

Remarks/Notes on 'The moduli space of genus 2 Riemann surfaces'
- Algebra and Topology Seminar: 15th October 2013

Notes from original talk on 24th March 2000 follow but they adopt a more classical trick of turning binary forms into differential operators rather than the equivalent more

$$\text{Diagram} = S \mapsto \hat{S} = \text{Diagram}$$

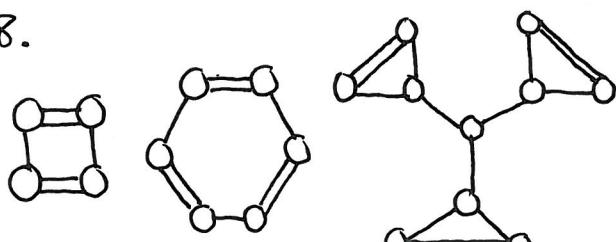
in terms of Roger Penrose's bug notation (cf. Clifford's handwritten notes, especially the page on the binary sextic).

Also, two cases were omitted in 2000. They were found by Alexander Isaev who went through the analysis carefully in 2011. They are x^4y^2 and $x^4(x^2+y^2)$.

For a full proof see Theorem 5.2 of

MGE and A.V. Isaev, Extracting invariants of isolated hypersurface singularities from their moduli algebras, *Math. Ann.* 356 (2013), 73–98.

For a discussion of



as the freely generating invariants of trace-free ternary cubics under $SO(3, \mathbb{C})$, see

MGE and V.V. Ezhov, A classification of non-degenerate homogeneous equiaffine hypersurfaces in four complex dimensions, *Asian Jour. Math.* 5 (2001), 721–740.

The Joy of Sextic

dg talk, Friday 24/3/2K

Motivation: Clear. More later (work with Volodya) if time.

$$S = ax^6 + 6bx^5y + 15cx^4y^2 + 20dx^3y^3 + 15ex^2y^4 + 6fy^5 + gy^6$$

$$\text{E.g. } 2x^6 + 18x^5 + 10x^3 - 1$$

$$Q = ax^2 + 2bxy + cy^2 \quad \text{zeroes} \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

better

$$x = \frac{-b \pm \sqrt{b^2 - ac}}{a}$$

better

$$ax = (-b \pm \sqrt{b^2 - ac})y$$

For Q either complete the square or

$$\left. \begin{array}{l} x \mapsto \hat{x} = -\partial/\partial y \\ y \mapsto \hat{y} = \partial/\partial x \end{array} \right\} \text{This is } \text{SL}(2, \mathbb{C}) \text{ invariant!}$$

Check

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}$$

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} s & -q \\ -r & p \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For quadratics:-

$$\hat{x}^2 = -x \partial/\partial y$$

$$\hat{2xy} = x \partial/\partial x - y \partial/\partial y$$

$$\hat{y}^2 = y \partial/\partial x$$

Also
 $\text{SL}(2, \mathbb{C})$ -invariant.

$$\hat{Q} x = bx + cy$$

$$\hat{Q} y = -ax - by$$

$$\hat{Q} \leftrightarrow \begin{bmatrix} b & c \\ -a & -b \end{bmatrix}$$

Can regard as
completely equivalent (in an $\text{SL}(2, \mathbb{C})$ -invariant way)!

charpoly of \hat{Q} is $\lambda^2 - b^2 + ac$ EASY!

E.g. eigenform $ax + (b - \sqrt{b^2 - ac})y$ with eigenvalue $\lambda = \sqrt{b^2 - ac}$

X choose any Y to go with (e.g. $Y = \frac{1}{a}y$)

$$\hat{Q} X = \lambda X \quad \text{NB: } \hat{2XY} X = X$$

$$\text{So } \hat{Q} - 2\lambda XY X = 0 \quad \text{So } Q - 2\lambda XY = \mu X^2$$

So $Q = \mu X^2 - 2\lambda XY = X(\mu X - 2\lambda Y) = uv$ Canonical Form.
(if roots are distinct)

As spin off: $X=0$ is a zero i.e. $\alpha x = (-b + \sqrt{b^2 - ac})y$ as before.

Cubic: Similar trick different details: $C = u^3 + v^3$ is Canonical Form.

Quartic: Same trick! Really quite different. ↑ (if roots are distinct)

Quintic: Same trick as Cubic Geometrically Clear
but may instead

Sextic: Same trick cf. Sextic

$$\widehat{x^6} = -x^3 \frac{\partial^3}{\partial y^3}$$

$$\widehat{6x^5y} = 3x^3 \frac{\partial^2}{\partial x \partial y^2} - 3x^2y \frac{\partial^3}{\partial y^3}$$

⋮ $SL(2, \mathbb{C})$ -invariant

So (suppressing an overall factor of 6):-

$$\begin{bmatrix} x^3 \\ x^2y \\ xy^2 \\ y^3 \end{bmatrix} \xrightarrow{\widehat{S}} \underbrace{\begin{bmatrix} d & 3e & 3f & g \\ -c & -3d & -3e & -f \\ b & 3c & 3d & e \\ -a & -3b & -3c & -d \end{bmatrix}}_{\text{charpoly}} \begin{bmatrix} x^3 \\ x^2y \\ xy^2 \\ y^3 \end{bmatrix}$$

$$\lambda^4 + (ag - 6bf + 15ec - 10d^2)\lambda^2 + (9d^4 + \dots)$$

a double quadratic. Easily solved so can find
→ explicit eigencubics.

Suppose \widehat{S} has one with distinct roots. Put in canonical form:-

$$\widehat{S}(X^3 + Y^3) = \lambda(X^3 + Y^3)$$

N.B. $\widehat{X^6 - Y^6}(X^3 + Y^3) = -(X^3 + Y^3) \approx$

$$\widehat{S + \lambda(X^6 - Y^6)}(X^3 + Y^3) = 0$$

(3)

$$S_0 \cdot S = C(X+Y)^6 + A(wX+w^2Y)^6 + B(w^2X+wY)^6 + D(X^6-Y^6)$$

$$S_0 \cdot S = Cu^6 + Av^6 + Bw^6 + Duvw(u-v)(v-w)(w-u)$$

where $u+v+w=0$. \uparrow Sylvester Canonical Form (1854).

Question: Which sextics cannot be put into SCF?

Answer: NEW!

$$X^2Y \rightsquigarrow AX^6 + 6BX^5Y + 20DX^3Y^3 + GY^6 *$$

$$X^3 \rightsquigarrow AX^6 + 6BX^5Y + 15CX^4Y^2 + 20DX^3Y^3$$

$$\begin{bmatrix} D & 0 & 0 & 0 \\ -C & -3D & 0 & 0 \\ & & & 0 \end{bmatrix} \quad \begin{array}{l} D=0 \Rightarrow X^2Y \text{ is eigen cubic} \\ D \neq 0 \Rightarrow X^2(CX+4DY) \text{ } \parallel \\ \text{so revert to } * \end{array}$$

* Can write down eigencubics:-

$$\lambda = -3D : X^2Y$$

$$\lambda = 3D : 8BD^2X^3 - B^2G X^2Y + 2D(AG + 8D^2)XY^2 + 2BDGY^3$$

$$\lambda = \pm \sqrt{D^2 - AG} : (D+\lambda)(3D+\lambda)X^3 - 3BGX^2Y + (3D+\lambda)GY^3$$

$$\begin{aligned} \text{Equ}'s \quad \text{discrim } (8BD^2X^3 \dots) &= 0 && \left. \begin{array}{l} \text{degree 12} \\ \text{degree 8} \\ \text{degree 8} \end{array} \right\} \text{Soluble!!} \\ \text{II} \quad ((D+\lambda)(3D+\lambda)X^3 + \dots) &= 0 \\ \text{II} \quad ((D-\lambda)(3D-\lambda)X^3 + \dots) &= 0 \\ D^2 - AG &= \lambda^2 \end{aligned}$$

Then have to consider
again and so on] : 5 Exceptions X^3Y^3 X^5Y $(X^5 + Y^5)Y$

They are distinct because
of location of zeroes and
of eigenvalues

$$\sqrt{3+\sqrt{7}}X^6 + 3(1+\sqrt{7})X^5Y + 10X^3Y^3 + 2\sqrt{3+\sqrt{7}}Y^6$$

EEEK!

Further discussion of these special cases

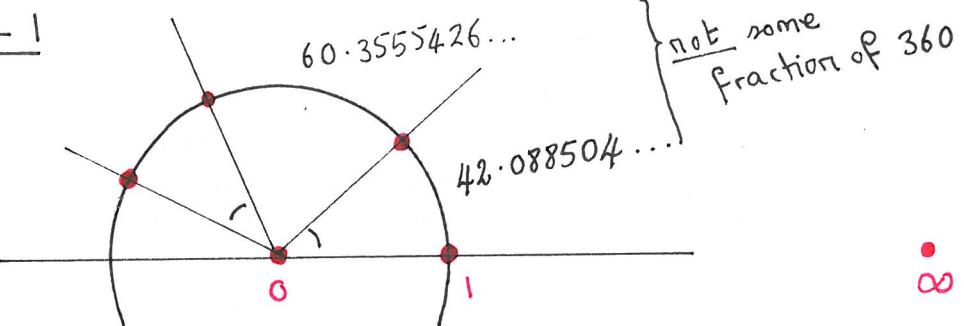
Where are the zeroes? x^3y^3 x^5y $(x^5+y^5)y$ easy.

↑ NB Five zeroes on a circle etcetera.

$$2x^6 + 18x^5 + 10x^3 - 1$$

discriminant

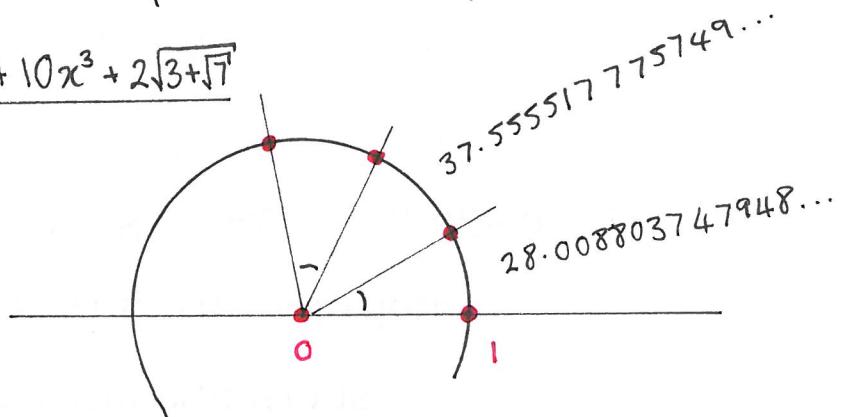
$$2^6 \cdot 3^{14} \cdot 7^2 \cdot 13!$$



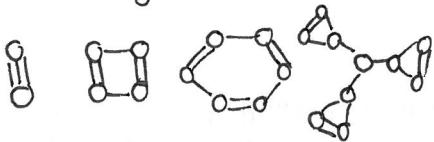
$$\sqrt{3+\sqrt{7}} x^6 + 3(1+\sqrt{7})x^5 + 10x^3 + 2\sqrt{3+\sqrt{7}}$$

discriminant

nowt special.



Invariants I, J, K, L
degrees 2 4 6 10



$$j = J/I^2 \quad k = K/I^3 \quad l = L/I^5$$

$$\begin{array}{lll} x^3 & 11/25 & 29/125 \\ x^5 & I = J = K = L = 0 \\ x^5 + 1 & I = J = K = 0 \quad L \neq 0 \end{array} \quad \frac{32}{3125}$$

$$\begin{array}{ccccc} 2x^6 + 18x^5 + 10x^3 - 1 & \frac{1}{3} & -\frac{7}{9} & \frac{16}{27} & || \\ \sqrt{3+\sqrt{7}} x^6 + 3(1+\sqrt{7})x^5 + 10x^3 + 2\sqrt{3+\sqrt{7}} & \frac{5}{7} & 1 & -\frac{64}{49} & || \end{array}$$

Alternative forms.

(fits with locations of zeroes)

double cubic
(and the cubic has discriminant $2^{10} 3^4 7^3$)

$$(-13 - 16\sqrt{2})x^6 + 105x^4 + 105x^2 - 13 + 16\sqrt{2}$$

double cubic

195100: There is a form for the fifth exceptional sextic with integral coefficients:-

$$186x^6 - 192x^5 - 300x^4 - 320x^3 - 150x^2 - 48x + 23 \quad (\text{and has discriminant } 2^{10} 3^4 7)$$

FURTHER MOTIVATION ON NORMAL FORMS, if time permits.