

**Conformal differential geometry  
and its interaction with representation theory**

# **Introduction to conformal differential geometry**

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# Motivation from physics

- GR: null geodesics are conformally invariant
- Maxwell's equations are conformally invariant

$g_{ab}$  = (pseudo-)metric on  $M$ , a smooth  $n$ -manifold

$\hat{g}_{ab} = \Omega^2 g_{ab}$  = conformally related metric (angles OK)

$g^2 : T^*M \rightarrow \mathbb{R} \rightsquigarrow dg^2 \rightsquigarrow X_{g^2} \rightsquigarrow$  geodesic spray

$\hat{g}^2 = \Omega^2 g^2 \rightsquigarrow d\hat{g}^2 = \Omega^2 dg^2 + g^2 d\Omega^2 \quad \therefore X_{\hat{g}^2}|_{g=0} \propto X_{g^2}|_{g=0}$

$g_{ab} \rightsquigarrow \epsilon_{ab\dots de}$  volume form (e.g.  $\epsilon^{ab\dots de} \epsilon_{ab\dots de} = n!$ )

$\therefore \hat{g}_{ab} = \Omega^2 g_{ab} \implies \hat{\epsilon}_{ab\dots de} = \Omega^n \epsilon_{ab\dots de}$

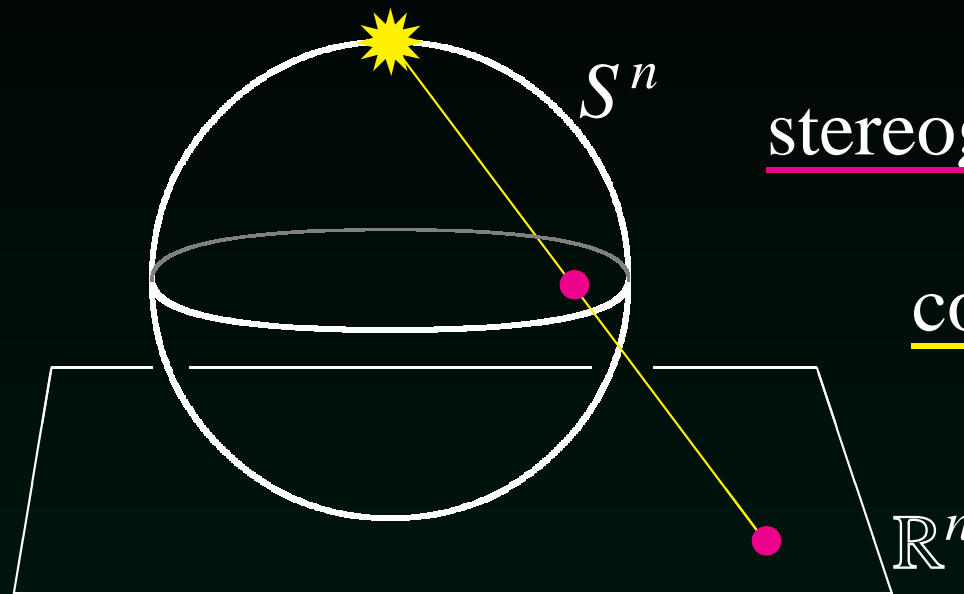
$\therefore \hat{\epsilon}_{ab}{}^{cd} = \epsilon_{ab}{}^{cd}$  when  $n = 4$

$\therefore F_{ab} \mapsto *F_{ab} \equiv \epsilon_{ab}{}^{cd} F_{cd}$  is invariant

$$dF = 0$$

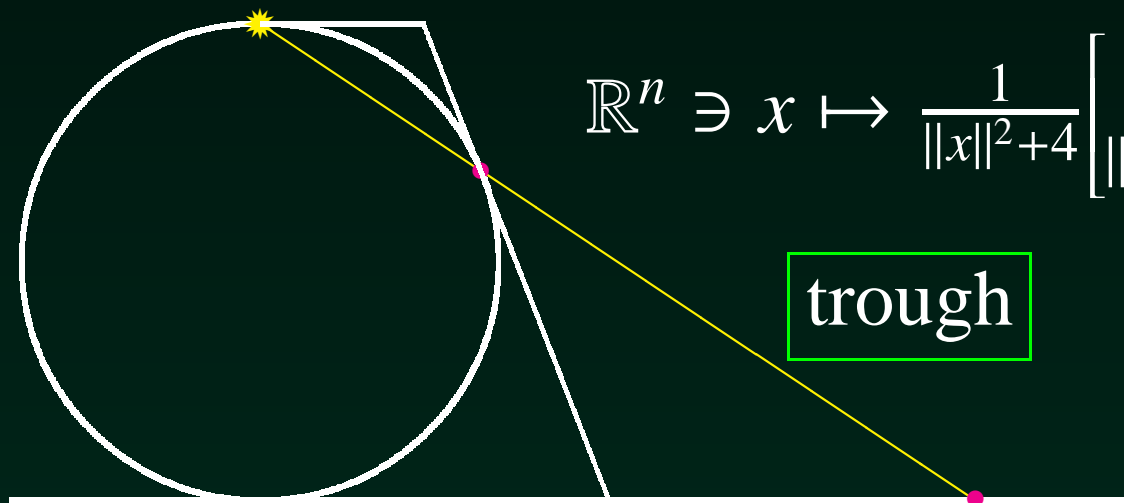
$$d*F = 0$$

# Motivation from geometry



stereographic projection

conformal

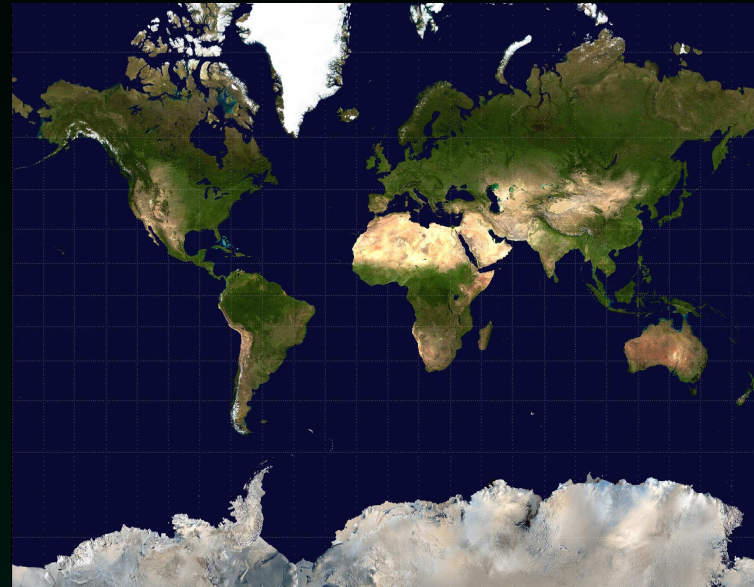


$$\mathbb{R}^n \ni x \mapsto \frac{1}{\|x\|^2 + 4} \begin{bmatrix} 4x \\ \|x\|^2 - 4 \end{bmatrix} \in S^n$$

trough

# Motivation from navigation

- Mercator  
(Cartographer) 1569



- Wright (Mathematician) 1599

$$S^2 \setminus \{\text{poles}\} \xrightarrow{\text{stereographic}} \mathbb{R}^2 \setminus \{0\} = \mathbb{C} \setminus \{0\} \xrightarrow{\log} \mathbb{C}$$

$$\text{Jac} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \iff \begin{matrix} u_x = v_y \\ v_x = -u_y \end{matrix} \quad \boxed{\text{Cauchy-Riemann}}$$

# Euclidean symmetries

$X$  = vector field

$$X = X^1 \frac{\partial}{\partial x_1} + X^2 \frac{\partial}{\partial x_2} + \dots + X^n \frac{\partial}{\partial x_n} = X^a \nabla_a$$

Infinitesimal Euclidean symmetry:  $\underbrace{\mathcal{L}_X \delta_{ab}}_{\text{Lie derivative}} = 0$ .

Compute

$$\begin{aligned} \mathcal{L}_X \delta_{ab} &= X^c \nabla_c \delta_{ab} + \delta_{cb} \nabla_a X^c + \delta_{ac} \nabla_b X^c \\ &= \nabla_a X_b + \nabla_b X_a \end{aligned}$$

$\therefore \mathcal{L}_X \delta_{ab} = 0 \iff \boxed{\nabla_{(a} X_{b)} = 0}$  Killing field

# Killing fields by prolongation

Killing operator:  $X_a \mapsto \nabla_{(a} X_b)$

Kernel in flat space:  $K_{ab} \equiv \nabla_a X_b$  is skew.

Claim:  $\nabla_a K_{bc} = 0$ .  
 $\nabla_a K_{bc} = \nabla_c K_{ba} - \nabla_b K_{ca}$   
 $= \nabla_c \nabla_b X_a - \nabla_b \nabla_c X_a$   
 $= 0$ , as required.

Hence,  $\nabla_{(a} X_b) = 0 \iff$

$$\begin{aligned} \nabla_a X_b &= K_{ab} \\ \nabla_a K_{bc} &= 0 \end{aligned}$$

Closed!

Conclusion:  $X_a = s_a + m_{ab} x^b$  where  $m_{ab} = -m_{ba}$ .

translations

rotations

# Conformal symmetries

trace-free part  $\nabla_{(a}X_{b)} = 0$  conformal Killing field

Rewrite as  $\nabla_a X_b = K_{ab} + \Lambda \delta_{ab}$  where  $K_{ab}$  is skew.

$$\begin{aligned}\nabla_a K_{bc} &= \nabla_c K_{ba} - \nabla_b K_{ca} \\ &= \nabla_c \nabla_b X_a - \nabla_b \nabla_c X_a - \delta_{ab} \nabla_c \Lambda + \delta_{ac} \nabla_b \Lambda \quad \text{so}\end{aligned}$$

$\nabla_a K_{bc} = \delta_{ab} Q_c - \delta_{ac} Q_b$  where  $\nabla_a \Lambda = -Q_c$  but

$$\begin{aligned}0 &= \delta^{ab} (\nabla_d \nabla_a K_{bc} - \nabla_a \nabla_d K_{bc}) \\ &= \delta^{ab} (\delta_{ab} \nabla_d Q_c - \delta_{ac} \nabla_d Q_b - \delta_{db} \nabla_a Q_c + \delta_{dc} \nabla_a Q_b) \\ &= (n-2) \nabla_d Q_c + \delta_{dc} \nabla^a Q_a \quad \text{whence}\end{aligned}$$

$\nabla_a Q_b = 0$  if  $n \geq 3$  Closed!!

# Conformal symmetries cont'd

Solve

$$\nabla_a X_b = K_{ab} + \Lambda \delta_{ab}$$

$$\nabla_a K_{bc} = \delta_{ab} Q_c - \delta_{ac} Q_b$$

$$\nabla_a \Lambda = -Q_c$$

$$\nabla_a Q_b = 0$$

$$Q_b = -r_b \quad \Lambda = \lambda + r^b x_b \quad K_{bc} = r_b x_c - r_c x_b - m_{bc}$$

$$X_a = s_a + m_{ab} x^b + \lambda x_a + r^b x_b x_a - \frac{1}{2} r_a x^b x_b$$

translation + rotation + dilation + inversion

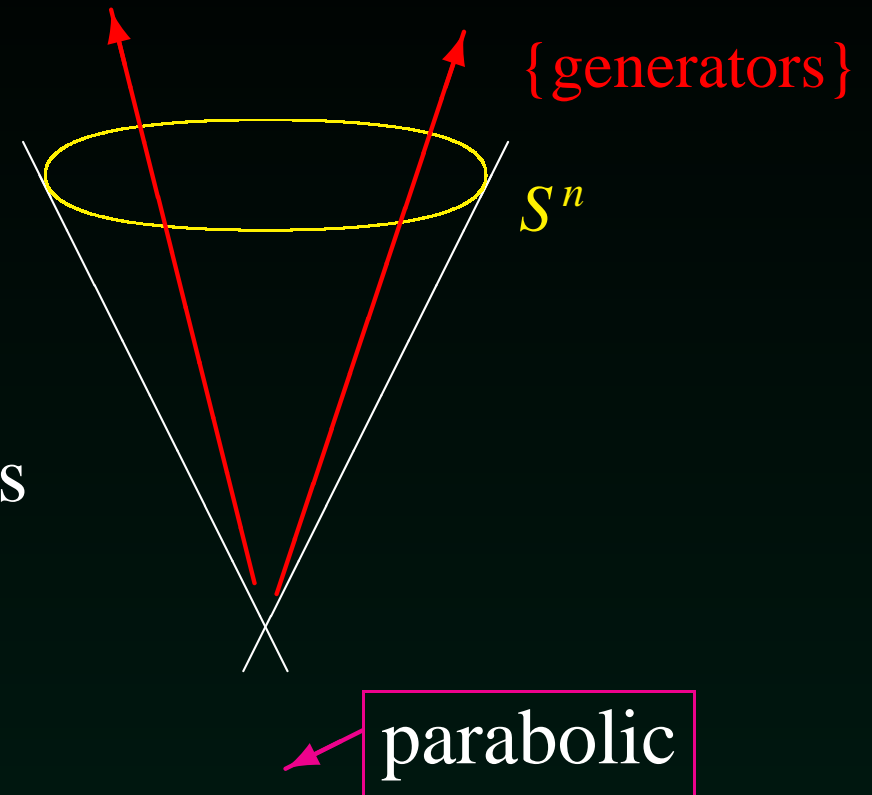
Integrate the inversions

$$x^a \mapsto \frac{x^a - \frac{1}{2} r^a \|x\|^2}{1 - r_a x^a + \frac{1}{4} \|r\|^2 \|x\|^2}$$



# Conformal group

$SO(n + 1, 1)$  acts on  $S^n$   
by conformal transformations



semisimple

parabolic

$$S^n = SO(n + 1, 1)/P$$

flat model of conformal differential geometry

$$\mathbb{R}^n = (SO(n) \ltimes \mathbb{R}^n)/SO(n)$$

flat model of Riemannian differential geometry

# A simple question on $\mathbb{R}^n$ , $n \geq 3$

Question: Which linear differential operators preserve harmonic functions? Answer on  $\mathbb{R}^3$ :—

Zeroth order  $f \mapsto \text{constant} \times f$  1

First order

$$\nabla_1 = \partial/\partial x_1 \quad \nabla_2 = \partial/\partial x_2 \quad \nabla_3 = \partial/\partial x_3 \quad \text{3}$$

$$x_1 \nabla_2 - x_2 \nabla_1 \quad \&c. \quad \text{3}$$

$$x_1 \nabla_1 + x_2 \nabla_2 + x_3 \nabla_3 \quad \boxed{+1/2} \quad \text{1}$$

$$(x_1^2 - x_2^2 - x_3^2) \nabla_1 + 2x_1 x_2 \nabla_2 + 2x_1 x_3 \nabla_3 + x_1 \quad \text{3}$$

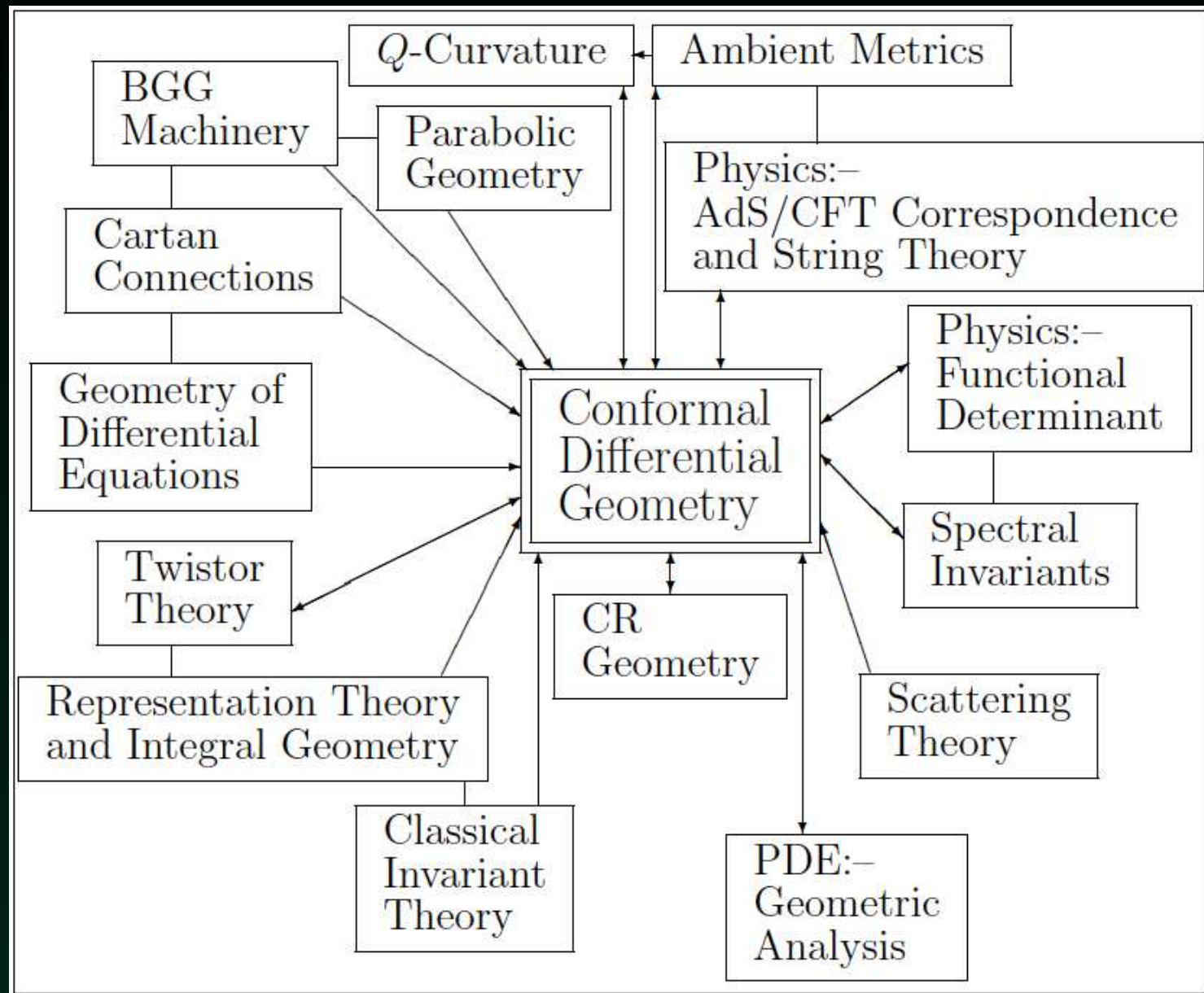
&c.

Dimensions ..... 10

$$[\mathcal{D}_1, \mathcal{D}_2] \equiv \mathcal{D}_1 \mathcal{D}_2 - \mathcal{D}_2 \mathcal{D}_1$$

Lie Algebra  $\cong \mathfrak{so}(4, 1) = \boxed{\text{conformal algebra}} \leftarrow \text{NB!}$

# Surroundings



# Next four talks

- What about conformally invariant operators? (Beyond Maxwell)
- What about higher order operators preserving harmonic functions? (Beyond first order)
- Bateman's formula and twistor theory
- The X-ray transform

## Further Reading

- M.G. Eastwood, Notes on conformal differential geometry, *Suppl. Rend. Circ. Mat. Palermo* **43** (1996) 57–76.
- R. Penrose and W. Rindler, *Spinors and space-time*, vols 1 and 2, Cambridge University Press 1984 and 1986.

THANK YOU

END OF PART ONE