COUNTING AND MEASURING I: WHAT'S THE DIFFERENCE?

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Abstract. Mathematics is a subject rife with connections. Here we give a few examples of the ways in which we can use techniques from the realm of measuring to solve number theoretic problems but the ideas within are not limited to analytic number theory.

What do we mean when we say that we can count something? Mathematically speaking, a set is countable if you can go through all its elements, one by one, and name every single one, everything. This is intuitive if we're dealing with finitely many things but it generalises nicely to infinite sets as well. Number theorists mostly deal with things that we can count, whether thet's prime numbers; integer solutions to equations; or a multitude of other similar problems.

On the other hand, we have problems that are very much in the realm of things we have to measure: problems of length which require a much more fine-grained set than we get with countable sets. However there is a strange "duality" between these concepts.

The field of analytic number theory is one of these bridges between these distinct fields and it uses techniques from analysis — a branch of maths largely dedicated to objects of a measurable[1] nature — to solve problems in number theory which, as we've already briefly discussed, is a much more countable discipline.

One of the biggest areas of analytic number theory was pioneered by the great British mathematicians G.H. Hardy and J.E. Littlewood and is known as the circle method. The rough idea, trivial as it may sound, is that if we go full circle then we haven't moved. To put it a little bit more concretely: suppose you're drawing circles and you have a fixed length you can draw in total. However you can vary the curvature of the circles you draw: they can either be small and so you'll draw lots of circles on top of each other or they can be large. However, no matter which you choose you will always end up where you started.

The only difference is the so-called *degenerate* case where the circle you draw is infinitely big. This translates into you drawing a straight line of length 1 and, in thise case, you don't end up where you started[2].

This all-or-nothing behaviour allows you to assign certain countable behaviour to the degenerate case that you'd really like to test for and generalise it to the non-degenerate cases. The treatment of circles is very much in the realm of measuring so we now have access to all the standard tools of analysis to deal with this problem originally rooted in counting.

There are other ways we can translate problems in the discrete realm to that of the continuum: the most well-known of which is probably the Riemann zeta function. This has a variety of interesting properties (and lies firmly within the realm of measuring) but these properties are controlled in their entirety by the properties of the whole numbers and, in particular, the properties of the prime numbers.

Prime numbers, along with being one of the most well-known objects in all of mathematics, are also not very well understood but we can see the effects of, for instance, how

[1]There is a technical definition of measurability in analysis but we use the term more loosely.

[2]The circles in question are the integrals of $e^{2xk\pi i}$ from $x = 0$ to $x = 1$. If $k = 0$ then the integrand is 1 and so the integral is 1, otherwise the integral goes around the full circle and you get lots of cancellation.

the primes are distributed in the Riemann zeta function. So, by studying the analytic properties of this function we can work backwards and discern what the primes must look like in order for these analytic properties to arise.

There's a theorem due to Dirichlet which states that every sensible[3] arithmetic progression has infinitely many primes. This is a very hard theorem to prove by conventional means but by encoding the information about the distribution of primes on arithmetic progressions in a close cousin of the Riemann zeta function — a Dirichlet L-function — and looking at how it behavious on a specific region[4]

There are countless similar objects and methods and I won't name them all but the take-home message is that things in mathematics are rarely self-contained, despite what librarians will tell you to the contrary. There will be numerous unforeseen and unforeseeable connections that enrich the subject matter greatly which, if we adopt the ideological entrenchments of discrete vs continuous (or any other such blanket categorisation), we will be blind to.

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[3]The arithmetic progressions we're talking about are those of the form $a + nd$ whenever a and d do not share any common prime divisors.

[4]Without going into masses of technical details, if $L(s, \chi)$ is a Dirichlet L-function (where s is a complex variable) then all we need is that $L(1, \chi) \neq 0$ for any non-principal Dirichlet character χ