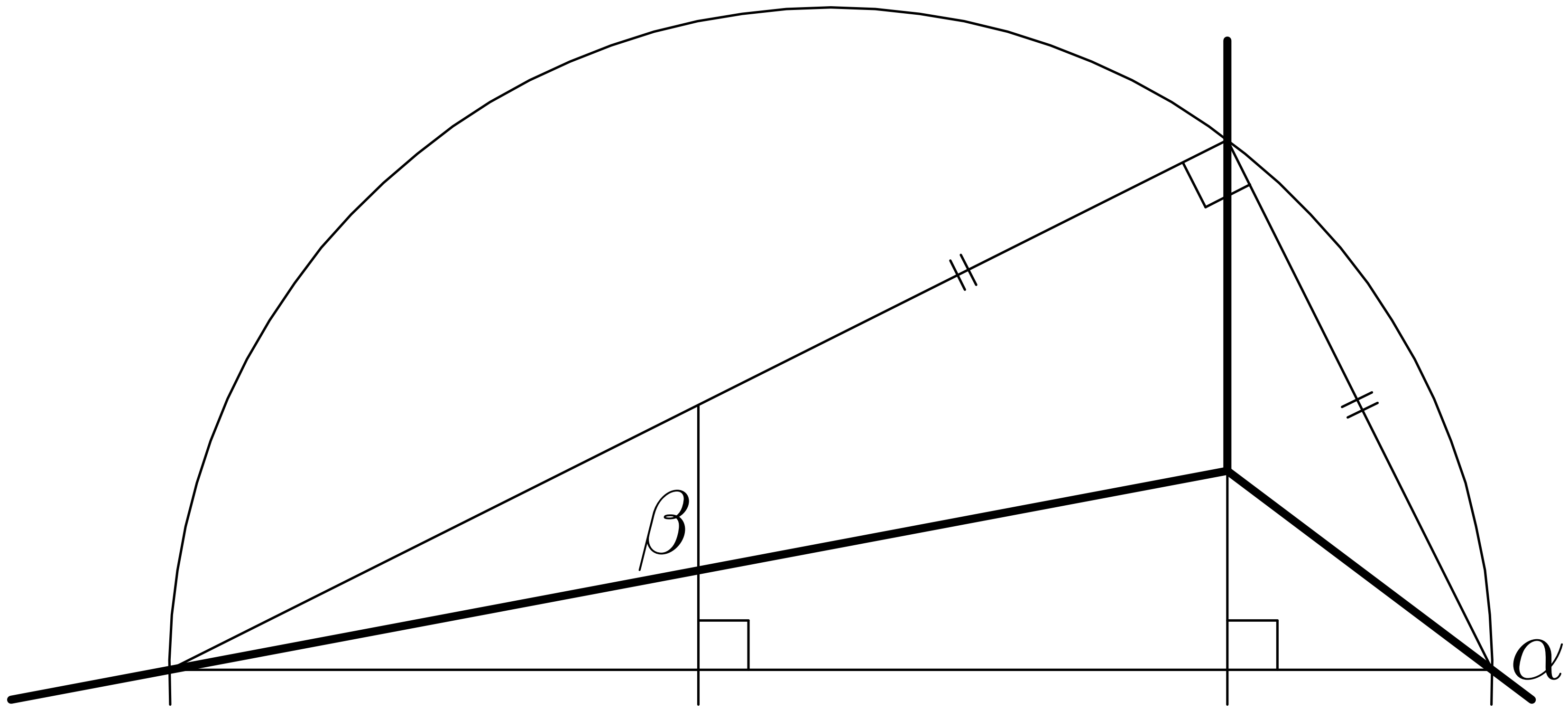


Expect: $\boxed{\text{2 equations!}}$

$$\boxed{\frac{\|\alpha\|^2}{\sin 2A} = \frac{\|\beta\|^2}{\sin 2B} = \frac{\|\gamma\|^2}{\sin 2C}}$$

Example: $\alpha = (2, -26) \quad \beta = (-23, 2) \quad \gamma = (14, 7)$

Schmid 1922



Gauß (Werke 1876)

Regard

$$\alpha = (x_1, y_1) \qquad \beta = (x_2, y_2) \qquad \gamma = (x_3, y_3)$$

as complex numbers

$$\alpha = x_1 + iy_1 \qquad \beta = x_2 + iy_2 \qquad \gamma = x_3 + iy_3$$

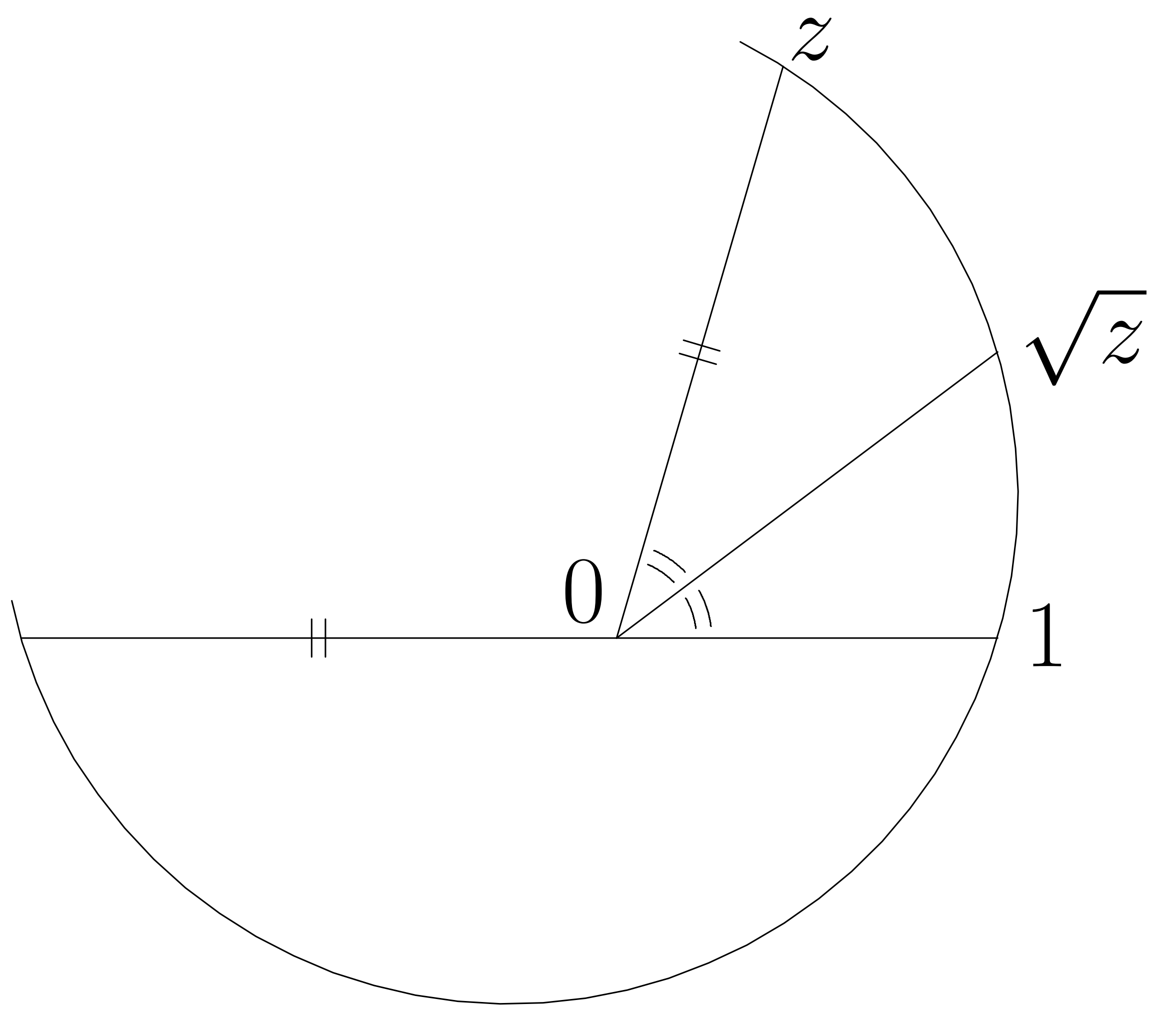
SHOCK HORROR!

Gauß's Fundamental Theorem of Axonometry

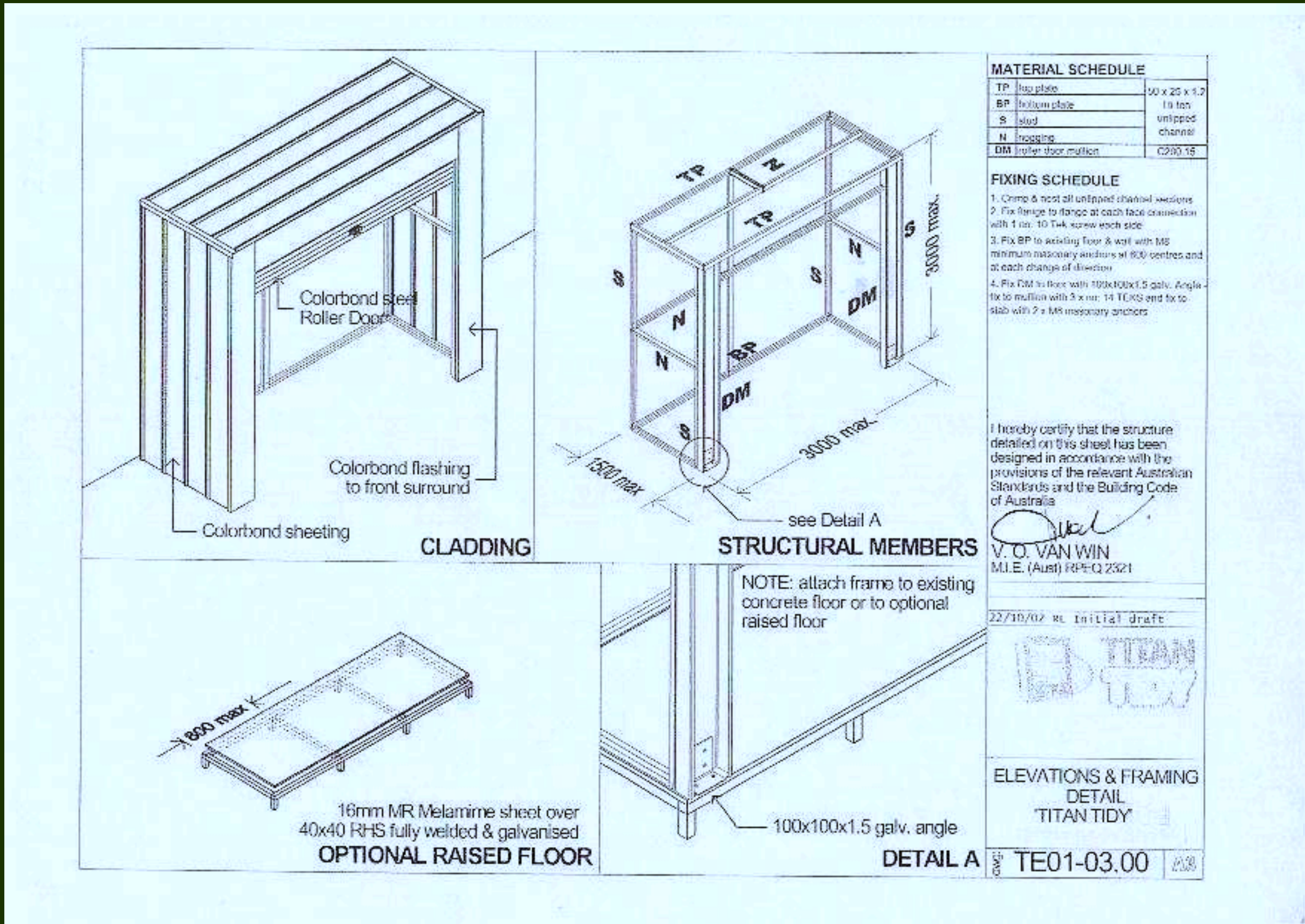
$$\boxed{\alpha^2 + \beta^2 + \gamma^2 = 0}$$

Example: $\alpha = 2 - 26i \quad \beta = -23 + 2i \quad \gamma = 14 + 7i$

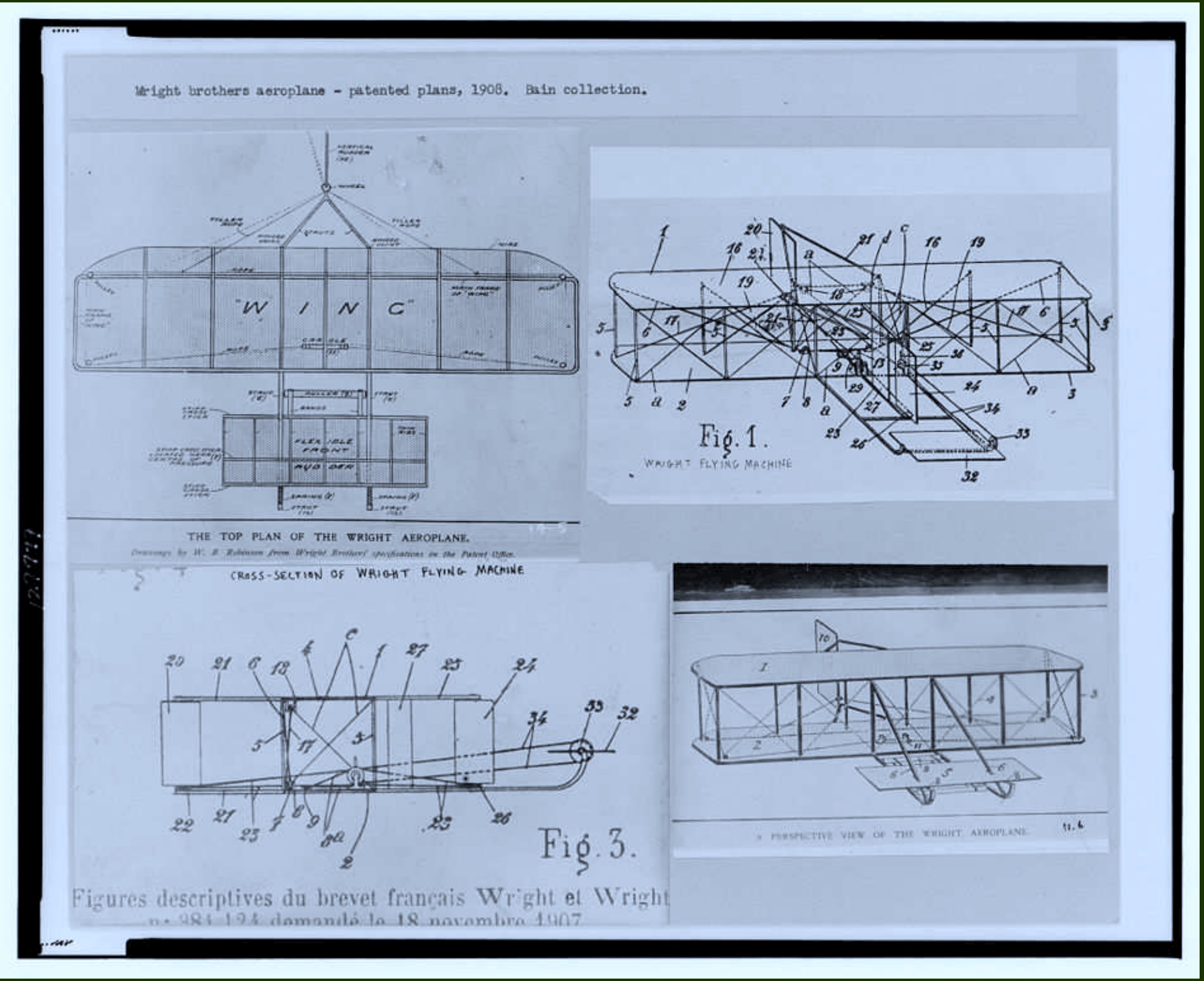
Square root



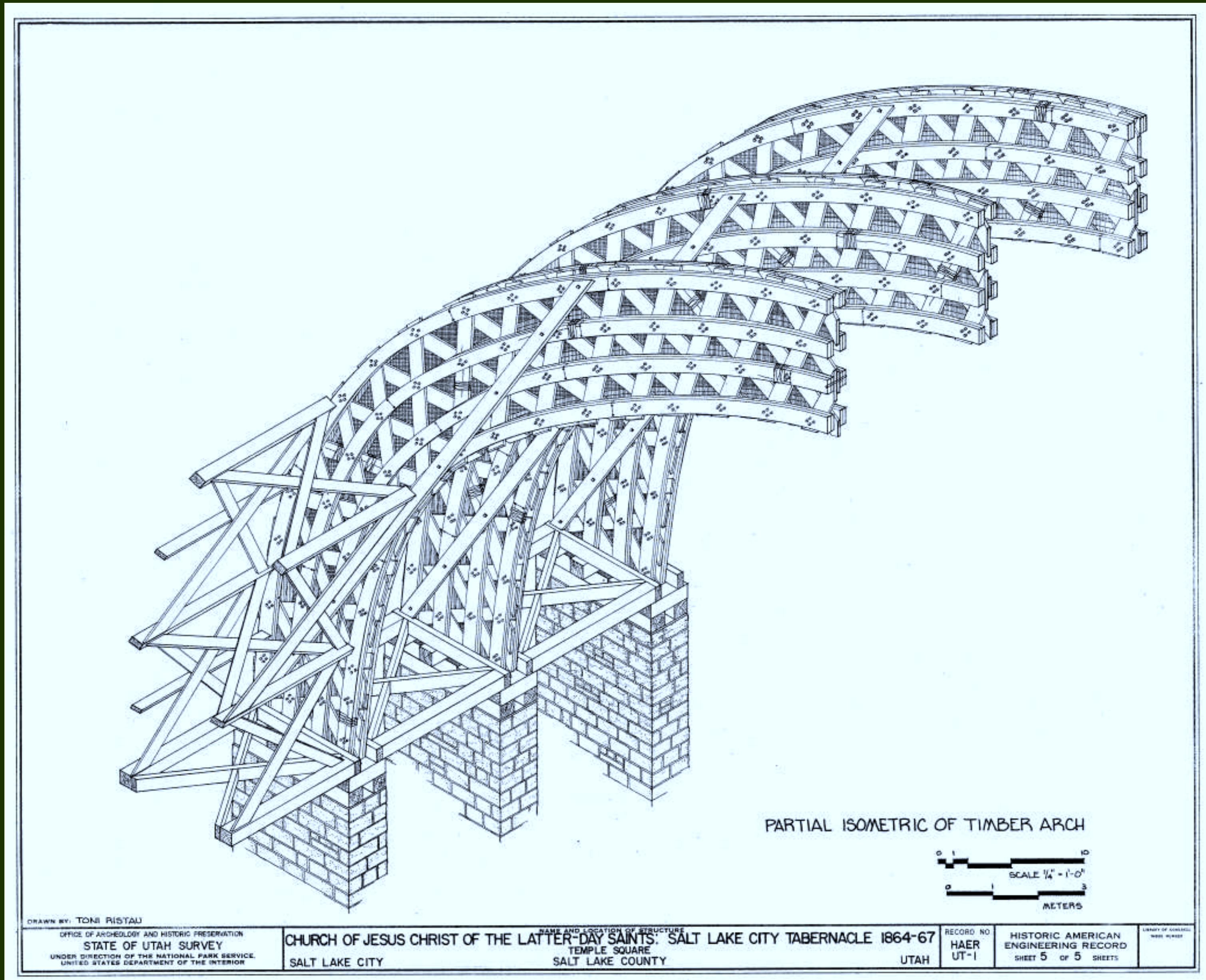
Engineering drawing: shed



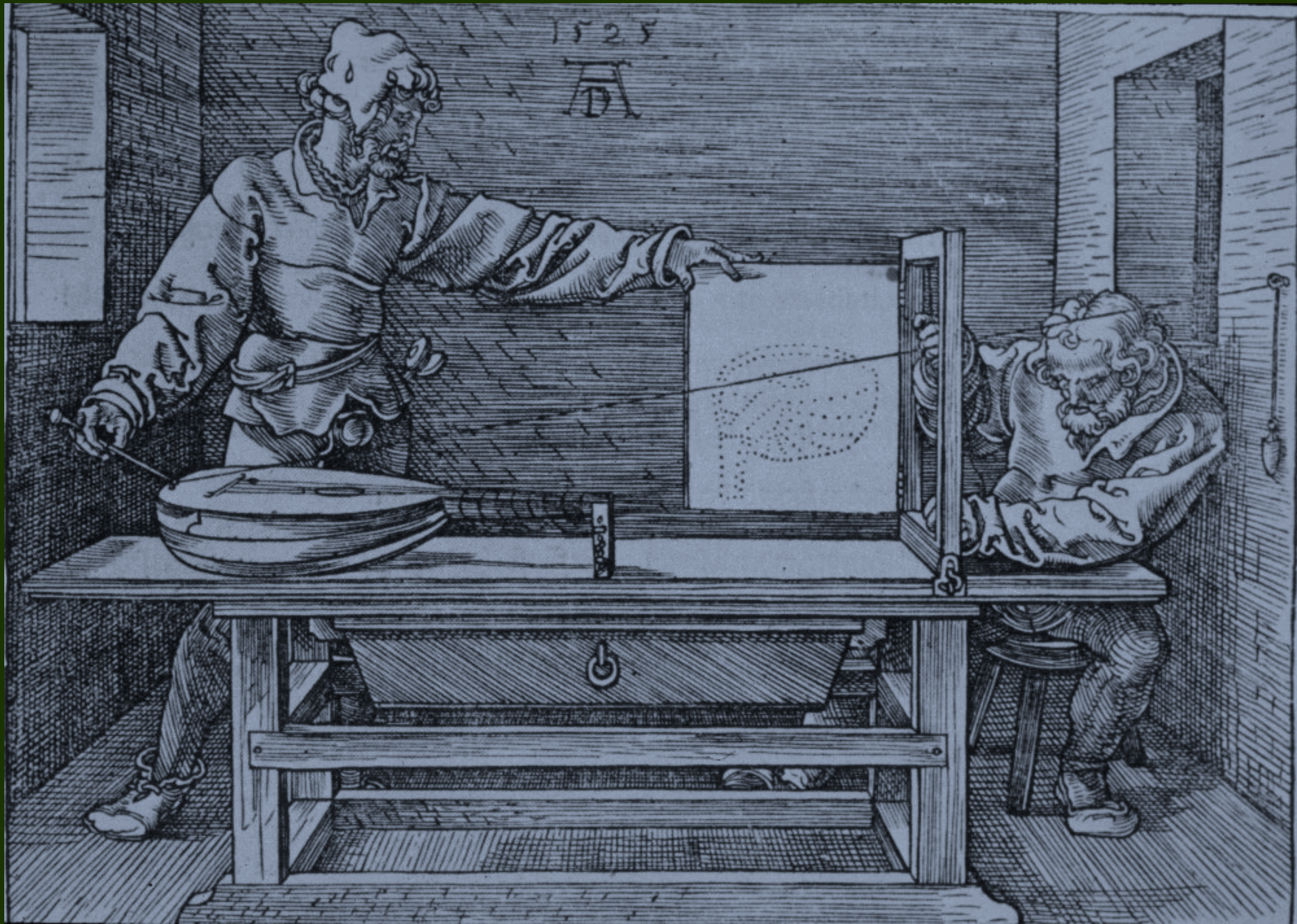
Engineering drawing: aeroplane



Engineering drawing: tabernacle



Perspective drawing: lute



Proof of Gauß's Theorem

Given
$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} \equiv X$$
Want

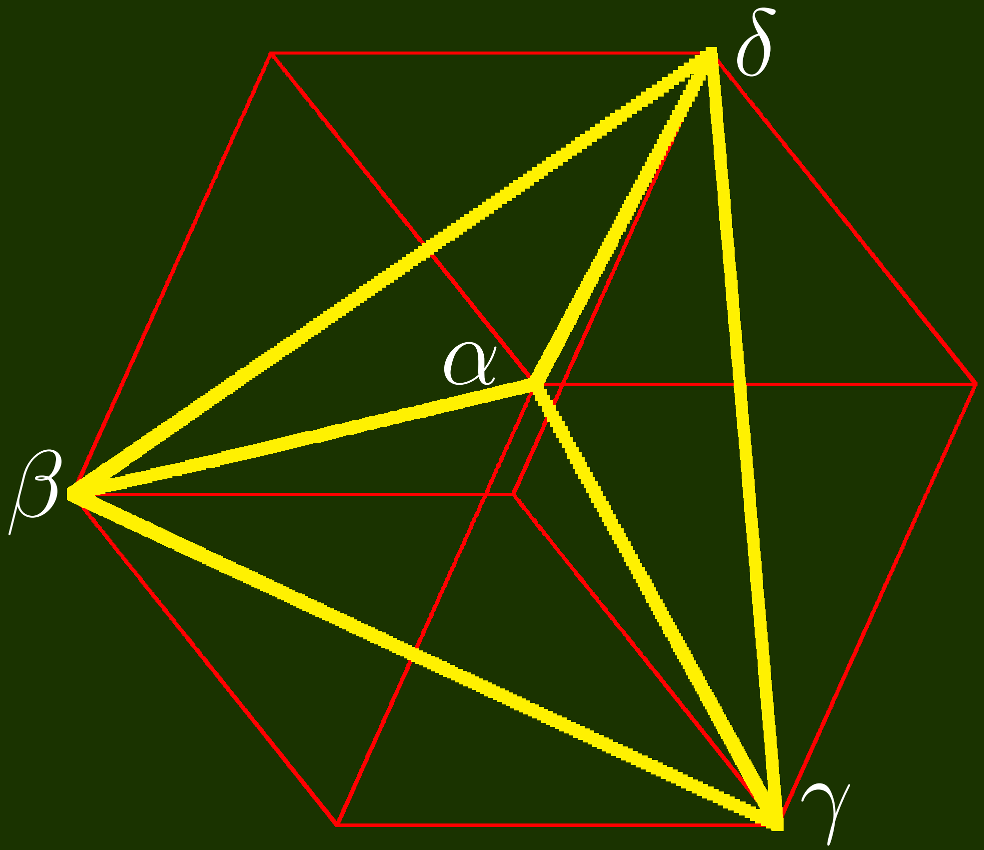
such that $XX^t = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$.

$$X^t = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

But $XX^t = \lambda \mathrm{Id} \iff X^t X = \lambda \mathrm{Id} \iff \boxed{\begin{array}{l} \|x\|^2 = \|y\|^2 \\ \langle x, y \rangle = 0 \end{array}}$

$$\iff \boxed{\alpha^2 + \beta^2 + \gamma^2 = 0}$$

Regular tetrahedron



$$\begin{aligned}\alpha &= 9+20i\\ \beta &= -12+15i\\ \gamma &= 20\\ \delta &= 17+35i\end{aligned}$$

ME & R. Penrose

$$\boxed{(\alpha+\beta+\gamma+\delta)^2 = 4(\alpha^2+\beta^2+\gamma^2+\delta^2)}$$

Any tetrahedron

Vertices $\leftrightarrow \begin{bmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ a & b & c & d \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix} \equiv A$ $\boxed{\text{NB}}$ AA^t is invertible

Theorem (ME & R. Penrose) Let $Q = A^t(AA^t)^{-2}A$

$$a \mapsto \alpha \quad b \mapsto \beta \quad c \mapsto \gamma \quad d \mapsto \delta$$

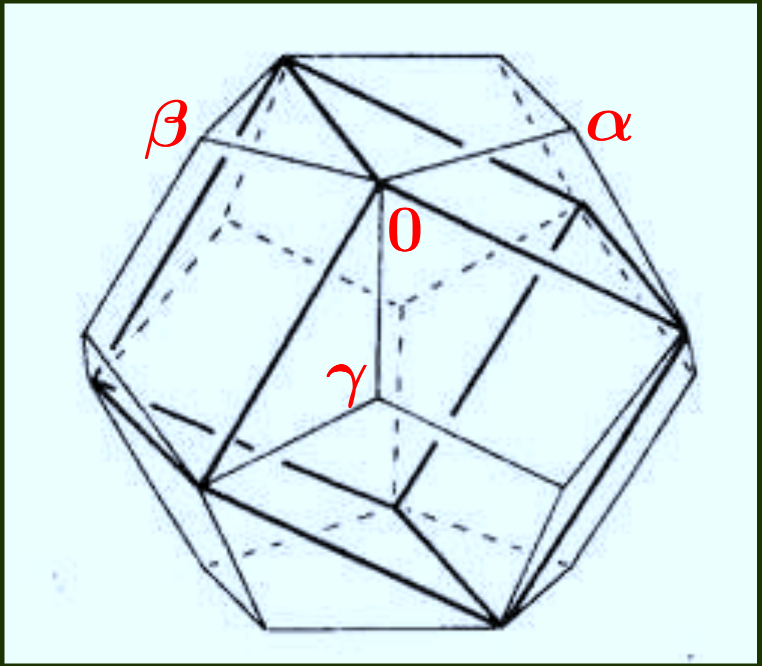
if and only if $\boxed{[\alpha,\beta,\gamma,\delta]\,Q\,[\alpha,\beta,\gamma,\delta]^t = 0}$

Applications

- CAD = Computer-aided Design
- Tracking

Example

Dodecahedron



$$\boxed{(\alpha+\beta+\gamma)^2 + (\sqrt{5}-1)(\alpha^2+\beta^2+\gamma^2) = 0}$$

Higher mathematics

- Complex numbers in three-dimensional geometry

$$\mathrm{SU}(2) \xrightarrow{\text{2–1}} \mathrm{SO}(3)$$

- Complex numbers in n-dimensional geometry

$$\begin{array}{c}\mathrm{Gr}_2^+(\mathbb{R}^n) \cong \ Q_{n-2} \ \subset \mathbb{CP}_{n-1} \\ \| \\ \{[z] \in \mathbb{CP}_{n-1} \mid {z_1}^2 + {z_2}^2 + \cdots + {z_n}^2 = 0\}\end{array}$$

Further reading

- M.G. Eastwood and R. Penrose, *Drawing with complex numbers*, Mathematical Intelligencer **22** (2000) 8–13.
- C.F. Gauss, *Werke II (page 309)*, Göttingen 1876.
- H. Hadwiger, *Über ausgezeichnete Vectorsterne und reguläre Polytope*, Commentarii Mathematici Helvetici **13** (1940) 90–108.
- R.P. Hoelscher and C.H. Springer, *Engineering Drawing and Geometry*, Wiley 1961.
- R.N. Roth and I.A. van Haeringen, *The Australian Engineering Drawing Handbook*, Australian Institute of Engineers 1988.

THANK YOU