Conformal geometry and twistor theory

Twistor theory and the harmonic hull

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[joint work with Feng Xu]

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Harmonic functions

The <u>Laplacian</u> on $\mathbb{R}^n$: $\Delta = \dfrac{\partial^2}{\partial x_1{}^2} + \dfrac{\partial^2}{\partial x_2{}^2} + \cdots + \dfrac{\partial^2}{\partial x_n{}^2}$

$\Delta u = 0 \implies u$ is <u>real-analytic</u>

$$\therefore \quad u(x_1, x_2, \cdots, x_n) \rightsquigarrow \widehat{u}(z_1, z_2, \cdots, z_n)$$

More precisely,

$$
\begin{array}{ccccc}
\mathbb{R}^n & \supseteq^{\text{open}} & U & \xrightarrow{u} & \mathbb{R} \text{ or } \mathbb{C} \\
\cap & & \cap & & \cap \\
\mathbb{C}^n & \supseteq^{\text{open}} & \widehat{U} & \xrightarrow{\widehat{u}} & \mathbb{C}
\end{array}
$$

<u>FAQ</u>: $\boxed{\text{Depends on } u?}$ $\boxed{\text{How big?}}$

2 dimensions

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = \left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2} \right) \underbrace{\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} \right)}_{\text{Cauchy-Riemann}}$$

- $\mathbb{R}^2 \supseteq^{\text{open}} U \xrightarrow{u} \mathbb{C}$ $\underline{\text{harmonic}}$
- U $\underline{\text{simply-connected}}$ $\Bigg\} \implies$

$$u(x_1, x_2) \;=\; f(x_1 + ix_2) \;+\; g(x_1 - ix_2)$$

$$\qquad\qquad\qquad\quad \underline{\text{holomorphic}} \qquad \underline{\text{holomorphic}}$$

$$\therefore \; \widehat{u}(z_1, z_2) = f(z_1 + iz_2) + g(z_1 - iz_2)$$

$\underline{\text{where it makes sense:}}$ $\boxed{z_1 + iz_2 \in U}$ and $\boxed{z_1 - iz_2 \in \overline{U}}$

2 dimensions cont'd

$\boxed{z_1 + iz_2 \in U}$ and $\boxed{z_1 - iz_2 \in \overline{U}}$ $\iff$

$\mathbb{C}^2$

(z_1, z_2)

$\boxed{\{(w_1 - z_1)^2 + (w_2 - z_2)^2 = 0\}}$

iy_1, iy_2

x_2

U

x_1

$\mathbb{R}^2$
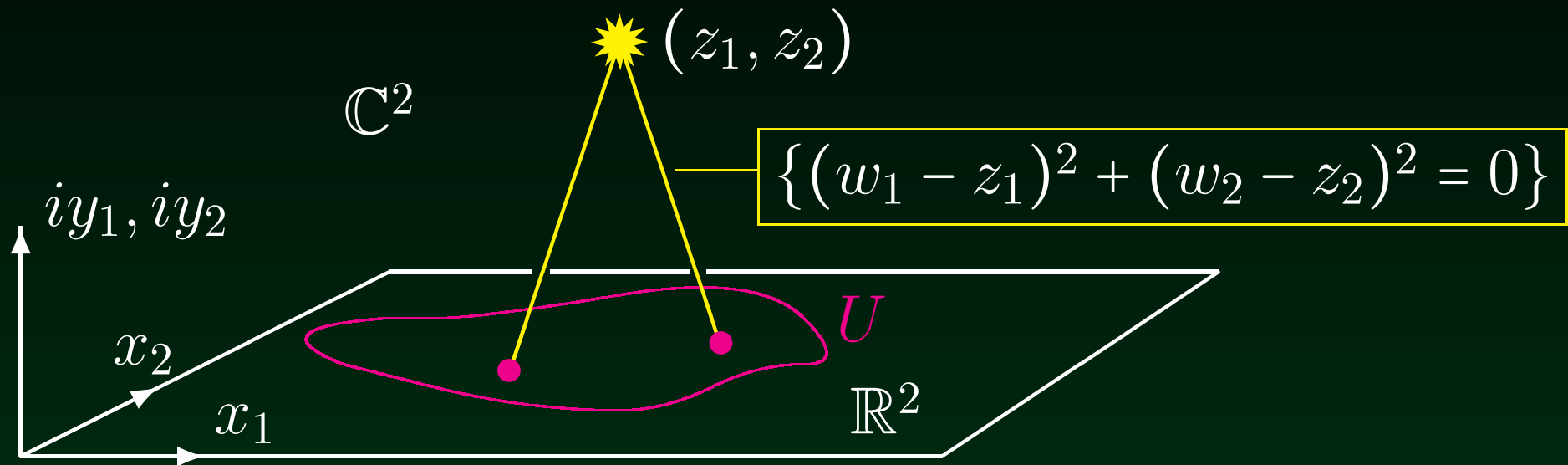
$\therefore$ <u>all</u> u harmonic on U extend to $\widetilde{u}$ holomorphic on

$$\widetilde{U} \equiv \{z \in \mathbb{C}^2 \text{ s.t. } \mathcal{N}(z) \cap \mathbb{R}^2 \subset U\}$$

where $\mathcal{N}(z) = \{w \in \mathbb{C}^2 \text{ s.t. } \|w - z\|^2 = 0\}$ $\boxed{\text{Null 'Cone'}}$

2 dimensions concl'd

If $U \subseteq \mathbb{R}^2$ is <u>simply-connected</u> then

- all harmonic functions on U extend to $\widetilde{U} \subseteq \mathbb{C}^2$,
- $\widetilde{U}$ is connected (and simply-connected),
- $\widetilde{U}$ is maximal in this respect ★.

$\widetilde{U}$ is called the <u>harmonic hull</u> of U.

★ consider $\log \|z - x\|^2$ for $x \in \partial U$.

If $U \subset \mathbb{R}^2$ is <u>multiply-connected</u>, it <u>does not have</u> a harmonic hull: consider

$$\log \|z - x\|^2 \quad \text{for } x \in \mathbb{R}^2 \smallsetminus U \text{ surrounded by } U.$$

4 dimensions

Bateman's formula

$$u(x) = \oint_{\gamma} f\left((x_1 + ix_2) + (ix_3 + x_4)\zeta, (ix_3 - x_4) + (x_1 - ix_2)\zeta, \zeta\right) d\zeta$$

a holomorphic function of 3 variables

a closed contour in the complex plane

Differentiation under the integral sign $\implies$ $\boxed{\Delta u = 0}$

FAQ (circa 1904–1980)

- where is f defined?
- where is γ located?
- which u arise in this way?

answer by means of Penrose transform

Suspend disbelief!

4 dimensions cont'd

Naïve use of Bateman

$$u(x) = \oint_\gamma f\left((x_1 + ix_2) + (ix_3 + x_4)\zeta, (ix_3 - x_4) + (x_1 - ix_2)\zeta, \zeta\right) d\zeta$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$u(z) = \oint_\gamma f\left((z_1 + iz_2) + (iz_3 + z_4)\zeta, (iz_3 - z_4) + (z_1 - iz_2)\zeta, \zeta\right) d\zeta$$

Perhaps we should insist that

$$\boxed{f(z, \zeta) \text{ be defined wherever } f(x, \zeta) \text{ is defined}} \text{ i.e.}$$

for each $z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4$,

$$L_z \equiv \left\{((z_1 + iz_2) + (iz_3 + z_4)\zeta, (iz_3 - z_4) + (z_1 - iz_2)\zeta, \zeta) \text{ s.t. } \zeta \in \mathbb{C}\right\}$$

$$\cap \ \underline{\text{WANT!}} \qquad \swarrow \text{open in } \mathbb{C}^3$$

$$\left\{((x_1 + ix_2) + (ix_3 + x_4)\zeta, (ix_3 - x_4) + (x_1 - ix_2)\zeta, \zeta) \text{ s.t. } x \in U, \zeta \in \mathbb{C}\right\}$$

What we want

Fix $z \in \mathbb{C}^4$. For any $\zeta \in \mathbb{C}$, we want to be able to solve

$$\begin{bmatrix} x_1 + ix_2 & ix_3 + x_4 \\ ix_3 - x_4 & x_1 - ix_2 \end{bmatrix} \begin{bmatrix} 1 \\ \zeta \end{bmatrix} = \begin{bmatrix} z_1 + iz_2 & iz_3 + z_4 \\ iz_3 - z_4 & z_1 - iz_2 \end{bmatrix} \begin{bmatrix} 1 \\ \zeta \end{bmatrix}$$

for some $x = (x_1, x_2, x_3, x_4) \in U \subseteq^{\mathrm{open}} \mathbb{R}^4$. Necessarily,

$$\det \left(\begin{bmatrix} x_1 + ix_2 & ix_3 + x_4 \\ ix_3 - x_4 & x_1 - ix_2 \end{bmatrix} - \begin{bmatrix} z_1 + iz_2 & iz_3 + z_4 \\ iz_3 - z_4 & z_1 - iz_2 \end{bmatrix} \right) = 0$$

i.e. $\boxed{(x_1 - z_1)^2 + (x_2 - z_2)^2 + (x_3 - z_3)^2 + (x_4 - z_4)^2 = 0}$

(also if $\zeta = \infty$). This is also sufficient: we require

$\boxed{\mathcal{N}(z) \cap \mathbb{R}^4 \subset U \quad \text{where } \mathcal{N}(z) = \text{null cone based at } z}$

4 dimensions concl'd

For <u>any</u> $U^{\text{open, connected}} \subseteq \mathbb{R}^4$, let

$$\widetilde{U} \equiv \left\{ z \in \mathbb{C}^4 \text{ s.t. } \mathcal{N}(z) \cap \mathbb{R}^4 \subset U \right\}^{\text{connected}}$$

where $\mathcal{N}(z) = \left\{ w \in \mathbb{C}^4 \text{ s.t. } \|w - z\|^2 = 0 \right\}$ $\boxed{\text{Null Cone}}$

- All harmonic functions on U extend to $\widetilde{U}$,
- $\widetilde{U}$ is maximal $\star$ (<u>harmonic hull</u>).

$\star$ Consider $1/\|z - x\|^2$ for $x \in \partial U$.

$\boxed{\text{NB}}$ $n = 4$ is <u>better</u> than $n = 2$ if <u>Bateman OK</u>.
<u>?????????????</u>

The twistor fibration

Looking at Bateman's formula, consider

$$\zeta \in \mathbb{C} \hookrightarrow \mathbb{C}^3 \quad \text{write } L_x \text{ for the range, where } x \in \mathbb{R}^4$$
$$\downarrow \qquad\qquad \cup$$
$$((x_1+ix_2)+(ix_3+x_4)\zeta,\, (ix_3-x_4)+(x_1-ix_2)\zeta,\, \zeta)$$

<u>Compactify!</u> $\quad \mathbb{CP}_1 \hookrightarrow \mathbb{CP}_3 \quad$ (still write range as L_x)

<u>Foliation</u> of $\ \mathbb{CP}_3 \smallsetminus \{[*,*,0,0]\} \ \hookrightarrow \ \mathbb{CP}_3$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow \tau \text{ submersion}$$

$$L_x = \tau^{-1}(x) \qquad \mathbb{R}^4 \qquad \overset{\hookrightarrow}{\underset{}{\longleftarrow}} \qquad S^4 \qquad \boxed{\text{cf. Hopf}}$$

$$\text{stereographic projection}$$

The Penrose transform

Theorem For any $U^{\mathrm{open}} \subseteq \mathbb{R}^4$

$$\mathcal{P} : H^1(\tau^{-1}(U), \mathcal{O}(-2)) \xrightarrow{\simeq} \{u : U \to \mathbb{C} \text{ s.t. } \Delta u = 0\}.$$
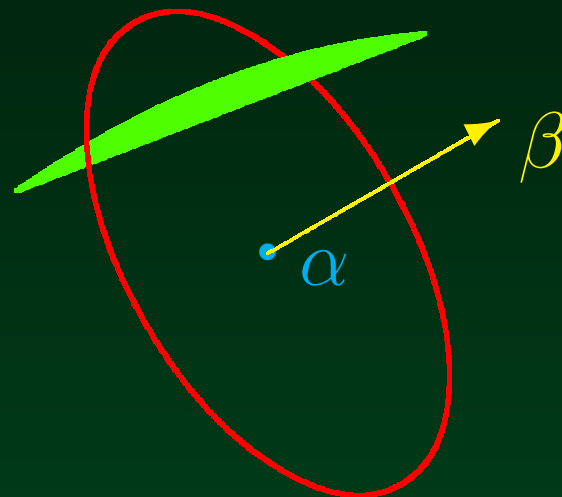
Interprets Bateman's formula and answers FAQ!

- $\mathcal{N}(z) \cap \mathbb{R}^4 \subset U \iff L_z \subset \tau^{-1}(U)$

- $\widetilde{U}$ = harmonic hull, as suspected

- For $U^{\mathrm{open}} \subseteq S^4, \Delta \rightsquigarrow \Delta - \frac{1}{6} R$ (Yamabe)

- OK in higher even dimensions (M.K. Murray).

Geometry of the harmonic hull

For $z = \alpha + i\beta \in \mathbb{C}^n$

$$\mathcal{N}(z) \cap \mathbb{R}^n = \{x \text{ s.t. } \|x - z\|^2 = 0\}$$
$$= \{x \text{ s.t. } |x - \alpha|^2 = |\beta|^2 \ \& \ (x - \alpha).\beta = 0\}$$

$\boxed{\text{NB}}$ $\mathcal{N}(z) \cap \mathbb{R}^n = \mathcal{N}(\bar{z}) \cap \mathbb{R}^n$

Odd dimensions

$U^{\text{open,connected}} \subseteq \mathbb{R}^{\text{odd}}$ need not have a harmonic hull!

Example $\mathbb{R}^3 \smallsetminus \{0\}$ Reasons

Recall $\mathbb{C}^3 \ni z \rightsquigarrow \mathcal{N}(z) \cap \mathbb{R}^3 \equiv \text{Circle}(z) \rightsquigarrow \text{Disc}(z)$

- OK extension to $\{z \text{ s.t. } \text{Disc}(z) \not\ni 0\}$
  Aronszajn, Creese, and Lipkin

- $f(x) \rightsquigarrow \frac{1}{\sqrt{1 - 2r.x + |r|^2 |x|^2}} f\left(\frac{x - r|x|^2}{1 - 2r.x + |r|^2 |x|^2}\right)$ harmonic $\checkmark$

- Hence should extend to $\mathbb{C}^3 \smallsetminus \{z_1^2 + z_2^2 + z_3^2 = 0\}$

- But $1/\sqrt{z_1^2 + z_2^2 + z_3^2}$ branches!!

Cf. Huygens, Kirchkoff, et cetera

Further Reading

- N. Aronszajn, T. Creese, and L. Lipkin, Polyharmonic Functions, Oxford University Press 1983.

- H. Bateman, The solution of partial differential equations by means of definite integrals, Proc. Lond. Math. Soc. **1** (1904) 451–458.

- M.G. Eastwood, The twistor construction and Penrose transform in split signature, Asian Jour. Math. **11** (2007) 103–112.

- B. Kroetz and H. Schlichtkrull, Holomorphic extension of eigenfunctions, arXiv:0812.0724.

- M.K. Murray, A Penrose transform for the twistor space of an even-dimensional conformally flat Riemannian manifold, Ann. Global Anal. Geom. **4** (1986) 71–88.

THANK YOU

THE END