Projective symmetries

Our objective is to derive the PDE satisfied by a vector field X^a in order that it be an infinitesimal symmetry of a projective structure. Recall that a projective structure is an equivalence class of torsion-free connections, where

$$\hat{\nabla}_a X^c = \nabla_a X^c + \Upsilon_a X^c + \delta_a{}^c \Upsilon_b X^b,$$

for some 1-form Υ_a or, in other words,

$$\hat{\nabla}_a X^c = \nabla_a X^c + \Gamma_{ab}{}^c X^b, \quad \text{where } \Gamma_{ab}{}^c = 2\delta_{(a}{}^c \Upsilon_{b)}.$$

An affine connection ∇_a is invariant under the flow generated by X^a if and only if the diagram

$$\begin{array}{ccc} TM & \xrightarrow{\nabla} & \Lambda^1 \otimes TM \\ \downarrow \mathcal{L}_X & & \downarrow \mathcal{L}_X \\ TM & \xrightarrow{\nabla} & \Lambda^1 \otimes TM \end{array}$$

commutes. Otherwise, the *Lie derivative of the connection* ∇ is defined to be the difference tensor $\Gamma : TM \to \Lambda^1 \otimes TM$, namely

$$\Gamma(Z) \equiv \mathcal{L}_X \nabla Z - \nabla \mathcal{L}_X Z$$

or, equivalently, $\Gamma : TM \otimes TM \to TM$ given by

$$\begin{aligned} \Gamma(Y,Z) &\equiv Y \lrcorner \big(\mathcal{L}_X \nabla Z - \nabla \mathcal{L}_X Z\big) \\ &= \mathcal{L}_X(Y \lrcorner \nabla Z) - (\mathcal{L}_X Y) \lrcorner \nabla Z - \nabla_Y \mathcal{L}_X Z \\ &= \mathcal{L}_X \nabla_Y Z - [X,Y] \lrcorner \nabla Z - \nabla_Y [X,Z] \\ &= [X, \nabla_Y Z] - [X,Y] \lrcorner \nabla Z - \nabla_Y [X,Z], \end{aligned}$$

which is usually how Γ is written. It is neater with abstract indices:-

$$\Gamma_{ab}{}^c Z^b = \mathcal{L}_X \nabla_a Z^c - \nabla_a \mathcal{L}_X Z^c.$$

Let us suppose that ∇_a is torsion-free. We may express the Lie derivative $\mathcal{L}_X$ in terms of any torsion-free affine connection and, in particular, use ∇_a itself for this purpose:-

$$\begin{aligned} \mathcal{L}_X \nabla_a Z^c &= X^d \nabla_d \nabla_a Z^c + (\nabla_a X^d) \nabla_d Z^c - (\nabla_d X^c) \nabla_a Z^d \\ \nabla_a \mathcal{L}_X Z^c &= \nabla_a (X^d \nabla_d Z^c - (\nabla_d X^c) Z^d) \\ &= X^d \nabla_a \nabla_d Z^c + (\nabla_a X^d) \nabla_d Z^c - (\nabla_d X^c) \nabla_a Z^d - (\nabla_a \nabla_d X^c) Z^d. \end{aligned}$$

Subtracting gives

$$\Gamma_{ab}{}^c Z^b = X^d (\nabla_d \nabla_a - \nabla_a \nabla_d) Z^c + (\nabla_a \nabla_b X^c) Z^b.$$

But the curvature of a torsion-free connection is characterised by

$$(\nabla_d \nabla_a - \nabla_a \nabla_d) Z^c = R_{da}{}^c{}_b Z^b.$$

Therefore

$$\Gamma_{ab}{}^c = X^d R_{da}{}^c{}_b + \nabla_a \nabla_b X^c.$$

By the Bianchi symmetry

$$R_{ab}{}^c{}_d = R_{db}{}^c{}_a - R_{da}{}^c{}_b = -2R_{d[a}{}^c{}_{b]}$$

so

$$R_{da}{}^c{}_b = R_{d(a}{}^c{}_{b)} + R_{d[a}{}^c{}_{b]} = R_{d(a}{}^c{}_{b)} - \tfrac{1}{2}R_{ab}{}^c{}_d.$$

Also,

$$\nabla_a\nabla_b X^c = \nabla_{(a}\nabla_{b)}X^c + \tfrac{1}{2}R_{ab}{}^c{}_d X^d.$$

Therefore,

$$\Gamma_{ab}{}^c = \nabla_a\nabla_b X^c + R_{da}{}^c{}_b X^d = \nabla_{(a}\nabla_{b)}X^c + R_{d(a}{}^c{}_{b)}X^d.$$

We recall the convenient curvature decomposition

$$R_{ab}{}^c{}_d = W_{ab}{}^c{}_d + \delta_a{}^c \mathrm{P}_{bd} - \delta_b{}^c \mathrm{P}_{ad},$$

where

$$W_{ab}{}^c{}_d \text{ is totally trace-free} \qquad \mathrm{P}_{ab} = \mathrm{P}_{(ab)}$$

in case the Ricci tensor is symmetric (which we may always suppose by a suitable projective change (calling such connections *special*)). In this case, we see that

$$R_{d(a}{}^c{}_{b)} = W_{d(a}{}^c{}_{b)} + \delta_d{}^c \mathrm{P}_{ab} - \delta_{(a}{}^c \mathrm{P}_{b)d}$$

so then

$$\Gamma_{ab}{}^c = \nabla_{(a}\nabla_{b)}X^c + \mathrm{P}_{ab}X^c + W_{d(a}{}^c{}_{b)}X^d - \delta_{(a}{}^c \mathrm{P}_{b)d}X^d.$$

Theorem The vector field X^a is an infinitesimal symmetry of a projective structure with special representative ∇_a if and only if

$$\left(\nabla_{(a}\nabla_{b)}X^c + \mathrm{P}_{ab}X^c + W_{d(a}{}^c{}_{b)}X^d\right)_\circ = 0$$

where $\circ$ means to take the trace-free part.

Proof In general, the tensor $\Gamma_{ab}{}^c$ breaks into two irreducible parts

$$\Gamma_{ab}{}^c = (\Gamma_{ab}{}^c)_\circ + \delta_{(a}{}^c \phi_{b)}$$

and a projective change is precisely when $\left(\Gamma_{ab}{}^c\right)_\circ = 0$. □

Remark The first BGG operator is

$$X^c \mapsto \left(\nabla_{(a}\nabla_{b)}X^c + \mathrm{P}_{ab}X^c\right)_\circ,$$

without the Weyl curvature (and both options are projectively invariant). However, as the only thing that really matters regarding the prolongation of this system is the symbol of this operator, this is a perfectly fine modification.

MikE, 8th February 2013