ON BOUNDED GAPS BETWEEN PRIMES AND MODERN SIEVE THEORY

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Contents

Abstract. We discuss the recent advances made towards the notorious twin prime conjecture and its generalisations. What follows is a lengthy exploration of the various techniques used in the field, both in their general setting and their specific application to the problem of gaps between prime numbers, along with a new result by the author on a conjecture of Polignac's.

1. Introduction

1.1. **The History of Twin Primes.** Euclid's proof of the infinitude of primes [6, Book IX, Proposition 20] is well-known and any professional mathematician can reproduce it from memory. It is an easy and intuitive result which raises the question as to how these numbers are distributed.

Looking through the sequence of primes we note that there seem to be rather a large number of primes whose difference is precisely two — 3 and 5; 11 and 13; 347 and 349 — but the earliest extant example of the question as to whether there are infinitely many of these so-called twin primes was first posed in a more general form by de Polignac [16].

Conjecture 1 (de Polignac, 1849). *Let k be any positive even integer and let p_n be the n^{th} prime number. Then, for infinitely many $n \in \mathbb{N}$, we have $p_{n+1} - p_n = k$.*

Any k which satisfies Polignac's conjecture is called a Polignac number and the twin prime conjecture simply states that 2 is a Polignac number.

This conjecture was partially resolved by Maynard [13] and Zhang [27] with the following theorem

Theorem 2 (Maynard-Zhang, 2013). *There exist infinitely many pairs of prime numbers whose difference is no more than* 600.

This bound was originally proved by Zhang [27] with 600 replaced by 7×10^7 which then got reduced to 4,680 by Polymath [17]. Maynard [13] then reduced it to 600 and we will give a further improvement in this report. This clearly shows that there exists at least one Polignac number $2 \leq k \leq 600$ and this, together with the following theorem from Pintz[15], asserts that there are infinitely many Polignac numbers.

Theorem 3 (Pintz, 2013). *There is an ineffective constant C such that every interval of the form $[m, m + C]$ contains a Polignac number. Therefore, if there is at least one Polignac number then there exist infinitely many Polignac numbers.*

Unfortunately this theorem is ineffective — it offers no way to generate future Polignac numbers — but it does give direction for future research, particularly some results by the author [10] on the behaviour of Polignac numbers and arithmetic progressions.

We might also like to ask whether we have similar bounded behaviour for non-consecutive primes and that question has also been resolved by Maynard [13]:

Theorem 4 (Maynard, 2013). *Let m and n be positive integers. Then there are infinitely many $n \in \mathbb{N}$ such that*

$$p_{n+m} - p_n \leq Am^3e^{4m} \tag{1}$$

for all sufficiently large m and some positive, real constant A.

This shows that we have bounded behaviour for prime pairs, whether consecutive or not.

There are some previous results in this direction, dealing with the normalised prime gaps

$$\frac{p_{n+1} - p_n}{\log n} \tag{2}$$

where the $\log n$ represents the expected gap. In 1931, Westzynthius [26] proved that

$$\limsup_{n\to\infty} \frac{p_{n+!} - p_n}{\log n} = \infty \tag{3}$$

and, in 2005, Goldston, Pintz and Yildirim [8] proved

$$\liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log n} = 0. \tag{4}$$

The Zhang result has the Golston-Pintz-Yildirim result as a corollary but, recently, Banks, Freiman and Maynard [1] have announced an improvement to these results which we will not have time to delve into but mention for the sake of completeness.

Theorem 5 (Banks-Freiman-Maynard, 2014). *For at least* 12.5% *of all positive real numbers x, the sequence of normalised prime gaps has a subsequence which tends to x.*

1.2. **Sieving.** Sieving as a technique for counting and/or generating primes has had a long and interesting history dating back to the Greeks and Eratosthenes. Unfortunately this is out of the scope of this work but much of the history can be found in [11].

The majority of the sieving in this report uses the large sieve and the Goldston-Pintz-Yildirim (GPY) version of the Selberg sieve: the large sieve gives us the Barban-Bombieri-Vinogradov theorem [2] and the GPY sieve [8] acts as a detector for multiple primes in an interval.

Informally, the Barban-Bombieri-Vinogradov theorem tells us that the average number of primes less than x on an arithmetic progression does not grow much faster than

$x^{1/2}(\log x)^{-A}$ for any positive A. More specifically, if we define the level of distribution of the primes by

Definition 6. Let A be a positive real number; (m, n) be the greatest common divisor of m and n; $\pi(x)$ be the prime counting function; $\pi(x; a, q)$ be the prime counting function on the arithmetic progression $n \equiv a \pmod q$ and φ be Euler's totient function. If, for all $\epsilon \in \mathbb{R}^+$,

$$\sum_{q \le x^{\theta-\epsilon}} \sup_{(a,q)=1} \left| \pi(x; a, q) - \frac{1}{\varphi(q)} \pi(x) \right| \le c_A x (\log x)^{-A} \tag{5}$$

holds for some positive real constant c_A, depending only on A, and all sufficiently large $x \in \mathbb{R}^+$, then $\theta \in (0, 1]$ is called the level of distribution of the primes.

The Barban-Bombieri-Vinogradov theorem is then equivalent to saying that the primes have level of distribution $1/2$.

1.3. **Open Problems and Conditional Results.** A generalisation of the Barban-Bombieri-Vinogradov theorem is the following conjecture:

Conjecture 7 (Elliott-Halberstam, 1970)**.** [5] *The primes have level of distribution* 1.

Under the assumption that the Elliott-Halberstam conjecture holds, Goldston, Pintz and Yildirim [8] proved the Maynard-Zhang theorem with a bound of 16. Maynard[13] himself has given an improvement of this bound to 12 for consecutive primes and a bound of 600 for $p_{n+2} - p_n$.

There is also a natural generalisation of Polignac's conjecture and Dirichlet's theorem of prime numbers in arithmetic progressions[4] which we would hope to treat using the methods of Zhang, Pintz and Maynard.

Conjecture 8 (Dickson, 1904)**.** *Let $a_1, \dots, a_k, b_1, \dots, b_k$ be a finite collection of integers with $b_i \ge 1$ for all $1 \le i \le k$ and let $f_i(n) = a_i + b_i n$ be a collection of linear forms. If there is no positive integer m dividing all the products $\prod_{i=1}^k f_i(n)$ for all integers n then there exist infinitely many natural numbers n such that all of the linear forms are prime* [3].

You can see that, by setting $k = 2$ and using the linear forms $n, n+2$, this implies the twin prime conjecture and proving Dickson with the linear forms $n, n+2k$ for all $k \in \mathbb{N}$ is equivalent to Polignac's conjecture. Finally, if we assume that there are infinitely many twin primes, there is the question of just how frequently they show up. Hardy and Littlewood [9] gave the following conjecture as to how many there should be:

Conjecture 9 (Hardy-Littlewood, 1923)**.** *Let $P_2(n)$ be the number of prime pairs less than n and let $\mathbb{P}$ be the set of all prime numbers. Then*

$$P_2(n) \sim 2C_2 \frac{n}{(\log n)^2} \tag{6}$$

where

$$C_2 = \prod_{p \in \mathbb{P} \setminus \{2\}} \left(1 - \frac{1}{(p-1)^2} \right) \approx 0.66. \tag{7}$$

2. Notation and Identities

2.1. **Notation.** We collect here a list of the symbols we will use in the rest of the report for easy reference.

x	a positive real number
p	a prime number
D_0	the value $\log\log\log N$
W	the primorial $\prod_{p\le D_0} p$
θ	level of distribution of the primes
$\mathbb{P}$	the set of all prime numbers
I	an interval
$\mathcal{S}_I$	the set of all squarefree numbers in I
$\mathcal{L}$	$\log x$
$[x]$	the largest integer less than or equal to x
(m,n)	the greatest common divisor of m and n
$\pi(x)$	the number of prime numbers less than or equal to x
$\phi(n)$	the Euler totient function, giving the number of $m<n$ such that $(m,n)=1$
$\Lambda(n)$	the von Mangoldt function
$\psi(x)$	the Chebyshev function $\sum\limits_{n\le x}\Lambda(n)$
$\vartheta(x)$	the Chebyshev function $\sum\limits_{p\le x}\log p$
$e(n)$	the exponential function $e^{2\pi in}$
$\sigma(n)$	the number-of-divisors function
$\sigma_k(n)$	the number of ways of writing n as the product of k natural numbers
$\Gamma(t)$	the gamma function $\int_0^\infty x^{t-1}e^{-x}dx$
g	the totally multiplicative function defined on primes by $g(p)=p-2$
χ	a Dirichlet character
$\tau(\chi)$	the Gauss sum $\sum\limits_{a\le q}\chi(a)e(a/q)$
$1_A(x)$	the characteristic function of the set A, occasionally written without the argument if the meaning is clear
$\Delta(f;a,q)$	the discrepancy $\sum_{a\in(\mathbb{Z}/q\mathbb{Z})^*}\left\lvert \sum\limits_{\substack{n\le x\\ n\equiv a \ (\mathrm{mod}\ q)}} f(n) - \frac{1}{\varphi(q)}\sum\limits_{\substack{n\le x\\ (n,q)=1}} f(n)\right\rvert$
$f\star g$	Dirichlet convolution of two arithmetic functions f and g

2.1.1. *Asymptotic Notation.* We will write $f(x)=O(g(x))$ or $f(x)\ll g(x)$ if and only if there is some positive constant M such that, for all sufficiently large x, $f(x)\le Mg(x)$. If the implied constant depends on some value A we will write $f(x)=O_A(g(x))$ or $f(x)\ll_A g(x)$. We will also write $f(x)=o(g(x))$ if and only if $\frac{f(x)}{g(x)}\to 0$ as $x\to\infty$.

2.2. **Identities.**

$$\varphi(mn)\le\varphi(m)\varphi(n) \tag{8}$$

$$\sum_{d\mid n}\mu(d)=1_{n=1} \tag{9}$$

$$\sum_{d\mid n}\frac{\mu(d)^2}{\varphi(d)}=\frac{n}{\varphi(n)} \tag{10}$$

$$\sum_{n\le Q}\frac{1}{\varphi(n)} \ll \log Q \tag{11}$$

$$\frac{1}{\varphi(z)}\sum_{\chi\,(\mathrm{mod}\,q)}\overline{\chi}(a)\chi(b) = 1_{a\equiv b\,(\mathrm{mod}\,q)}, \text{if } (a,q)=1 \tag{12}$$

$$\overline{\chi}(n)\tau(\chi) = \sum_{a\le q}\chi(a)e(an/q),\ \text{ if } \chi \text{ is a primitive character} \tag{13}$$

$$|\tau(\chi)| = \sqrt{q},\ \text{ if } \chi \text{ is a primitive character} \tag{14}$$

$$\frac{1}{N}\sum_{n<N}\sigma_k(n) \ll (\log N)^k \tag{15}$$

$$\frac{1}{[m,n]} = \frac{1}{mn}\sum_{d|(m,n)}\varphi(d) \tag{16}$$

$$\frac{1}{\varphi([m,n])} = \frac{1}{\varphi(m)\varphi(n)}\sum_{d|mn} g(d) \tag{17}$$

$$\sum_{p\le n}\frac{\log p}{p} \sim \log n \tag{18}$$

where q is always taken to be the modulus of the Dirichlet character

3. The Large Sieve Inequality and the Barban-Bombieri-Vinogradov Theorem

In order to prove Maynard's theorem we will first need some information on prime numbers, specifically about their growth on arithmetic progressions. We say that the primes have level of distribution θ if

$$\sum_{q\le x^{\theta-o(1)}}\ \sum_{a\in(\mathbb{Z}/q\mathbb{Z})^*}\left|\sum_{\substack{n\equiv a\,(\mathrm{mod}\,q)\\ n<x}} 1_{\mathbb{P}}(n) - \frac{1}{\varphi(q)}\sum_{\substack{(n,q)=1\\ n<x}} 1_{\mathbb{P}}(x)\right| \ll_A x\mathcal{L}^{-A} \tag{19}$$

for some sufficiently slowly decaying $o(1)$ and any $A\in\mathbb{R}^+$. The Barban-Bombieri-Vinogradov theorem will show that the primes have level of distribution at least $1/2$ and that is enough for our purposes.

Theorem 10 (Large Sieve Inequality)**.** *For any complex sequence $(a_n)_{n\in\mathbb{N}}$; any $M,N\in\mathbb{N}$; and all real $Q>1$*

$$\sum_{q\le Q}\ \sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)}\left|\sum_{n=M+1}^{M+N} a_n\chi(n)\right|^2 \ll (Q^2+N)\sum_{n=M+1}^{M+N}|a_n|^2 \tag{20}$$

where $\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)}$ *represents a sum of primitive Dirichlet characters modulo q.*

Proof. Let χ be a primitive Dirichlet character modulo q. If $\tau(\chi)$ is the Gauss sum of χ then, by (13),

$$\sum_{n=M+1}^{M+N} a_n\chi(n) = \frac{1}{\tau(\overline{\chi})}\sum_{a\le q}\overline{\chi}(a)\sum_{n=M+1}^{M+N} a_n e(an/q). \tag{21}$$

Define $S(a/q)=\sum_{n=M+1}^{M+N} a_n e(an/q)$. Then

$$\sum_{\chi \pmod q}^{*} \left|\sum_{n=M+1}^{M+N} a_n \chi(n)\right|^2 = \sum_{\chi \pmod q}^{*} \frac{1}{|\tau(\overline{\chi})|^2} \left|\sum_{a=1}^{q} \overline{\chi}(a) S\left(\frac{a}{q}\right)\right|^2.$$

So, by 14, this is no larger than

$$\begin{aligned}\frac{1}{q}\sum_{\chi \pmod q} \left|\sum_{a=1}^{q} \overline{\chi}(a) S\left(\frac{a}{q}\right)\right|^2 &\leq \frac{1}{q}\sum_{\chi \pmod q}\sum_{a=1}^{q} |\overline{\chi}(a)|^2 \left|S\left(\frac{a}{q}\right)\right|^2 \\ &= \frac{1}{q}\sum_{a=1}^{q} \left|S\left(\frac{a}{q}\right)\right|^2 \sum_{\chi \pmod q} |\overline{\chi}(a)|^2. \end{aligned} \tag{22}$$

As non-zero Dirichlet characters are roots of unity and non-zero only when $(a,q)=1$, $\sum_{\chi \pmod q} |\overline{\chi}(a)|^2$ equals $\varphi(q)$ whenever $(a,q)=1$ and is 0 otherwise. Therefore (22) is less than or equal to

$$\frac{\varphi(q)}{q}\sum_{\substack{a\leq q\\(a,q)=1}} \left|S\left(\frac{a}{q}\right)\right|^2. \tag{23}$$

Hence we have shown that

$$\sum_{q\leq Q}\frac{q}{\varphi(q)}\sum_{\chi \pmod q}^{*} \left|\sum_{n=M+1}^{M+N} a_n\chi(n)\right|^2 \leq \sum_{q\leq Q}\sum_{\substack{a\leq q\\(a,q)=1}} \left|\sum_{n=M+1}^{M+N} a_n e\left(\frac{an}{q}\right)\right|^2. \tag{24}$$

We can view the double sum as a sum over all positive proper fractions whose denominators do not exceed q. Moreover, as $(a,q)=1$, the fractions are all distinct so we can apply Gallagher's large sieve inequality (Theorem 31) to show that this is less than or equal to

$$(\pi N+\delta^{-1})\sum_{n=M+1}^{M+N} |a_n|^2 \ll (N+\delta^{-1})\sum_{n=M+1}^{M+N} |a_n|^2. \tag{25}$$

Finally note that

$$\delta = \min_{\frac{a_1}{q_1}\neq\frac{a_2}{q_2}} \left\|\frac{a_1}{q_1}-\frac{a_2}{q_2}\right\| \geq \min_{\frac{a_1}{q_1}\neq\frac{a_2}{q_2}} \frac{1}{q_1q_2} \geq \frac{1}{Q^2} \tag{26}$$

completing the proof. □

We will immediately apply this to proving that primes have level of distribution 1/2.

Theorem 11 (Barban-Bombieri-Vinogradov)**.** *Let* $\epsilon\in(0,1/2)$*. Then*

$$\sum_{q\leq x^{1/2-\epsilon}}\sup_{a\in(\mathbb{Z}/q\mathbb{Z})^*}\left|\sum_{\substack{n\leq x\\ n\equiv a \pmod q}} 1_{\mathbb{P}}(n) - \frac{1}{\varphi(q)}\sum_{\substack{n\leq x\\(n,q)=1}} 1_{\mathbb{P}}(n)\right| \ll x\mathcal{L}^{-A} \tag{27}$$

for any fixed $A\in\mathbb{R}^+$*.*

Sketch of proof. Consider the Chebyshev ψ, ϑ functions and define

$$\pi(x;a,q) = \sum_{\substack{n\le x\\ n\equiv a \pmod q}} 1_{\mathbb{P}}(n) \tag{28}$$

$$\psi(x;a,q) = \sum_{\substack{n\le x\\ n\equiv a \pmod q}} \Lambda(n) \tag{29}$$

$$\vartheta(x;a,q) = \sum_{\substack{p\le x\\ p\equiv a \pmod q}} \log p. \tag{30}$$

Using partial summation along with the facts that $\pi(x;a,1) = \pi(x)$ and $\vartheta(x;a,1) = \vartheta(x)$ we see

$$\begin{aligned}
|\Delta(\pi;a,q)| &= \left|\frac{\vartheta(x;a,q)}{\log x} + \int_2^x \frac{\vartheta(t;a,q)}{t(\log t)^2}dt - \frac{1}{\varphi(q)}\left(\frac{\vartheta(x;a,1)}{\log x} + \int_2^x \frac{\vartheta(t;a,1)}{t(\log t)^2}dt\right)\right| \\
&\le \left|\frac{\vartheta(x;a,q)}{\log x} - \frac{\vartheta(x;a,1)}{\varphi(q)\log x}\right| + \left|\int_2^x \frac{\vartheta(t;a,q)}{t(\log t)^2}dt - \frac{1}{\varphi(q)}\int_2^x \frac{\vartheta(t;a,1)}{t(\log t)^2}dt\right| \\
&\le \frac{1}{\log x}|\Delta(\vartheta 1_{[2,x]};a,q)| + \max_{2\le t\le x}\left|\Delta(\vartheta 1_{[2,x]};a,q)\right| \left|\left[\frac{1}{\log t}\right]_{t=2}^{t=x}\right| \\
&\ll |\Delta(\vartheta 1_{[2,x]};a,q)| + \max_{2\le t\le x}|\Delta(\vartheta 1_{[2,t]};a,q)|.
\end{aligned} \tag{31}$$

Furthermore

$$\begin{aligned}
\psi(x;a,q) &= \sum_{\substack{p\le x\\ p\equiv a \pmod q}} \log p + \sum_{\substack{p\le x^{1/2}\\ p^2\equiv a \pmod q}} \log p + \sum_{\substack{p\le x^{1/3}\\ p^2\equiv a \pmod q}} \log p + \cdots \\
&= \vartheta(x;a,q) + O\left(\vartheta(x^{1/2}) + \vartheta(x^{1/3}) + \cdots\right).
\end{aligned} \tag{32}$$

Note that the sum in the 'O' term is finite as $x^{1/\log_2 x} = 2$ and so $\vartheta(x^{1/n}) = 0$ for all $n > \log_2 x$. Also, by the prime number theorem,

$$\vartheta(x^{1/n}) \le \frac{x^{1/n}}{\log x^{1/n}} \log x^{1/n} = x^{1/n} \tag{33}$$

and so

$$\vartheta(x^{1/2}) + \vartheta(x^{1/3}) + \cdots \ll x^{1/2} + x^{1/3}\log_2 x \ll x^{1/2}. \tag{34}$$

Id est

$$\psi(x;a,q) = \vartheta(x;a,q) + O\left(x^{1/2}\right). \tag{35}$$

Then

$$|\Delta(\vartheta 1_{[2,x]};a,q)| \le |\Delta(\psi 1_{[2,x]};a,q)| + O(x^{1/2}) \tag{36}$$

so it is sufficient to show that

$$\sum_{q\le x^{1/2-\epsilon}} \sup_{a\in(\mathbb{Z}/q\mathbb{Z})^*} \left|\psi(x;a,q) - \frac{1}{\varphi(q)}\psi(x)\right| \ll x\mathcal{L}^{-A}. \tag{37}$$

This can be achieved by decomposing the von Mangoldt function via the Heath-Brown identity (Lemma 32) and noting that all of the items we are convolving are coefficient sequences and that log obeys the Siegel-Walfisz condition, allowing us to apply the generalised Barban-Bombieri-Vinogradov theorem (see Theorem 39 in Appendix B for details). $\square$

4. The Selberg Sieve

4.1. **General GPY-Selberg Sieve Manipulations.** We will now look at the Selberg lambda-squared sieve [11] and its applications to the problem of bounded gaps. The Selberg sieve is any sieve of the form

$$\sum_n a_n \left(\sum_{d|n} \lambda_d \right)^2 \tag{38}$$

for some arithmetic function λ_n. This will let us weight a detector function for primes in constellations. We will show the particular sieve to use later in this report but we need to develop a few notions first.

Let $\mathcal{H} = \{h_1, h_2, \ldots, h_k\}$ be some finite set. We say that $\mathcal{H}$ is admissible if, for all primes p, there is some integer a such that $h_i \not\equiv a \pmod{p}$ for all $h_i \in \mathcal{H}$. We choose to study these sets as these are the sets which possess the potential to have infinitely many translates $n + \mathcal{H}$, all of whose elements are prime. From here on we will consider $\mathcal{H}$ to be some fixed admissible set.

If we take some prime p_j then there must be some a_j such that $h \not\equiv a_j \pmod{p_j}$ for all $h \in \mathcal{H}$. By the Chinese remainder theorem there must be a unique solution to the system of congruence equations

$$v_0 \equiv -a_j \pmod{p_j} \tag{39}$$

for all $p_j \leq D_0$, where $D_0 = \log\log\log x$. Then, as these p_j are precisely the prime divisors of the value W, we must have

$$v_0 + h \equiv h - a_j \not\equiv 0 \pmod{p_j}. \tag{40}$$

Thus $v_0 + h$ is prime to W.

These definitions allow us to define the particular detector function we will use. So let us define the GPY sieve sum

$$S = S_2 - \rho S_2 \tag{41}$$

where

$$S_1 = \sum_{\substack{N \leq n < 2N \\ n \equiv v_0 \pmod{W}}} \left(\sum_{(\forall i)(d_i | n + h_i)} \lambda_{d_1, \ldots, d_k} \right)^2 \tag{42}$$

$$S_2 = \sum_{\substack{N \leq n < 2N \\ n \equiv v_0 \pmod{W}}} \left(\sum_{i=1}^{k} 1_{\mathbb{P}}(n + h_i) \right) \left(\sum_{(\forall i)(d_i | n + h_i)} \lambda_{d_1, \ldots, d_k} \right)^2. \tag{43}$$

From now on we will omit the $(\forall i)$ on the subscript of the summation for ease of reading.

Note that if $S > 0$ then at least one of the terms in the outer sum must be positive. As squared factors don't affect the sign of a term this implies that

$$\sum_{i=1}^{k} 1_{\mathbb{P}}(n + h_i) - \rho > 0 \tag{44}$$

for some n satisfying the previous conditions. Thus there must be at least ρ primes in the translate $n + \mathcal{H}$.

Let F be some fixed piecewise-differentiable function supported on

$$\mathcal{R}_k = \left\{ (x_1, \ldots, x_k) \in [0,1]^k : \sum_{i=1}^{k} x_i \leq 1 \right\}. \tag{45}$$

Also let $R = N^{\theta/2-\delta}$ for some small, fixed, $\delta \in \mathbb{R}^+$. Then we want our Selberg weight function to be of the form

$$\lambda_{d_1,\ldots,d_k} = \left(\prod_{i=1}^{k} \mu(d_i) d_i\right) \sum_{\substack{r_1,\ldots,r_k \\ d_i|r_i \\ (r_i,W)=1}} \frac{\mu\left(\prod_{i=1}^{k} r_i\right)^2}{\prod_{i=1}^{k} \varphi(r_i)} F\left(\frac{\log r_1}{\log R}, \ldots, \frac{\log r_k}{\log R}\right). \tag{46}$$

whenever $\left(\prod_{i=1}^{k} d_i, W\right) = 1$, when $\prod_{i=1}^{k} d_i < R$ and when $\prod_{i=1}^{k} d_i$ is square-free. Let $\lambda_{d_1,\ldots,d_k} = 0$ otherwise.

We proceed by finding asymptotic equations for S_1 and S_2.

Lemma 12. *Let*

$$y_{r_1,\ldots,r_k} = \left(\prod_{i=1}^{k} \mu(r_i)\varphi(r_i)\right) \sum_{\substack{d_1,\ldots,d_k \\ r_i|d_i}} \frac{\lambda_{d_1,\ldots,d_k}}{\prod_{i=1}^{k} d_i} \tag{47}$$

and $y_{\max} = \sup_{r_1,\ldots,r_k} |y_{r_1,\ldots,r_k}|$. *Then*

$$S_1 = \frac{N}{W} \sum_{r_1,\ldots,r_k} \frac{y_{r_1,\ldots,r_k}^2}{\prod_{i=1}^{k} \varphi(r_i)} + O\left(\frac{y_{\max}^2 \varphi(W)^k N (\log R)^k}{W^{k+1} D_0}\right) \tag{48}$$

Proof. Expanding the square and switching the order of summation gives

$$\begin{aligned} S_1 &= \sum_{\substack{N \le n < 2N \\ n \equiv v_0 \ (\mathrm{mod}\ W)}} \sum_{d_i | n+h_i} \lambda_{d_1,\ldots,d_k} \sum_{e_i | n+h_i} \lambda_{e_1,\ldots,e_k} \\ &= \sum_{\substack{d_1,\ldots,d_k \\ e_1,\ldots,e_k}} \lambda_{d_1,\ldots,d_k} \lambda_{e_1,\ldots,d_k} \sum_{\substack{N \le n < 2N \\ n \equiv v_0 \ (\mathrm{mod}\ W) \\ [d_i,e_i] | n+h_i}} 1. \end{aligned} \tag{49}$$

We will suppose that $D_0 \geq \max \mathcal{H}$ and conjecture that W and the $[d_i, e_i]$ are coprime. If not then there exists some d such that either $d|([d_i, e_i], [d_j, e_j])$ or $d|(W, [d_i, e_i])$ for all $i \neq j$. In the first case d must divide $n + h_i$ and $n + h_j$, both of which are coprime to W so d also divides $h_i - h_j \leq \max(\mathcal{H})$, giving us that $(h_i - h_j, W) = h_i - h_j$. Thus $(d, W) \neq 1$, contradicting our earlier statement.

On the other hand $\lambda_{d_1,\ldots,d_k}$ is only supported when the d_i and W are coprime so we can suppose that $W, [d_1, e_1], \ldots, [d_k, e_k]$ are pairwise coprime. This means that, by the Chinese remainder theorem, we can turn the inner sum in (49) into a sum over a single residue class $q = W \prod_{i=1}^{k} [d_i, e_i]$ and this sum has a value of $N/q + O(1)$. So

$$S_1 = \frac{N}{W} \sideset{}{'}\sum_{\substack{d_1,\ldots,d_k \\ e_1,\ldots,e_k}} \frac{\lambda_{d_1,\ldots,d_k} \lambda_{e_1,\ldots,d_k}}{\prod_{i=1}^{k} [d_i, e_i]} + O\left(\sideset{}{'}\sum_{\substack{d_1,\ldots,d_k \\ e_1,\ldots,e_k}} |\lambda_{d_1,\ldots,d_k} \lambda_{e_1,\ldots,e_k}|\right) \tag{50}$$

where $\sum'$ represents the restriction that $W, [d_1, e_1] \ldots, [d_k, e_k]$ are pairwise coprime.

Let $\lambda_{\max} = \sup_{d_1,\ldots,d_k} |\lambda_{d_1,\ldots d_k}|$. Then, as $\lambda_{d_1,\ldots,d_k}$ is non-zero only if $\prod_{i=1}^k d_i < R$,

$$\begin{aligned}\sideset{}{'}\sum_{\substack{d_1,\ldots,d_k\\ e_1,\ldots,e_k}} |\lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k}| &\le |\lambda_{\max}|^2 \left(\sum_{\prod_{i=1}^k d_i<R} 1\right)^2 \\ &= \lambda_{\max}^2 \left(\sum_{d<R} \sigma_k(d)\right)^2 \\ &\ll \lambda_{\max}^2 R^2 (\log R)^{2k}. \end{aligned} \tag{51}$$

Now we rewrite the main term using (16) as

$$\frac{N}{W} \sideset{}{'}\sum_{\substack{d_1,\ldots,d_k\\ e_1,\ldots,e_k}} \lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k} \prod_{i=1}^k \frac{1}{d_i e_i} \sum_{u_i|(d_i,e_i)} \varphi(u_i). \tag{52}$$

Recall that $\lambda_{d_1,\ldots,d_k}$ is supported only when the product $d_1 \cdots d_k$ is squarefree so the d_i must be pairwise coprime and also recall that this product and W must also be coprime. Therefore $[d_i, e_i]$ is coprime to W. Then the above equals

$$\frac{N}{W} \sum_{u_1,\ldots,u_k} \left(\prod_{i=1}^k \varphi(u_i)\right) \sideset{}{'}\sum_{\substack{d_1,\ldots,d_k\\ e_1,\ldots,e_k\\ u_i|(d_i,e_i)}} \frac{\lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k}}{\left(\prod_{i=1}^k d_i\right)\left(\prod_{i=1}^k e_i\right)} \tag{53}$$

with the restriction on the sum simply now being that $(d_i, e_j) = 1$ for all $i \ne j$ but this can be removed by multiplying through by $\sum_{s_{i,j}|(d_i,e_j)} \mu(s_{i,j})$. This turns the main term into

$$\frac{N}{W} \sum_{u_1,\ldots,u_k} \left(\prod_{i=1}^k \varphi(u_i)\right) \sum_{\substack{d_1,\ldots,d_k\\ e_1,\ldots,e_k\\ u_i|(d_i,e_i)}} \left(\prod_{\substack{s_{i,j}\\ i\ne j}} \sum_{\substack{l\ne m\\ s_{l,m}|(d_l,e_m)}} \mu(s_{l,m})\right) \frac{\lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k}}{\left(\prod_{i=1}^k d_i\right)\left(\prod_{i=1}^k e_i\right)} \tag{54}$$

or

$$\frac{N}{W} \sum_{u_1,\ldots,u_k} \left(\prod_{i=1}^k \varphi(u_i)\right) \sum_{\substack{s_{i,j}\\ i\ne j}} \left(\prod_{\substack{1\le l,m,\le k\\ l\ne m}} \mu(s_{l,m})\right) \sum_{\substack{d_1,\ldots,d_k\\ e_1,\ldots,e_k\\ u_i|(d_i,e_i)\\ s_{i,j}|(d_i,e_j),\forall i\ne j}} \frac{\lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k}}{\left(\prod_{i=1}^k d_i\right)\left(\prod_{i=1}^k e_i\right)}. \tag{55}$$

We can furthermore require that $s_{i,j}$ is coprime to both u_i and u_j as, if $(s_{i,j}, u_i) \ne 1$, we would have some common divisor between (d_i, e_i) and (d_i, e_j). This would mean that e_i and e_j share some common factor but then $\lambda_{e_1,\ldots,e_k} = 0$ and so these terms do not contribute to the sum.

Similarly we can further restrict the sum so that $s_{i,j}$ is coprime to $s_{i,a}$ and $s_{b,j}$ for all $a \ne j$ and $b \ne i$. We denote this restricted sum by $\sum^*$.

Write

$$y_{r_1,\ldots,r_k} = \left(\prod_{i=1}^k \mu(r_i)\varphi(r_i)\right) \sum_{\substack{d_1,\ldots,d_k\\ r_i|d_i}} \frac{\lambda_{d_1,\ldots,d_k}}{\prod_{i=1}^k d_i}. \tag{56}$$

If we fix $d_1, \ldots, d_k$ with $\mu\left(\prod_{i=1}^k d_i\right) \ne 0$ we find

$$\begin{aligned}
\sum_{\substack{r_1,\dots,r_k\\ d_i|r_i}} \frac{y_{r_1,\dots,r_k}}{\prod_{i=1}^k \varphi(r_i)} &= \sum_{\substack{r_1,\dots,r_k\\ d_i|r_i}} \left(\prod_{i=1}^k \mu(r_i)\right) \sum_{\substack{e_1,\dots,e_k\\ r_i|e_i}} \frac{\lambda_{e_1,\dots,e_k}}{\prod_{i=1}^k e_i} \\
&= \sum_{e_1,\dots,e_k} \frac{\lambda_{e_1,\dots,e_k}}{\prod_{i=1}^k e_i} \sum_{\substack{r_1,\dots,r_k\\ d_i|r_i|e_i}} \prod_{i=1}^k \mu(r_i) \\
&= \sum_{e_1,\dots,e_k} \frac{\lambda_{e_1,\dots,e_k}}{\prod_{i=1}^k e_i} \sum_{\substack{r_1,\dots,r_k\\ r_i|e_i/d_i}} \prod_{i=1}^k \mu(d_i r_i) \\
&= \prod_{i=1}^k \mu(d_i) \sum_{e_1,\dots,e_k} \frac{\lambda_{e_1,\dots,e_k}}{\prod_{j=1}^k e_j} \sum_{r_i|e_i/d_i} \mu(r_i). \qquad (57)
\end{aligned}$$

If any of the e_i/d_i do not equal 1 then one of the divisor sums of the Möbius function will be 0. Therefore we can write $e_i = d_i$ for all i and so

$$\sum_{\substack{r_1,\dots,r_k\\ d_i|r_i}} \frac{y_{r_1,\dots,r_k}}{\prod_{i=1}^k \varphi(r_i)} = \frac{\lambda_{d_1,\dots,d_k}}{\prod_{i=1}^k \mu(d_i) d_i}. \qquad (58)$$

Thus any choice of $y_{r_1,\dots,r_k}$ which is supported on $r_1,\dots,r_k$ such that $r = \prod_{i=1}^k r_i < R$ is square-free and $(r, W) = 1$ will give us some $\lambda_{d_1,\dots,d_k}$. Now let

$$y_{\max} = \sup_{r_1,\dots,r_k} |y_{r_1,\dots,r_k}| \qquad (59)$$

then we find that, by taking $r' = \prod_{i=1}^k \frac{r_i}{d_i}$

$$\begin{aligned}
\lambda_{\max} &= \sup_{d_1,\dots,d_k} |\lambda_{d_1,\dots,d_k}| \\
&= \sup_{d_1,\dots,d_k} \left| \left(\prod_{i=1}^k \mu(d_i) d_i\right) \sum_{\substack{r_1,\dots,r_k\\ d_i|r_i\\ r<R\\ \mu(r)\neq 0}} \frac{y_{r_1,\dots,r_k} \prod_{i=1}^k \mu(r_i)^2}{\prod_{i=1}^k \varphi(r_i)} \right| \\
&= y_{\max} \sup_{\substack{d_1,\dots,d_k\\ \mu\left(\prod_{i=1}^k d_i\right)\neq 0}} \left| \left(\prod_{i=1}^k d_i\right) \sum_{\substack{r'<R/\prod_{i=1}^k d_i\\ \mu\left(\prod_{i=1}^k d_i r'\right)\neq 0}} \frac{\mu\left(\prod_{i=1}^k d_i r'\right)^2 \sigma_k(r')}{\varphi\left(\prod_{i=1}^k d_i r'\right)} \right| \\
&= y_{\max} \sup_{\substack{d_1,\dots,d_k\\ \mu\left(\prod_{i=1}^k d_i\right)\neq 0}} \left| \left(\prod_{i=1}^k \frac{d_i}{\varphi(d_i)}\right) \sum_{\substack{r'<R/\prod_{i=1}^k d_i\\ \mu\left(\prod_{i=1}^k d_i r'\right)\neq 0}} \frac{\mu\left(\prod_{i=1}^k d_i r'\right)^2 \sigma_k(r')}{\varphi\left(\prod_{i=1}^k d_i r'\right)} \right|
\end{aligned}$$

$$= y_{\max} \sup_{\substack{d_1,\dots,d_k \\ \mu\left(\prod_{i=1}^k d_i\right)\neq 0}} \left| \left(\sum_{d|\prod_{i=1}^k d_i} \frac{\mu(d)^2}{\varphi(d)} \right) \sum_{\substack{r'<R/\prod_{i=1}^k d_i \\ \mu\left(\prod_{i=1}^k d_i r'\right)\neq 0}} \frac{\mu\left(\prod_{i=1}^k d_i r'\right)^2 \sigma_k(r')}{\varphi\left(\prod_{i=1}^k d_i r'\right)} \right| \tag{60}$$

by equation (10). Setting $u = dr'$ and noting that $\sigma_k(dr') \geq \sigma_k(r')$ shows

$$\lambda_{\max} \leq y_{\max} \sum_{u<R} \frac{\mu(u)^2 \sigma_k(u)}{\varphi(u)} \ll y_{\max}(\log R)^k \tag{61}$$

by standard estimates on the mean values of arithmetic functions.

We can use this to bound our error term $O(\lambda_{\max}^2 R^2 (\log N)^{2k})$ by $O(y_{\max}^2 R^2 (logN)^{4k})$. Substituting our expression for $y_{r_1,\dots,r_k}$ into equation (55) and using our new estimate for the error term we get

$$S_1 = \frac{N}{W} \sum_{u_1,\dots,u_k} \left(\prod_{i=1}^k \varphi(u_i) \right) \sum_{\substack{s_{i,j} \\ i\neq j}}^{*} \left[\left(\prod_{\substack{1\leq l,m\leq k \\ l\neq m}} \mu(s_{l,m}) \right) \left(\prod_{l=1}^k \frac{\mu(a_l)\mu(b_l)}{\varphi(a_l)\varphi(b_l)} y_{a_1,\dots,a_k} y_{b_1,\dots,b_k} \right) \right]$$
$$+ O(y_{\max}^2 R^2 (\log R)^{4k}) \tag{62}$$

where we define $a_m = u_m \prod_{l\neq m} s_{m,l}$ and $b_m = u_m \prod_{l\neq m} s_{l,m}$. As u_m is coprime to all $s_{l,m}$ we note that $\mu(a_m), \mu(b_m) \neq 0$ and the main term becomes

$$\frac{N}{W} \sum_{u_1,\dots,u_k} \left(\prod_{i=1}^k \frac{\mu(u_i)^2}{\varphi(u_i)} \right) \sum_{\substack{s_{i,j} \\ i\neq j}}^{*} \left(\prod_{\substack{1\leq l,m\leq k \\ l\neq m}} \frac{\mu(s_{l,m})}{\varphi(s_{l,m})^2} \right) y_{a_1,\dots,a_k} y_{b_1,\dots,b_k} \tag{63}$$

Using the fact that $y_{d_1,\dots,d_k}$ is supported only on $\left(\prod_{i=1}^k d_i, W\right) = 1$ it must be that the $s_{i,j}$ are coprime to W. Thus we only need to consider $s_{i,j} = 1$ and $s_{i,j} > D_0$. The contribution when $s_{i,j} > D_0$ is

$$\ll \frac{y_{\max}^2 N}{W} \left(\sum_{\substack{u<R \\ (u,W)=1}} \frac{\mu(u)^2}{\varphi(u)} \right)^k \left(\sum_{s_{i,j}>D_0} \frac{\mu(s_{i,j})^2}{\varphi(s_{i,j})^2} \right) \left(\sum_{s\geq 1} \frac{\mu(s)^2}{\varphi(s)^2} \right)^{k^2-k-1}$$
$$\ll \frac{y_{\max}^2 \varphi(W)^k N (\log R)^k}{W^{k+1} D_0} \tag{64}$$

by standard estimates on the mean values of multiplicative functions.

We can now just restrict our attention to the case where $s_{i,j} = 1$ for all $i \neq j$. This gives

$$S_1 = \frac{N}{W} \sum_{u_1,\dots,u_k} \frac{y_{u_1,\dots,u_k}^2}{\prod_{i=1}^k \varphi(u_i)} + O\left(\frac{y_{\max}^2 \varphi(W)^k N (\log R)^k}{W^{k+1} D_0} + y_{\max}^2 R^2 (\log R)^{4k} \right) \tag{65}$$

but $R^2 = N^{\theta-2\epsilon} \ll N^{1-2\epsilon}$, $W \ll N^\epsilon$ and $\log R \sim \log N \ll N^\epsilon$ for all $\epsilon \in \mathbb{R}^+$, giving the required result. □

We derive an asymptotic bound for S_2 in a similar way but we first need to break it up. Define

$$S_2^{(m)} = \sum_{\substack{N\leq n<2N \\ n\equiv v_0 \pmod W}} 1_{\mathbb{P}}(n+h_m) \left(\sum_{\substack{d_1,\dots,d_k \\ d_i|n+h_i}} \lambda_{d_1,\dots,d_k} \right)^2 \tag{66}$$

All we have done is to break up the set $\mathcal{H}$ as the parameter to the GPY sieve term S_2. Clearly we have $S_2 = \sum_{m=1}^{k} S_2^{(m)}$ and our next lemma gives us a bound for these summands.

Lemma 13. *Let*

$$y^{(m)}_{r_1,\ldots,r_m} = \left(\prod_{i=1}^{k} \mu(r_i) g(r_i)\right) \sum_{\substack{d_1,\ldots,d_r \\ r_i | d_i \\ d_m=1}} \frac{\lambda_{d_1,\ldots,d_k}}{\prod_{i=1}^{k} \varphi(d_i)} \tag{67}$$

where g is the totally multiplicative function defined on primes by $g(p) = p - 2$. Also let $y^{(m)}_{\max} = \sup_{r_1,\ldots,r_k} |y^{(m)}_{r_1,\ldots,r_k}|$. Then, for any fixed $A \in \mathbb{R}^+$, we have

$$S_2^{(m)} = \frac{N}{\varphi(W)\log N} \sum_{r_1,\ldots,r_k} \frac{\left(y^{(m)}_{r_1,\ldots,r_k}\right)^2}{\prod_{i=1}^{k} g(r_i)} + O\left(\frac{(y^{(m)}_{\max})^2 \varphi(W)^{k-2} N (\log N)^{k-2}}{W^{k-1} D_0}\right)$$
$$+ O\left(\frac{y^2_{\max} N}{(\log N)^A}\right). \tag{68}$$

Proof. As before we expand out the square and swap the order of summation in $S_2^{(m)}$ to give

$$S_2^{(m)} = \sum_{\substack{d_1,\ldots,d_k \\ e_1,\ldots,d_k}} \lambda_{d_1,\ldots,d_k} \lambda_{e_!,\ldots,d_k} \sum_{\substack{N \le n < 2N \\ n \equiv v_0 \ (\mathrm{mod}\ W) \\ [d_i,e_i] | n+h_i}} 1_{\mathbb{P}}(n + h_m). \tag{69}$$

As with the estimation of S_1 we can use the Chinese remainder theorem to write the inner sum as a sum over a single primitive residue class modulo $q = W \prod_{i=1}^{k} [d_i, e_i]$ if $W, [d_1, e_1], \ldots, [d_k, e_k]$ are pairwise coprime. Also note that the inner sum equals 0 unless $[d_m, e_m] = 1$. Then the inner sum contributes

$$\begin{aligned}
\sum_{\substack{N \le n < 2N \\ n \equiv a_0 \ (\mathrm{mod}\ q)}} 1_{\mathbb{P}}(n + h_m) &\le \sum_{\substack{N \le n < 2N \\ n \equiv a_0 \ (\mathrm{mod}\ q)}} 1_{\mathbb{P}}(n) + O(1) \\
&= \frac{1}{\varphi(q)} \sum_{N \le n < 2N} 1_{\mathbb{P}}(n) + \sum_{\substack{N \le n < 2N \\ n \equiv a_0 \ (\mathrm{mod}\ q)}} 1_{\mathbb{P}}(n) - \frac{1}{\varphi(q)} \sum_{N \le n < 2N} 1_{\mathbb{P}}(n) \\
&\quad + O(1) \\
&\le \frac{1}{\varphi(q)} \sum_{N \le n < 2N} 1_{\mathbb{P}}(n) \\
&\quad + \sup_{(a,q)=1} \left| \sum_{\substack{N \le n < 2N \\ n \equiv a \ (\mathrm{mod}\ q)}} 1_{\mathbb{P}}(n) - \frac{1}{\varphi(q)} \sum_{N \le n < 2N} 1_{\mathbb{P}}(n) \right| + O(1) \\
&= \frac{X_N}{\varphi(q)} + O(E(N, q)).
\end{aligned} \tag{70}$$

where

$$X_N = \sum_{N \le n < 2N} 1_{\mathbb{P}}(n); \tag{71}$$

$$E(N, q) = 1 + \sup_{(a,q)=1} |\Delta(1_{\mathbb{P}} 1_{[N,2N)}; a, q)|. \tag{72}$$

Thus

$$S_2^{(m)} = \frac{X_N}{\varphi(W)} \sideset{}{^*}\sum_{\substack{d_1,\dots,d_k \\ e_1,\dots,e_k \\ d_m=e_m=1}} \frac{\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}}{\prod_{i=1}^k \varphi([d_i,e_i])} + O\left(\sum_{\substack{d_1,\dots,d_k \\ e_1,\dots,e_k}} |\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}| E(N,q) \right). \tag{73}$$

Looking at the error term we see that, by the restricted support of $\lambda_{d_1,\dots,d_k}$ we only need to worry about $\prod_{i=1}^k d_i < R$. I.E. when $q \leq \prod_{i=1}^k d_i e_i < R^2 W$ with q being square-free.

Combinatorially, if r is square-free then there are no more than $\sigma_{3k}(r)$ choices of $d_1, \dots, d_k, e_1, \dots, e_k$ for which $W \prod_{i=1}^k [d_i, e_i] = r$. Then the error term contributes

$$\ll y_{\max}^2 (\log R)^{2k} \sum_{r < R^2 W} \mu(r)^2 \sigma_{3k}(r) E(N,r). \tag{74}$$

Trivially we see that $E(N,r) \leq 1 + \frac{2N}{\varphi(q)} \ll \frac{N}{\varphi(q)}$. Using Cauchy-Schwarz and the fact that the primes have level of distribution θ this error term is

$$\begin{aligned}
&\ll y_{\max}^2 (\log R)^{2k} \left(\sum_{r < R^2 W} \mu(r)^2 \sigma_{3k}(r)^2 \frac{N}{\varphi(r)} \right)^{1/2} \left(\sum_{r < R^2 W} \mu(r)^2 E(N,r) \right)^{1/2} \\
&\ll y_{\max}^2 (\log R)^{2k} \times N^{1/2} \log(R^2 W)^{O(1)} \times \frac{N^{1/2}}{(\log N)^A} \\
&\ll \frac{y_{\max}^2 N}{(\log N)^A}
\end{aligned} \tag{75}$$

for any fixed $A \in \mathbb{R}^+$ with the penultimate step coming from standard multiplicative function means.

We treat the main term similarly to in the previous lemma and rewrite the condition $(d_i, e_i) = 1$ by multiplying through by $\sum_{s_{i,j} | (d_i, e_j)} \mu(s_{i,j})$. Again we can restrict the $s_{i,j}$ to be coprime to $u_i, u_j, s_{i,a}$ and $s_{b,j}$ for all $a \neq i$ and $b \neq j$ and denote the summation with respect to these restrictions by $\sum^*$. Using the fact that d_i, e_i are square-free we get a main term of

$$\frac{X_N}{\varphi(W)} \sum_{u_1,\dots,u_k} \left(\prod_{i=1}^k g(u_i) \right) \sideset{}{^*}\sum_{\substack{s_{i,j} \\ i \neq j}} \left(\prod_{\substack{1 \leq l,m \leq k \\ l \neq m}} \mu(s_{l,m}) \right) \sum_{\substack{d_1,\dots,d_k \\ e_1,\dots,e_k \\ u | (d_i,e_i) \\ s_{i,j} | (d_i,e_j), \forall i \neq j \\ d_m=e_m=1}} \frac{\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}}{\prod_{i=1}^k \varphi(d_i)\varphi(e_i)} \tag{76}$$

where the product of the $g(u_i)$ comes from equation (17). Now let

$$y_{r_1,\dots,r_k}^{(m)} = \left(\prod_{i=1}^k \mu(r_i) g(r_i) \right) \sum_{\substack{d_1,\dots,d_r \\ r_i | d_i \\ d_m=1}} \frac{\lambda_{d_1,\dots,d_k}}{\prod_{i=1}^k \varphi(d_i)} \tag{77}$$

and note that this will be 0 unless $r_m = 1$ and unless the r_i are all square-free. Substituting this into (76) we get

$$\frac{X_N}{\varphi(W)} \sum_{u_1,\dots,u_k} \left(\prod_{i=1}^k \frac{\mu(u_i)^2}{g(u_i)} \right) \sideset{}{^*}\sum_{\substack{s_{i,j} \\ i \neq j}} \left(\prod_{\substack{1 \leq l,m \leq k \\ l \neq m}} \frac{\mu(s_{i,j})}{g(s_{i,j})} \right) y_{a_1,\dots,a_k}^{(m)} y_{b_1,\dots,b_k}^{(m)} \tag{78}$$

where $a_j = u_j \prod_{i\neq j} s_{j,i}$ and $b_j = u_j \prod_{i\neq j} s_{i,j}$ for each $1 \leq j, \leq k$ as before. Arguing as in the preceding lemma we find that the contribution from $s_{i,j} \neq 1$ is of size

$$\ll \frac{\left(y^{(m)}_{\max}\right)^2 N}{\varphi(W)\log N} \left(\sum_{\substack{u<R\\(u,W)=1}} \frac{\mu(u)^2}{g(u)} \right)^{k-1} \left(\sum_{s_{i,j}>D_0} \frac{\mu(s_{i,j})^2}{g(s_{i,j})^2} \right) \left(\sum_{s\geq 1} \frac{\mu(s)^2}{g(s)^2} \right)^{k^2-k-1}$$

$$\ll \frac{\left(y^{(m)}_{\max}\right)^2 \varphi(W)^{k-2} N (\log R)^{k-1}}{W^{k-1} D_0 \log N}. \tag{79}$$

Thus

$$S_2^{(m)} = \frac{X_N}{\varphi(W)} \sum_{u_1,\ldots,u_k} \frac{\left(y^{(m)}_{u_1,\ldots,u_k}\right)^2}{\prod_{i=1}^k g(u_i)} + O\left(\frac{\left(y^{(m)}_{\max}\right)^2 \varphi(W)^{k-2} N (\log R)^{k-1}}{W^{k-1} D_0} \right) + O\left(\frac{y^2_{\max} N}{(\log N)^A} \right) \tag{80}$$

for all $A \in \mathbb{R}^+$. Finally, by the prime number theorem, $X_N = \frac{N}{\log N} + O\left(\frac{N}{(\log N)^2}\right)$. This error term contributes

$$\ll \frac{\left(y^{(m)}_{\max}\right)^2 N}{\varphi(W)(\log N)^2} \left(\sum_{\substack{u<R\\(u,W)=1}} \frac{\mu(u)^2}{g(u)} \right)^{k-1} \ll \frac{\left(y^{(m)}_{\max}\right)^2 \varphi(W)^{k-2} N (\log R)^{k-3}}{W^{k-3}} \tag{81}$$

which is absorbed in the first error term above due to $\log R$ being significantly bigger than D_0, finishing the proof. □

Lemma 14. *If $r_m = 1$ then*

$$y^{(m)}_{r_1,\ldots,r_k} = \sum_{a_m} \frac{y_{r_1,\ldots,r_{m-1},a_m,r_{m+1},\ldots,r_k}}{\varphi(a_m)} + O\left(\frac{y_{\max}\varphi(W)\log R}{W D_0} \right) \tag{82}$$

Proof. Substituting (58) into the definition of $y^{(m)}_{r_1,\ldots,r_k}$ we get

$$y^{(m)}_{r_1,\ldots,r_k} = \left(\prod_{i=1}^k \mu(r_i) g(r_i) \right) \sum_{\substack{d_1,\ldots,d_k\\ r_i|d_i\\ d_m=1}} \left(\prod_{i=1}^k \frac{\mu(d_i) d_i}{\varphi(d_i)} \right) \sum_{\substack{a_1,\ldots,a_k\\ d_i|a_i}} \frac{y_{a_1,\ldots,a_k}}{\prod_{i=1}^k \varphi(a_i)}. \tag{83}$$

Now, by changing the order of summation, the sum over $d_1, \ldots, d_k$ equals

$$\sum_{\substack{a_1,\ldots,a_k\\ r_i|a_i}} \frac{y_{a_1,\ldots,a_k}}{\prod_{i=1}^k \varphi(a_i)} \sum_{\substack{d_1,\ldots,d_k\\ r_i|d_i|a_i\\ d_m=1}} \left(\prod_{i=1}^k \frac{\mu(d_i) d_i}{\varphi(d_i)} \right) \tag{84}$$

or, by replacing d_i by $d_i r_i$

$$\sum_{\substack{a_1,\ldots,a_k\\ r_i|a_i}} \frac{y_{a_1,\ldots,a_k}}{\prod_{i=1}^k \varphi(a_i)} \sum_{\substack{d_1,\ldots,d_k\\ d_i|a_i/r_i\\ d_m=1}} \left(\prod_{i=1}^k \frac{\mu(d_i r_i) d_i r_i}{\varphi(d_i r_i)} \right). \tag{85}$$

Note that, due to multiplicativity and the fundamental theorem of arithmetic

$$\begin{aligned}\sum_{d|r}\frac{\mu(d)d}{\varphi(d)} &= \sum_{d|r}\prod_{p|d}\frac{\mu(p)p}{\varphi(p)} \\ &= \prod_{p|r}\left(1+\frac{\mu(p)p}{\varphi(p)}\right) \\ &= \prod_{p|r}\frac{(p-1)-p}{p-1} \\ &= \prod_{p|r}\frac{-1}{p-1}. \end{aligned} \tag{86}$$

Looking specifically at the sum over d we see that, by switching order of summation and product, (85) equals

$$\begin{aligned}\prod_{i\neq m}\sum_{d_i|a_i/r_i}\frac{\mu(d_ir_i)d_ir_i}{\varphi(d_ir_i)} &= \prod_{i\neq m}\frac{\mu(r_i)r_i}{\varphi(r_i)}\left(\prod_{p|a_i/r_i}\frac{-1}{p-1}\right) \\ &= \prod_{i\neq m}\frac{\mu(r_i)r_i}{\varphi(r_i)}\frac{\mu(a_i/r_i)}{\varphi(a_i/r_i)} \\ &= \prod_{i\neq m}\frac{\mu(a_i)r_i}{\varphi(a_i)} \end{aligned} \tag{87}$$

where the splitting and recombining of the multiplicative functions holds due to the fact that the a_i are squarefree (else the contribution is 0). Substituting this back into (85) and then (83) gives

$$y^{(m)}_{r_1,\dots,r_k} = \left(\prod_{i=1}^k \mu(r_i)g(r_i)r_i\right)\sum_{\substack{a_1,\dots,a_k\\ r_i|a_i}}\frac{y_{a_1,\dots,a_k}}{\prod_{i=1}^k\varphi(a_i)}\prod_{i\neq m}\frac{\mu(a_i)}{\varphi(a_i)} \tag{88}$$

By the restricted support of $y_{a_1,\dots,a_k}$ we can specify that $(a_i,W)=1$. This implies that $(a_i/r_i,W)=1$ which, in turn, implies that either $a_i=r_i$ or $a_i>D_0r_i$.

For $j\neq m$ the total contribution from $a_j\neq r_j$ is

$$\begin{aligned} &\ll y_{\max}\left(\prod_{i=1}^k g(r_i)r_i\right)\left(\sum_{a_j>D_0r_j}\frac{\mu(a_j)^2}{\varphi(a_j)^2}\right)\left(\sum_{\substack{a_m<R\\(a_m,W)=1}}\frac{\mu(a_j)^2}{\varphi(a_j)}\right)\prod_{\substack{1\leq i\leq k\\ i\neq j,m}}\left(\sum_{r_i|a_i}\frac{\mu(a_i)^2}{\varphi(a_i)^2}\right) \\ &\ll \frac{y_{\max}\varphi(W)\log R}{WD_0}\prod_{i=1}^k\frac{g(r_i)r_i}{\varphi(r_i)^2} \\ &\ll \frac{y_{\max}\varphi(W)\log R}{WD_0} \end{aligned} \tag{89}$$

by standard results on the mean values of multiplicative functions and noting that $\prod_{p|r_i}\frac{(p-2)p}{(p-1)^2} = \prod_{p|r_i}\frac{p^2-2p}{p^2-2p+1}\leq 1$. Thus the main contribution is when $a_j=r_j$ for all $j\neq m$. This gives

$$y^{(m)}_{r_1,\dots,r_k} = \left(\prod_{i=1}^k\frac{g(r_i)r_i}{\varphi(r_i)^2}\right)\sum_{a_m}\frac{y_{r_1,\dots,r_{m-1},a_m,r_{m+1},\dots,r_k}}{\varphi(a_m)} + O\left(\frac{y_{\max}\varphi(W)\log R}{WD_0}\right) \tag{90}$$

due to the $\mu(r_i)^2$ term getting absorbed into $g(r_i)$. Note that

$$\frac{g(p)p}{\varphi(p)^2} = 1 - \frac{1}{p^2 - 2p + 1} = 1 + O\left(p^{-2}\right), \tag{91}$$

implying that the whole product is equal to $1 + O\left(p^{-2}\right)$. Furthermore, as the contribution is zero unless $\prod_{i=1}^k r_i$ is coprime to W we must have $p > D_0$. This then gives a second error term of size

$$\begin{aligned} &\ll \frac{y_{\max}}{D_0^2} \sum_{a_m \le R} \frac{1}{\varphi(a_m)} \\ &\ll \frac{y_{\max} \log R}{D_0^2} \\ &\ll \frac{y_{\max} W \log R}{W D_0}, \end{aligned} \tag{92}$$

completing the proof. □

4.2. **Choosing y.** We will now define

$$y_{r_1,\ldots,r_k} = F\left(\frac{\log r_1}{\log R}, \ldots, \frac{\log r_k}{\log R}\right) \tag{93}$$

whenever $r = \prod_{i=1}^k r_i$ satisfies $(r, W) = 1$ and $\mu(r)^2 = 1$ for some piecewise-differentiable function F supported on the k-simplex

$$\mathcal{R}_k = \left\{ (x_1, \ldots, x_k) \in [0,1]^k : \sum_{i=1}^k x_i \le 1 \right\}. \tag{94}$$

It is not hard to see that this choice of y satisfies all of our support conditions and we can now develop some slightly more explicit bounds for S_1 and S_2 which is what we spend the rest of this chapter doing.

Lemma 15. *Define*

$$F_{\max} = \sup_{(t_1,\ldots,t_k) \in [0,1]^k} \left(|F(t_1, \ldots, t_k)| + \sum_{i=1}^k \left| \frac{\partial F}{\partial t_i}(t_1, \ldots, t_k) \right| \right). \tag{95}$$

Then we have

$$S_1 = \frac{\varphi(W)^k N (\log R)^k}{W^{k+1}} I_k(F) + O\left(\frac{F_{\max}^2 \varphi(W)^k N (\log R)^k}{W^{k+1} D_0} \right). \tag{96}$$

where

$$I_k = \int_0^1 \cdots \int_0^1 F(t_1, \ldots, t_k)^2 dt_1, \ldots, dt_k \tag{97}$$

Proof. By our definition of y we clearly see that

$$y_{\max} = \sup_{(t_1,\ldots,t_k) \in [0,1]^k} |F(t_1, \ldots, t_k)| \le F_{\max} \tag{98}$$

so, substituting our choice of y into the expression for S_1 given by Lemma 12, we get

$$\begin{aligned} S_1 = \frac{N}{W} &\sum_{\substack{u_1,\ldots,u_k \\ (u_i,u_j)=1 \\ (u_i,W)=1}} \left(\prod_{i=1}^k \frac{\mu(u_i)^2}{\varphi(u_i)} \right) F\left(\frac{\log u_1}{\log R}, \ldots, \frac{\log u_k}{\log R} \right)^2 \\ &+ O\left(\frac{F_{\max}^2 \varphi(W)^k N (\log R)^k}{W^{k+1} D_0} \right). \end{aligned} \tag{99}$$

Any integers a and b with $(a, W) = (b, W) = 1$ but $(a, b) \neq 1$ must share some common prime factor which is greater than D_0. Therefore we can drop the condition $(u_i, u_j) = 1$ by introducing an error of size

$$
\begin{aligned}
&\ll \frac{F_{\max}^2 N}{W} \sum_{p > D_0} \sum_{\substack{u_1,\dots,u_k < R \\ p \mid u_i, u_j \\ (u_l, W) = 1}} \prod_{l=1}^{k} \frac{\mu(u_l)^2}{\varphi(u_l)} \\
&\ll \frac{F_{\max}^2 N}{W} \sum_{p > D_0} \frac{1}{(p-1)^2} \left(\sum_{\substack{u < R \\ (u, W) = 1}} \frac{\mu(u)^2}{\varphi(u)} \right)^2 \\
&\ll \frac{F_{\max}^2 N}{W} \sum_{p > D_0} \frac{\varphi(W)^k (\log R)^k}{(p-1)^2 W^k} \\
&\ll \frac{F_{\max}^2 \varphi(W)^k N (\log R)^k}{W^{k+1} D_0}. \qquad (100)
\end{aligned}
$$

by standard estimates on the mean values of multiplicative functions.

We now deal with the sum

$$
\sum_{\substack{u_1,\dots,u_k \\ (u, W) = 1}} \left(\prod_{i=1}^{k} \frac{\mu(u_i)^2}{\varphi(u_i)} \right) F\left(\frac{\log u_1}{\log R}, \dots, \frac{\log u_k}{\log R} \right)^2 \qquad (101)
$$

Let $\kappa = 1$ and suppose that

$$
\gamma(p) = \begin{cases} 1 & , p \nmid W \\ 0 & , p \mid W \end{cases} \qquad (102)
$$

$$
L = \sum_{p \mid W} \frac{\log p}{p} + O(1) \ll \log D_0. \qquad (103)
$$

Then $h(p) = \frac{\gamma(p)}{p - \gamma(p)} = \varphi(p)^{-1}$ whenever $(p, W) = 1$ and 0 otherwise so, when extended multiplicatively, $h(n) = \varphi(n)^{-1}$ whenever $(n, W) = 1$ and 0 otherwise. Also note that $L > 0$ and, by Chebychev's inequality (18),

$$
\sum_{\substack{w \le p \le z \\ p \nmid W}} \frac{\log p}{p} - \log \frac{z}{w} \sim \log \frac{z}{w} - \log \frac{z}{w} + O(1) = O(1). \qquad (104)
$$

All the conditions of Lemma 40 are thus satisfied and we can apply it k times to (101) which then equals

$$
\frac{\varphi(W)^k (\log R)^k}{W^k} I_k(F) + O\left(\frac{F_{\max}^2 \varphi(W)^k (\log D_0)(\log R)^{k-1}}{W^k} \right). \qquad (105)
$$

as we would have

$$
\begin{aligned}
\mathfrak{S} &= \prod_p \left(1 - \frac{\gamma(p)}{p} \right)^{-1} \left(1 - \frac{1}{p} \right) \\
&= \prod_p \frac{p}{p - \gamma(p)} \times \frac{\varphi(p)}{p} \\
&= \frac{\varphi(W)}{W} \qquad (106)
\end{aligned}
$$

Substituting this into (99) and using the fact that $\log D_0 \ll \frac{\log R}{D_0}$ gives the result. $\square$

We get a similar result for S_2

Lemma 16. *Using the notation from the previous lemma*

$$S_2^{(m)} = \frac{\varphi(W)^k N(\log R)^{k+1}}{W^{k+1}\log N} J_k^{(m)}(F) + O\left(\frac{F_{\max}^2 \varphi(W)^k N(\log R)^k}{W^{k+1}D_0}\right) \tag{107}$$

where

$$J_k^{(m)}(F) = \int_0^1 \cdots \int_0^1 \left(\int_0^1 F(t_1,\ldots,t_k)dt_m\right)^2 dt_1\ldots dt_{m-1}dt_{m+1}\ldots dt_k. \tag{108}$$

Proof. By substituting our choice of y into the expression given by Lemma 14 we get

$$y^{(m)}_{r_1,\ldots,r_k} = \sum_{\substack{u<R \\ (u,W\prod_{i=1}^k r_i)=1}} \frac{\mu(u)^2}{\varphi(u)} F\left(\frac{\log r_1}{\log R},\ldots,\frac{\log r_{m-1}}{\log R},\frac{\log u}{\log R},\frac{\log r_{m+1}}{\log R},\ldots,\frac{\log r_k}{\log R}\right) \tag{109}$$

We can use estimates of multiplicative functions on this to bound $y^{(m)}_{\max}$ asymptotically above by $F_{\max}\varphi(W)(\log R)/W$.

Now set $\kappa = 1$ and

$$\gamma(p) = \begin{cases} 1 & ,p \nmid W\prod_{i=1}^k r_i \\ 0 & ,p \mid W\prod_{i=1}^k r_i \end{cases} \tag{110}$$

$$\begin{aligned} L = O(1) + \sum_{p \mid W\prod_{i=1}^k r_i} \frac{\log p}{p} &\ll \sum_{p<\log R} \frac{\log p}{p} + \sum_{\substack{p\mid W\prod_{i=1}^k r_i \\ p>\log R}} \frac{\log\log R}{\log R} \\ &\ll \log\log N \end{aligned} \tag{111}$$

as the $p < \log R$ sum will dominate over the second and $R \ll N$. Also $h(n)$, defined in the same way as in the previous lemma as $\frac{\gamma(p)}{p-\gamma(p)}$ and extended multiplicatively, is now equal to $\frac{\mu(n)}{\varphi(n)}$ whenever n is coprime to $W\prod_{i=1}^k r_i$..

As in the previous lemma we note that $\sum_{w\le p\le z}(\log p)/p - \log(z/w)$ is bounded in magnitude by a constant and so all the conditions of Lemma 40 are satisfied and we apply it once to (109) to get

$$y^{(m)}_{r_1,\ldots,r_k} = \log R \frac{\varphi(W)}{W}\left(\prod_{i=1}^k \frac{\varphi(r_i)}{r_i}\right) F^{(m)}_{r_1,\ldots,r_k} + O\left(\frac{F_{\max}\varphi(W)\log R}{WD_0}\right) \tag{112}$$

as, similarly,

$$\mathfrak{S} = \frac{\varphi(W\prod_{i=1}^k r_i)}{W\prod_{i=1}^k r_i}. \tag{113}$$

Now, substituting this back into the result from Lemma 13, remembering that gives

$$\begin{aligned} S_2^{(m)} = &\frac{\varphi(W)N(\log R)^2}{W^2\log N} \sum_{\substack{r_1,\ldots,r_k \\ (r_i,W)=1 \\ (r_i,r_j)=1 \\ r_m=1}} \prod_{i=1}^k \frac{\mu(r_i)^2\varphi(r_i)^2}{g(r_i)r_i^2}\left(F^{(m)}_{r_1,\ldots,r_k}\right)^2 \\ &+ O\left(\frac{F_{\max}^2\varphi(W)^k N(\log R)^k}{W^{k+1}D_0}\right). \end{aligned} \tag{114}$$

Similarly to before, we can remove the condition $(u_i, u_j) = 1$ with an error

$$
\begin{aligned}
&\ll \frac{F_{\max}^2 N}{W} \sum_{p>D_0} \sum_{\substack{u_1,\ldots,u_k<R\\ p|u_i,u_j\\ (u_l,W)=1}} \prod_{l=1}^{k} \frac{\mu(u_l)^2\varphi(u_l)^2}{g(u_l)u_l^2} \\
&\ll \frac{F_{\max}^2 N}{W} \sum_{p>D_0} \left(\frac{(p-1)^4}{(p-2)^2p^4}\right) \left(\sum_{\substack{u<R\\ (u,W)=1}} \frac{\mu(u)^2\varphi(u)^2}{g(u)^2u^2} \right)^{k-1} \\
&\ll \frac{F_{\max}^2 \varphi(W)^k N(\log N)^k}{W^{k+1}D_0} \qquad (115)
\end{aligned}
$$

again, by multiplicative function estimates. We now look at the sum

$$
\sum_{\substack{r_1,\ldots,r_{m-1},r_{m+1},\ldots,r_k\\ (r_i,W)=1}} \left(\prod_{\substack{1\le i\le k\\ i\neq j}} \frac{\mu(r_i)^2\varphi(r_i)^2}{g(r_i)r_i^2} \right) (F^{(m)}_{r_1,\ldots,r_k})^2 \qquad (116)
$$

and estimate it using Lemma 40 $k-1$ times, on each of them $r_1, \ldots, r_{m-1}, r_{m+1}, \ldots, r_k$ in turn. We do this, as a slight correction to [13], with $\kappa = 1$ and set

$$
\gamma(p) = \begin{cases} 1 - \frac{p^2-3p+1}{p^3-p^2-2p+1} & , p \nmid W \\ 0 & , p \mid W \end{cases} \qquad (117)
$$

$$
L = O(1) + \sum_{p|W} \frac{\log p}{p} \ll \log D_0. \qquad (118)
$$

so $h(p) = \frac{\gamma(p)}{p-\gamma(p)} = \frac{\varphi(p)^2}{g(p)p^2}$ whenver $p \nmid W$ and is 0 otherwise. So $h(n) = \frac{\varphi(n)^2}{g(n)n^2}$ whenever $(n, W) = 1$. Also

$$
\begin{aligned}
\mathfrak{S} &= \prod_{p} \left(1 - \frac{\gamma(p)}{p}\right)^{-1} \left(1 - \frac{1}{p}\right) \\
&= \prod_{p\nmid W} \left(1 - \frac{1}{p} + \frac{p^2-2p+1}{p^4-p^3-2p^2+1}\right) \left(1 - \frac{1}{p}\right) \times \frac{\varphi(W)}{W} \\
&\ll \frac{\varphi(W)}{W} \qquad (119)
\end{aligned}
$$

Then, as before

$$
\begin{aligned}
\sum_{\substack{w\le p\le z\\ p\nmid W}} \frac{\gamma(p)\log p}{p} - \log\frac{z}{w} &= \left(\sum_{\substack{w\le p\le z\\ p\nmid W}} \frac{\log p}{p} - \log\frac{z}{w} \right) - \sum_{\substack{w\le p\le z\\ p\nmid W}} \frac{\log p}{p} \times \frac{p^2-3p+1}{p^3-p^2-2p+1} \\
&\ll 1 + \sum_{n\in\mathbb{N}} n^{-2+\epsilon} \\
&= 1 + \zeta(2-\epsilon) \\
&\ll 1 \qquad (120)
\end{aligned}
$$

for all small enough $\epsilon \in \mathbb{R}^+$. So we get

$$S_2^{(m)} = \frac{\varphi(W)^k N(\log R)^{k+1}}{W^{k+1}\log N} J_k^{(m)} + O\left(\frac{F_{\max}^2 \varphi(W)^k N(\log N)^k}{W^{k+1}D_0}\right) \tag{121}$$

where

$$J_k = \int_0^1 \cdots \int_0^1 \left(\int_0^1 F(t_1, \ldots, t_k) dt_m\right)^2 dt_1, \ldots, dt_{m-1}, dt_{m+1}, \ldots, dt_k \tag{122}$$

as required. □

Putting these results all together while noticing that $1/D_0 \to 0$ yields

Proposition 17. *With the notation as before, there exists some choice of $\lambda_{d_1,\ldots,d_k}$ such that*

$$S_1 = \frac{(1+o(1))\varphi(W)^k N(\log R)^k}{W^{k+1}} I_k(F) \tag{123}$$

$$S_2 = \frac{(1+o(1))\varphi(W)^k N(\log R)^{k+1}}{W^{k+1}\log N} \sum_{m=1}^{k} J_k^{(m)}(F) \tag{124}$$

provided that neither $I_k(F)$ and $J_k^{(m)}$ are zero for any m.

5. Bounded Gaps between Primes

We are almost in a position to prove the main results of Maynard's paper but we first need a preparatory result

Proposition 18. *Let the primes have level of distribution $\theta \in (0,1)$ and let S_k denote the set of piecewise-differentiable functions $F : [0,1]^k \to \mathbb{R}$ supported on the k-simplex $\mathcal{R}_k$ with $I_k(F) \neq 0$ and $J_k^{(m)} \neq 0$ for all m. Also let*

$$M_k = \sup_{F \in S_k} \left(\frac{\sum_{m=1}^k J_k^{(m)}(F)}{I_k(F)}\right), \qquad r_k = \left\lceil \frac{\theta M_k}{2} \right\rceil. \tag{125}$$

Then there are infinitely many integers n such that at least r_k of the elements of $n + \mathcal{H}$ are prime. In particular,

$$\liminf_{n\to\infty}(p_{n+r_k-1} - p_n) \leq \max_{1\leq i,j\leq k}(h_i - h_j) \tag{126}$$

Proof. Let $R = N^{\theta/2-\delta}$ for some $\delta \in \mathbb{R}^+$, as before. Then, by definition, we can select some $F_0 \in S_k$ such that

$$\sum_{m=1}^{k} J_k^{(m)}(F_0) > (M_k - \delta) I_k(F_0) \tag{127}$$

Now let ρ be some positive real number and recall that $S = S_2 - \rho S_2$. Then, using Proposition 17, we can choose $\lambda_{d_1,\ldots,d_k}$ such that

$$\begin{aligned} S &= \frac{\varphi(W)^k N(\log R)^k}{W^{k+1}} \left(\frac{\log R}{\log N} \sum_{m=1}^{k} J_k^{(m)}(F_0) - \rho I_k(F_0) + o(1)\right) \\ &\geq \frac{\varphi(W)^k N(\log R)^k I_k(F_0)}{W^{k+1}} \left(\left(\frac{\theta}{2} - \delta\right)(M_k - \delta) - \rho + o(1)\right). \end{aligned} \tag{128}$$

If we set $\rho = \frac{\theta M_k}{2} - \epsilon$, for some $\epsilon \in \mathbb{R}$, then

$$\left(\frac{\theta}{2} - \delta\right)(M_k - \delta) - \rho = \epsilon - \delta\left(\frac{\theta}{2} - M_k + 1\right) \tag{129}$$

which is positive for small enough δ. Then, as the fraction in (128) is also strictly positive we must have that $S > 0$. So, as noted previously, there must exist an $N \le n < 2N$ such that at least $\lfloor \rho + 1 \rfloor$ primes in $n + \mathcal{H}$. Also note that $\lfloor \rho + 1 \rfloor = \lceil \frac{\theta M_k}{2} \rceil$ as long as ϵ is sufficiently small.

As this holds for any interval $[N, 2N)$ we obtain the lim inf result. □

The M_k are a ratio of k-fold integration so, as one might expect, their calculation is rooted deep within the study of the calculus of variations. This is beyond the scope of this work so we will have to content ourselves with taking other people's calculations [13] [24] at face value. Our best known values are

- $M_5 > 2$;
- $M_{59} > 4$;
- $M_k > \log k + 2 \log\log k - 2$ for sufficiently large k.

Then

Theorem 19 (Maynard-Polymath)**.** *Unconditionally we have that*

$$\liminf_{n\to\infty}(p_{n+1} - p_n) \le 300 \tag{130}$$

and, on assuming that the primes have level of distribution $\theta = 1 - \epsilon$ for every $\epsilon \in \mathbb{R}^+$,

$$\liminf_{n\to\infty}(p_{n+1} - p_n) \le 12 \tag{131}$$

$$\liminf_{n\to\infty}(p_{n+2} - p_n) \le 300. \tag{132}$$

Proof. By Theorem 11 we can take $\theta = 1/2 - \epsilon$ for all $\epsilon \in \mathbb{R}^+$. Then

$$\frac{\theta M_{59}}{2} = \frac{M_{59}}{4} - \frac{\epsilon M_{59}}{2} \tag{133}$$

can be made to exceed 1 by choosing ϵ to be sufficiently small, as $M_{59} > 4$. By Proposition 18 this gives

$$\liminf_{n\to\infty}(p_{n+1} - p_n) \le \max_{1\le i,j\le 59}(h_i - h_j) \tag{134}$$

for some 59 element admissible set. There exists such an admissible set $\mathcal{H}$ [1] whose diameter is 300, proving the first part.

The results for the primes having level of distribution $\theta = 1$ are almost identical except that you get

$$\frac{\theta M_5}{2} = \frac{M_5}{2} - \frac{\epsilon M_5}{2} > 1. \tag{135}$$

The required 5 element admissible set is given by $\{0, 2, 6, 8, 12\}$ □

In generality, we can look for a result for general prime pairs, consecutive or nonconsecutive. Unfortunately we are currently unable to obtain an effective inequality so we must content ourselves with an asymptotic one.

Theorem 20 (Maynard)**.** *Let $m \in \mathbb{N}$. Then*

$$\liminf_{n\to\infty}(p_{n+m} - p_n) \ll m^3 e^{4m}. \tag{136}$$

[1] An example is given by $\{0, 4, 6, 16, 18, 28, 30, 34, 36, 46, 48, 58, 60, 64, 66, 70, 84, 88, 90, 94, 106, 108, 114, 118, 126, 130, 136, 144, 148, 150, 156, 160, 168, 174, 178, 184, 190, 196, 198, 204, 210, 220, 226, 228, 234, 238, 240, 244, 246, 256, 268, 270, 276, 280, 286, 288, 294, 298, 300\}$

Proof. Consider the case where k is large. For this proof any constants implied by asymptotic notation will be independent of k.

By the Barban-Bombieri-Vinogradov theorem, we can take $\theta = 1/2 - \epsilon$ and so, by our explicit expression for M_k we have, for sufficiently large k:

$$\frac{\theta M_k}{2} \geq \left(\frac{1}{4} - \frac{\epsilon}{2}\right)(\log k - 2\log\log k - 2). \tag{137}$$

Set $\epsilon = 1/k$ and then, by Lemma 41,

$$\frac{\theta M_k}{2} > m \tag{138}$$

if $k \geq Cm^2e^{4m}$ for some C, independent of k and m. Then, for any admissible set $\mathcal{H} = h_1, \ldots, h_k$ with $k \geq Cm^2e^{4m}$, at least $m+1$ of the $n+h_i$ must be prime for infinitely many integers n.

We can choose our set $\mathcal{H}$ to be equal to $\{p_{\pi(k)+1} + \cdots + p_{\pi(k)+k}\}$. These are all prime and not divisible by primes less than k so, for primes $p < k$, $p_{\pi(k)+i} \not\equiv 0 \pmod{p}$. Also note that $\mathcal{H}$ only has k elements so there must be an open residue class modulo any prime greater than or equal to k. Therefore this choice of $\mathcal{H}$ is indeed admissible and has diameter

$$p_{\pi(k)+k} - p_{\pi(k)+1} \ll k\log k \tag{139}$$

by the prime number theorem.

Thus, if we take $k = \lceil Cm^2e^{4m}\rceil$, we get

$$\liminf_{n\to\infty}(p_{n+m} - p_n) \ll k\log k \ll m^2e^{4m}(2\log m + 4m) \ll m^3e^{4m} \tag{140}$$

as required. □

With a very different tone — combinatorial rather than sieve theoretic — to the rest of his results, Maynard proved one final result concerning the density of these prime tuples. We will denote by $\mathcal{P}_m$ the set of all m-tuples $h_1, h_2, \ldots, h_m$ such that all of $n+h_1, n+h_2, \ldots, n+h_m$ are simultaneously prime infinitely often. Then we have the following result.

Theorem 21 (Maynard)**.** *Let $m \in \mathbb{N}$ and let $r \in \mathbb{N}$ be sufficiently large, depending on m. Also let $\mathcal{A}$ be the set of all tuples of r distinct integers. Then*

$$\frac{\#\{(h_1, \ldots, h_m) \in \mathcal{A} : (h_1, \ldots, h_m) \in \mathcal{P}_m\}}{\#\{(h_1, \ldots, h_m)\}} \gg_m 1. \tag{141}$$

That is to say: a positive proportion of m-tuples are simultaneously prime infinitely often.

Proof. Let m be fixed and define $k = \lceil Cm^2e^{4m}\rceil$, as before. If $\{h_1, \ldots, h_k\}$ is an admissible set then there is a subset $\{h'_1, \ldots, h'_m\} \subseteq \{h_1, \ldots, h_m\}$ such that there are infinitely many integers n for which all of the $n + h'_i$ are prime.

Now let $\mathcal{A}_2$ be the set starting with $\mathcal{A}$ and then, for every $p \leq k$, removing all elements of the residue class modulo p with the fewest integers. Then, for each prime, at most $1/p$ elements will be removed. Thus

$$\#\mathcal{A}_2 \geq r\prod_{p\leq k}\left(1 - \frac{1}{p}\right) \gg_m r. \tag{142}$$

as the number of primes less than or equal to k will depend only on m. Moreover, any subset of $\mathcal{A}_2$ of size k will be admissible as it cannot cover all residue classes modulo p for any prime $p \leq k$. Write $s = \#\mathcal{A}_2$ and, as r is sufficiently large, we may assume that $s > k$.

Now there are $\binom{s}{k}$ sets $\mathcal{H} \subseteq \mathcal{A}_2$ of size k. Each of these is admissible due to the fact that $\mathcal{A}_2$ is admissible. Therefore these must contain at least one subset $\{h'_1, \dots, h'_m\}$ which are simultaneously prime infinitely often.

Conversely, any admissible set $\mathcal{B} \subseteq \mathcal{A}_2$ of size m is contained in $\binom{s-m}{k-m}$ sets $\mathcal{H} \subseteq \mathcal{A}_2$ of size k. Thus there are at least

$$\binom{s}{k}\binom{s-m}{k-m}^{-1} = \frac{s!(k-m)!(s-k)!}{k!(s-k)!(s-m)!} = \prod_{i=1}^{m} \frac{s-i}{k-i} \gg_m s^m \gg_m r^m \tag{143}$$

admissible sets of size m which are simultaneously prime infinitely often.

As there are $\binom{r}{m} \le r^m$ sets $\{h_1, \dots, h_m\} \subseteq \mathcal{A}$ we then get

$$\#\mathcal{A}_2 \ge r \prod_{p \le k}\left(1 - \frac{1}{p}\right) \gg_m \frac{r^m}{r^m} \gg_m r. \tag{144}$$

□

6. Beyond Bounded Gaps

So we have these beautiful results but what now? The Polymath project has further improved upon these bounds, getting 246 as the bound for consecutive primes and

$$\liminf_{n\to\infty} |p_{n+m} - p_n| \ll m e^{(4-52/283)m} \tag{145}$$

unconditionally[18]. Furthermore, if we consider how we arrived at the Barban-Bombieri-Vinogradov theorem then there is the following natural generalisation to the Elliott-Halberstam conjecture:

Conjecture 22 (Generalised Elliott-Halberstam). *Let $x^\epsilon \ll M, N \ll x^{1-\epsilon}$ be such that $x \ll MN \ll x$ and let α, β be coefficient sequences at scale M and N respectively. Then*

$$\sum_{q \le x^\theta} \sup\,(a,q) = 1\, |\Delta(\alpha \star \beta; a, q)| \ll x(\log x)^{-A} \tag{146}$$

for any fixed $A \in \mathbb{R}^+$ and for all $\theta \in (0,1)$.

Under the assumption that this holds, Polymath have proven that we have bounded gaps between primes of size 6 [18].

Also, if we suppose that we have proved that any admissible set of size k has infinitely many translates that contain ρ primes then we can verify Conjecture 1 in a few specific cases.

Lemma 23. *Let $k \ge 59$ and $d = \prod_{p \le k} p$. Then, for every $N \in \mathbb{N}$,*

$$\{dN, 2dN, \dots, (k-1)dN\} \tag{147}$$

contains at least one Polignac number.

Proof. Consider the set

$$\mathcal{H} = \{0, dN, 2dN, \dots, (k-1)dN\}. \tag{148}$$

If we take any prime $p \le k$ then, for all $h \in \mathcal{H}$, we have $p|h$. Id est

$$h \equiv 0 \pmod{p} \tag{149}$$

for all $p \le k$. Furthermore, if $p > k$ then $\mathcal{H}$ cannot occupy all residue classes modulo p as there are only k elements. Therefore $\mathcal{H}$ is admissible.

As k is large enough we know that there are infinitely many translates of $\mathcal{H}$ which contain two primes infinitely often. Therefore there is some pair (n_1dN, n_2dN) which are simultaneously prime infinitely often, and the difference between these must be one of the elements of the set (147). $\square$

Let $q \in \mathbb{N}$ and consider the sequence given by $n \equiv 0 \pmod q$. This has a subsequence given by $n \equiv 0 \pmod{qd}$ where $d = 59\#$. This, in turn, has the subsequence

$$(qd, 2qd, \dots, 58qd), (59qd, 2 \times 59qd, \dots, 58 \times 59qd), \dots \tag{150}$$

where each of the parenthesised terms contains a Polignac number by the above theorem so we have proven [10]:

Corollary 24. *There are infinitely many Polignac numbers on any arithmetic progression of the form*

$$q, 2q, \dots \tag{151}$$

for all $q \in \mathbb{N}$.

As there is only one even prime number we cannot have odd Polignac numbers, and we will call the arithmetic progressions containing only odd numbers trivial, but we then have the question as to whether the following statement is true.

Conjecture 25. *Every non-trivial arithmetic progression contains infinitely many Polignac numbers.*

Theorem 23 implies the conclusion of Pintz' (Theorem 3) but in a more effective form. We can reduce the value of k on improvement in our knowledge of the M_k values and, specifically, under the assumption of the Elliott-Halberstam conjecture we know that one of $30, 60, 90$ or 120 is a Polignac number and, under the generalised Elliott-Halberstam conjecture, one of 6 or 12 is a Polignac number. This very clearly shows.

Theorem 26. *Polignac's conjecture is true if and only if every admissible tuple of size* 2 *is prime infinitely often.*

By using Pintz' result that every interval of the form $[m, m+C]$ contains a Polignac number if C is large enough we can calculate a lower bound for the upper asymptotic density of the set of Polignac numbers $\mathcal{P}$.

$$\begin{aligned}\overline{\sigma}(\mathcal{P}) &= \limsup_{n\to\infty} \frac{|\mathcal{P} \cap [0,n]|}{n} \\ &\geq \limsup_{n\to\infty} \frac{\left|\bigcup_{i=1}^{n/c} \mathcal{P} \cap [(i-1)C, iC]\right|}{n}.\end{aligned} \tag{152}$$

All of the intersections above are non-empty but they might count some Polignac numbers twice so this is

$$\begin{aligned}&\geq \limsup_{n\to\infty} \frac{\sum_{i=1}^{n/c} 1}{2n} \\ &\geq \limsup_{n\to\infty} \frac{n/c - 1}{2n} \\ &= \frac{1}{2c}.\end{aligned} \tag{153}$$

So the upper asymptotic density is positive and we can apply Szemerédi's theorem [19] [25] to show [10]

Theorem 27 (Hanson, 2014). *The set of Polignac numbers contains arbitrarily long arithmetic progressions.*

This has the following natural generalisation:

Conjecture 28. *The Polignac numbers contain infinitely long arithmetic progressions.*

One might also ask whether the Selberg sieve techniques used in the proof of bounded gaps have applications to other problems related to the distribution of primes such as the binary Goldbach conjecture. Analytic number theory is certainly advancing quickly and only time can tell us what the limits of these techniques are.

A. Large Sieve Theorems

Here we gather together a collection of results needed to prove the large sieve inequality and the Barban-Bombieri-Vinogradov theorem.

Lemma 29 (Sobolev-Gallagher Inequality). *Let g have a continuous derivative on $[0,1]$. Then*

$$g\left(\frac{1}{2}\right) \leq \int_0^1 \left(|g(t)| + \frac{1}{2}|g'(t)|\right) dt. \tag{154}$$

Proof. Using integration by parts

$$\int_0^1 g(t)dt = [tg(t)]_0^1 - \int_0^1 tg'(t)dt = g(t) - \int_0^1 tg'(t)dt \tag{155}$$

and, by the fundamental theorem of calculus

$$g(x) + \int_x^1 g'(t)dt = g(x) + g(1) - g(x) = g(1) \tag{156}$$

for $x \in [0,1]$. Substituting out $g(1)$ and rearranging to put things in terms of $g(x)$ we find that

$$g(x) = \int_0^1 g(t)dt + \int_0^x tg'(t)dt + \int_x^1 (t-1)g'(t)dt. \tag{157}$$

Setting $x = 1/2$ gives

$$\begin{aligned} \left|g\left(\frac{1}{2}\right)\right| &\leq \int_0^1 |g(t)|dt + \int_0^{1/2} t|g'(t)|dt + \int_{1/2}^1 (t-1)|g'(t)|dt \\ &\leq \int_0^1 |g(t)|dt + \int_0^{1/2} \frac{1}{2}|g'(t)|dt + \int_{1/2}^1 \frac{1}{2}|g'(t)|dt \\ &\leq \int_0^1 |g(t)|dt + \frac{1}{2}\int_0^1 |g'(t)|dt \\ &\leq \int_0^1 \left(|g(t)| + \frac{1}{2}|g'(t)|\right) dt \end{aligned} \tag{158}$$

due to the fact that t and $1-t$ are less than or equal to $1/2$ on $[0,1/2]$ and $[1/2,1]$ respectively. □

Lemma 30 (Parseval's Identity). *Let f be a function with Fourier series $f(x) = \frac{1}{2a}\sum_{n=-\infty}^{\infty} c_n e(nx)$ where $c_n = \int_{-a}^{a} f(x)e(-nx)dx$. Then*

$$\frac{1}{2a}\sum_{n=-\infty}^{\infty} |c_n|^2 = \int_{-a}^{a} |f(x)|^2 dx \tag{159}$$

whenever $a = q/2$ for some $q \in \mathbb{Z}$.

Proof. Let $S_N(x) = \frac{1}{2a}\sum_{n=-N}^{N} c_n e(nx)$. Then, if $\|\cdot\|_2$ is the norm over $L^2[-a,a]$:

$$\begin{aligned}\|f - S_N\|^2 &= \langle f, f\rangle - \langle S_N, f\rangle^* - \langle S_N, f\rangle + \langle S_N, S_N\rangle \\ &= \|f\|^2 - \int_{-a}^{a} \frac{1}{2a}\sum_{n=-N}^{N} \overline{c_n} e(-nx) f(x)dx - \int_{-a}^{a} \frac{1}{2a}\sum_{n=-N}^{N} c_n e(nx)\overline{f}(x)dx \\ &\quad + \int_{-a}^{a} \frac{1}{2a}\sum_{m=-N}^{N} c_m e(mx) \frac{1}{2a}\sum_{n=-N}^{N} \overline{c_n} e(-nx). \end{aligned} \tag{160}$$

As all of our sums are finite they converge and we can exchange the order of summation and integration to give

$$\|f\|^2 - \frac{1}{2a}\sum_{n=-N}^{N} c_n\overline{c_n} - \frac{1}{2a}\sum_{n=-N}^{N} c_n\overline{c_n} + \frac{1}{4a^2}\sum_{m=-N}^{N}\sum_{n=-N}^{N} c_m\overline{c_n}\int_{-a}^{a} e((m-n)x)dx. \tag{161}$$

If $m \neq n$ then $m - n = A \in \mathbb{Z}^*$ and

$$\int_{-a}^{a} e(Ax)dx = \frac{1}{2\pi i A}[e(Ax)]_{x=-a}^{a} = 0 \tag{162}$$

whenever a is of the form $q/2$ for some integral q. However, if $m = n$, then the integrand is 1 and so the integral is $2a$ giving

$$\|f - S_N\| = \int_{-a}^{a} |f(x)|^2 - \frac{1}{2a}\sum_{n=-N}^{N} |c_n|^2. \tag{163}$$

To finish the proof we simply note that $S_N \to f$ as $n \to \infty$ implying that $\|f - S_N\| \to 0$. $\square$

Theorem 31 (Gallagher's Large Sieve Inequality)**.** *Let $\alpha_1, \alpha_2, \ldots, \alpha_J \in \mathbb{R}$ and $a_1, a_2, \ldots, a_N \in \mathbb{C}$ be two finite sequences with*

$$\delta := \min_{j\neq k} \|\alpha_j - \alpha_k\| \neq 0 \tag{164}$$

where $\|a\|$ is the distance from a to the nearest integer. Then

$$\sum_{j=1}^{J}\left|\sum_{n=M+1}^{M+N} a_n e(n\alpha_j)\right|^2 \leq \left(\pi N + \delta^{-1}\right)\sum_{n=M+1}^{M+N} |a_n|^2. \tag{165}$$

Proof. Let $g(t) = f(\alpha + \delta(t - \frac{1}{2}))$ and suppose that f has a continuous derivative. Then g has a continuous derivative and, by the Sobolev-Gallagher inequality:

$$\begin{aligned} f(\alpha) = g\left(\frac{1}{2}\right) &\leq \int_0^1 \left(\left|f\left(\alpha + \delta\left(t - \frac{1}{2}\right)\right)\right| + \frac{1}{2}\left|\delta f'\left(\alpha + \delta\left(t - \frac{1}{2}\right)\right)\right| dt\right) \\ &= \int_{\alpha-\delta/2}^{\alpha+\delta/2} \left(\delta^{-1}|f(u)| + \frac{1}{2}|f'(u)|\right) du. \end{aligned} \tag{166}$$

Let $S(\alpha) = \sum_{n=-N}^{N} a_n e(\alpha n)$ and set $f(\alpha) = S(\alpha)^2$. This has a continuous derivative so

$$\begin{aligned} \sum_{n=1}^{J} |S(\alpha_j)|^2 &\leq \sum_{n=1}^{J} \int_{\alpha_j-\delta/2}^{\alpha+\delta/2} (\delta^{-1}|S(u)|^2 + |S(u)S'(u)|)du \\ &\leq \int_0^1 (\delta^{-1}|S(u)|^2 + |S(u)S'(u)|)du \end{aligned} \tag{167}$$

as the intervals $[\alpha_j - \delta/2, \alpha_j + \delta/2]$ are non-overlapping modulo 1 and S is periodic with period 1. By the Cauchy-Schwarz inequality

$$\int_0^1 |S(u)S'(u)|du \leq \left(\int_0^1 |S(u)|^2 du\right)^{1/2} \left(\int_0^1 |S'(u)|^2 du\right)^{1/2}. \tag{168}$$

On changing variables we see that

$$\int_0^1 |S(u)|^2 du = \int_{-1/2}^{1/2} \left|S\left(u + \frac{1}{2}\right)\right|^2 du. \tag{169}$$

Calculating directly we get

$$S\left(u + \frac{1}{2}\right) = \sum_{n=-N}^{N} c_n e\left(\left(u + \frac{1}{2}\right) n\right) = \sum_{n=-N}^{N} \left(a_n e\left(\frac{1}{2} n\right)\right) e(un) \tag{170}$$

and

$$S'\left(u + \frac{1}{2}\right) = \sum_{n=-N}^{N} 2\pi i n \left(a_n e\left(\frac{1}{2} n\right)\right) e(un). \tag{171}$$

Using Parseval's identity with $a = 1/2$ gives

$$\int_{-1/2}^{1/2} \left|S\left(u + \frac{1}{2}\right)\right|^2 = \sum_{n=-N}^{N} \left|a_n e\left(\frac{1}{2} n\right)\right|^2 = \sum_{n=-N}^{N} |a_n|^2 \tag{172}$$

and

$$\int_{-1/2}^{1/2} \left|S'\left(u + \frac{1}{2}\right)\right|^2 = \sum_{n=-N}^{N} \left|2\pi i n a_n e\left(\frac{1}{2} n\right)\right|^2 = 4\pi^2 \sum_{n=-N}^{N} |n a_n|^2. \tag{173}$$

Putting these values back into (167) and using Parseval's identity again we get

$$\begin{aligned}
\sum_{j=1}^{J} |S(\alpha_j)|^2 &\leq \int_{-1/2}^{1/2} \delta^{-1} |S(u)|^2 du + \left(\int_{-1/2}^{1/2} \left|S\left(u + \frac{1}{2}\right)\right|\right)^{1/2} \left(\int_{-1/2}^{1/2} \left|S'\left(u + \frac{1}{2}\right)\right|\right)^{1/2} \\
&\leq \delta^{-1} \sum_{n=-N}^{N} |a_n|^2 + 2\pi \left(\sum_{n=-N}^{N} |a_n|^2\right)^{1/2} \left(\sum_{n=-N}^{N} |n a_n|^2\right)^{1/2} \\
&\leq (\delta^{-1} + 2\pi N) \sum_{n=-N}^{N} |a_n|^2. \qquad (174)
\end{aligned}$$

Note that $2[N/2] \leq N$ so, defining any extra a_n to be 0 if necessary:

$$\begin{aligned}
\sum_{j=1}^{J} \left|\sum_{n=M+1}^{M+N} a_n e\left(\left(\alpha_j - [N/2] - M - 1\right) n\right)\right|^2 &= \sum_{j=1}^{J} \left|\sum_{n=-[N/2]}^{[N/2]} a_{n+[N/2]+M+1} e(n\alpha_j)\right| \\
&\leq (\delta^{-1} + \pi N) \sum_{n=-[N/2]}^{[N/2]} |a_{n+[N/2]+M+1}|^2 \\
&\leq (\delta^{-1} + \pi N) \sum_{n=M+1}^{M+N} |a_n|^2 \qquad (175)
\end{aligned}$$

completing the proof. $\square$

Lemma 32 (Heath-Brown Identity). *Let K be some natural number. Then*

$$\Lambda = \sum_{j=0}^{K} (-1)^j \binom{K}{j} \mu^{\star K-j} \star \mu_{\leq}^{j} \star 1^{\star K-1} \star \log \tag{176}$$

on $[1, 2x]$, where $\mu_{\leq}$ is the Möbius function restricted to $[1, (2x)^{1/K}]$ and $f^{\star n}$ is the n-fold convolution of f with itself.

Proof. Using the identities that $\Lambda = \mu \star \log$ and $\delta = \mu \star 1_{\mathbb{N}}$, where $\delta(n) = 1_{n=1}$ is the identity for Dirichlet convolutions, we deduce that

$$\Lambda = \mu^{\star K} \star 1_{\mathbb{N}}^{\star K-1} \star \log . \tag{177}$$

Writing $\mu = \mu_{\leq} + \mu_{>}$ where $\mu_{\leq}$ and $\mu_{>}$ represent the restriction of the Möbius function to $\left[1, (2x)^{1/K}\right]$ and $\left((2x)^{1/K}, \infty\right)$ respectively, we notice that the K-fold convolution $\mu_{>}^{\star K}$ vanishes on $[1, 2x]$ and, in particular,

$$\mu_{>}^{\star K} \star 1_{\mathbb{N}}^{\star K-1} \star \log = 0 \tag{178}$$

on $[1, 2x]$. Now, as Dirichlet convolutions distribute over addition, we can generalise the binomial theorem to Dirichlet convolutions. Specifically:

$$\begin{aligned} \mu_{>}^{\star K} &= (\mu - \mu_{\leq})^{\star K} \\ &= \sum_{j=0}^{K} (-1)^j \binom{K}{j} \mu^{\star K-j} \star \mu_{\leq}^{\star j}. \end{aligned} \tag{179}$$

Putting this back into equation (178) we get

$$0 = \sum_{j=0}^{K} (-1)^j \binom{K}{j} \mu^{\star K-j} \star \mu_{\leq}^{j} \star 1_{\mathbb{N}}^{\star K-1} \star \log \tag{180}$$

on $[1, 2x]$. But the $j = 0$ term is simply Λ so the lemma follows by using the fact that Dirichlet convolution is a symmetric operation and remembering the cancellation identity $\mu \star 1_{\mathbb{N}} = \delta$. $\square$

B. Coefficient Sequences

Definition 33. A coefficient sequence is a finitely-supported sequence $\alpha : \mathbb{N} \to \mathbb{R}$ such that

$$|\alpha(n)| \ll \sigma(n)^{O(1)} \mathcal{L}^{O(1)} \tag{181}$$

Definition 34. If α is a coefficient sequence and $a \pmod q$ is a primitive residue class then the signed discrepancy $\Delta(\alpha; a, q)$ of α in the sequence $a \pmod q$ is given by

$$\Delta(\alpha; a, q) = \sum_{n \equiv a \ (\mathrm{mod}\ q)} \alpha(n) - \frac{1}{\varphi(q)} \sum_{(n,q)=1} \alpha(n) \tag{182}$$

Definition 35. A coefficient sequence α is said to be at scale N for some $N \geq 1$ if it is supported on some interval of the form $[(1 - O(\mathcal{L}^{-A_0}))N, (1 + O(\mathcal{L}^{-A_0}))N]$.

Definition 36. A coefficient sequence α at scale N is said to obey a Siegel-Walfisz theorem if

$$\Delta(\alpha 1_{(\cdot,q)=1}; a, r) \ll \sigma(qr)^{O(1)} N \mathcal{L}^{-A} \tag{183}$$

for any $q, r \geq 1$, $A \in \mathbb{R}^{+}$ and any primitive residue class $a \pmod r$.

Lemma 37. *Let α be a coefficient sequence and let $C \in \mathbb{R}^+$. Then*

$$\sum_{d \leq x^C} |\alpha(d)|^C \ll x^C \mathcal{L}^x. \tag{184}$$

Proof. Beyond the scope of this work but can be found in [23]. □

Proposition 38. *Let α, β be coefficient sequences and let a be a primitive residue class modulo q. Then*

$$\Delta(\alpha \star \beta; a, q) = \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_0 \pmod q} \overline{\chi}(a) \left(\sum_{m=1}^{\infty} (\alpha\chi)(m) \right) \left(\sum_{n=1}^{\infty} (\beta\chi)(n) \right). \tag{185}$$

Proof. By expanding out the sums and remembering that Dirichlet characters are totally multiplicative:

$$\left(\sum_{m=1}^{\infty} (\alpha\chi)(m) \right) \left(\sum_{n=1}^{\infty} (\beta\chi)(n) \right) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \alpha(m)\beta(n)\chi(mn). \tag{186}$$

Changing the first sum to be over $k = mn$ this is equal to

$$\sum_{k=1}^{\infty} \chi(k) \sum_{d|k} \alpha(d)\beta(k/d) = \sum_{k=1}^{\infty} (\alpha \star \beta)(k)\chi(k). \tag{187}$$

Thus the right-hand side of (185) becomes

$$\frac{1}{\varphi(q)} \sum_{\chi \pmod q} \overline{\chi}(a) \sum_{k=1}^{\infty} (\alpha \star \beta)(k)\chi(k) - \frac{1}{\varphi(q)} \sum_{k=1}^{\infty} (\alpha \star \beta)(k)\chi_0(k). \tag{188}$$

The primitive Dirichlet character modulo q is equal to $1_{(\cdot,q)}$ so the right-most term is

$$\frac{1}{\varphi(q)} \sum_{(k,q)=1} (\alpha \star \beta)(k) \tag{189}$$

and the left-most term is, upon switching the order of summation and recalling the orthogonality relations for Dirichlet characters

$$\sum_{k=1}^{\infty} (\alpha \star \beta)(k) \frac{1}{\varphi(q)} \sum_{\chi \pmod q} \overline{\chi}(a)\chi(k) = \sum_{k \equiv a \pmod q} (\alpha \star \beta)(k). \tag{190}$$

Summing these last two equalities gives the required result. □

We will use these results to prove a generalised form of the Barban-Bombieri-Vinogradov theorem

Theorem 39 (Generalised BBV)**.** *Let M and N be such that $x \ll MN \ll x$ and $x^\epsilon \ll M$, $N \ll x^{1-\epsilon}$ for some fixed $\epsilon \in (0,1)$. Let α, β be coefficient sequences at scale M, N respectively. Suppose also that β obeys a Siegel-Walfisz theorem, then*

$$\sum_{q \leq x^{1/2-o(1)}} \sup_{a \in (\mathbb{Z}/q\mathbb{Z})^*} |\Delta(\alpha \star \beta; a, q)| \ll x\mathcal{L}^{-A} \tag{191}$$

for some sufficiently slowly decaying $o(1)$ and any $A \in \mathbb{R}^+$.

Proof. Set $Q = x^{1/2-\epsilon}$ then, by the overspill principle (Lemma **??**), it is sufficient to show that

$$\sum_{q \leq Q} \sup_{a \in (\mathbb{Z}/q\mathbb{Q})^*} |\Delta(\alpha \star \beta; a, q)| \ll x\mathcal{L}^{-A}. \tag{192}$$

Using Proposition 38, the left-hand side is simply

$$\sum_{q\le Q}\sup_{a\in(\mathbb{Z}/q\mathbb{Z})^*}\left|\frac{1}{\varphi(q)}\sum_{\chi\ne\chi_0 \pmod q}\overline{\chi}(a)\left(\sum_{m=1}^{\infty}(\alpha\chi)(m)\right)\left(\sum_{n=1}^{\infty}(\beta\chi)(n)\right)\right|. \tag{193}$$

By the definition of imprimitivity we can rewrite every character in the above sum in the form $\chi(n)=\chi'(n)1_{(n,e)=1}$ where $q=de$ and χ' is a primitive Dirichlet character with conductor $d>1$. Then the above expression becomes

$$\sum_{e\le Q}\sum_{1<d\le Q/e}\sup_{a\in(\mathbb{Z}/q\mathbb{Z})^*}\left|\frac{1}{\varphi(de)}\sideset{}{^*}\sum_{\chi\ne\chi_0 \pmod d}\overline{\chi}(a)\left(\sum_{m=1}^{\infty}(\alpha\chi 1_{(\cdot,e)=1})(m)\right)\left(\sum_{n=1}^{\infty}(\beta\chi 1_{(\cdot,e)=1})(n)\right)\right|. \tag{194}$$

Now, using $\varphi(de)\ge\varphi(d)\varphi(e)$ we can split up the totient function term. Furthermore, if we define S_e to be equal to

$$\sum_{1<d\le Q/e}\sup_{a\in(\mathbb{Z}/q\mathbb{Z})^*}\left|\frac{1}{\varphi(d)}\sideset{}{^*}\sum_{\chi\ne\chi_0 \pmod d}\overline{\chi}(a)\left(\sum_{m=1}^{\infty}(\alpha\chi 1_{(\cdot,e)=1})(m)\right)\left(\sum_{n=1}^{\infty}(\beta\chi 1_{(\cdot,e)=1})(n)\right)\right| \tag{195}$$

then (194) is equal to

$$\sum_{e\le Q}\frac{1}{\varphi(e)}S_e \ll \mathcal{L}\sup_{e\le Q}S_e. \tag{196}$$

Working with S_e itself shows that, by the triangle inequality and the fact that the absolute value of any Dirichlet character is at most 1, S_e is no more than

$$\sum_{1<d\le Q/e}\frac{1}{\varphi(d)}\sideset{}{^*}\sum_{\chi\ne\chi_0 \pmod d}\left|\left(\sum_{m=1}^{\infty}(\alpha\chi 1_{(\cdot,e)=1})(m)\right)\left(\sum_{n=1}^{\infty}(\beta\chi 1_{(\cdot,e)=1})(n)\right)\right|. \tag{197}$$

We now break apart the sum over d by dyadic decomposition. I.E. using the observation that

$$\begin{aligned}\sum_{1<d\le Q/e}a_d &\le \sum_{k=0}^{[\log_2 Q/e]}\sum_{2^k<d\le 2^{k+1}}a_d\\ &\le \mathcal{L}\sup_{D\le Q}\sum_{D<d\le 2D}a_d\end{aligned} \tag{198}$$

Putting these all together shows that (194) is asymptotically less than

$$\mathcal{L}^{O(1)}\sup_{e,D\le Q}\sum_{D<d\le 2D}\frac{1}{\varphi(d)}\sideset{}{^*}\sum_{\chi\ne\chi_0 \pmod d}\left|\left(\sum_{m=1}^{\infty}(\alpha\chi 1_{(\cdot,e)=1})(m)\right)\left(\sum_{n=1}^{\infty}(\beta\chi 1_{(\cdot,e)=1})(n)\right)\right|. \tag{199}$$

Fix some $C \in \mathbb{R}^+$ and suppose that $D \le \mathcal{L}^C$. Then, splitting the sum over the primitive residue classes

$$\begin{aligned}\left|\sum_n \beta(n)\chi(n)1_{(n,e)=1}\right| &\le \left|\sum_{a\in(\mathbb{Z}/d\mathbb{Z})^*}\sum_{n\equiv a \ (\mathrm{mod}\ d)} \beta(n)\chi(n)1_{(n,e)=1} - \frac{1}{\varphi(d)}\sum_{(n,d)=1}\beta(n)\chi(n)1_{(n,e)=1}\right| \\ &\quad + \left|\sum_{(n,d)=1}\beta(n)\chi(n)1_{(n,e)=1}\right| \\ &\le \varphi(d)\max_{a\in(\mathbb{Z}/d\mathbb{Z})^*}\left|\Delta(\beta\chi 1_{(\cdot,e)=1};a,d)\right| + \left|\sum_{(n,d)=1}\beta(n)\chi(n)1_{(n,e)=1}\right|. \end{aligned} \tag{200}$$

We treat the first term using the Siegel-Walfisz condition (183) along with the fact that $\varphi(d) \ll d \ll \mathcal{L}^C$ and the second using Lemma 37 to obtain

$$\left|\sum_n \beta(n)\chi(n)1_{(n,e)=1}\right| \ll \sigma(de)^{O(1)} N\mathcal{L}^{-A'+O(C)}. \tag{201}$$

for any $A' \in \mathbb{R}^+$. Note that, because A' can take any positive value, we can ignore any bounded contribution of $\mathcal{L}$. So, multiplying this through by

$$\left|\sum_n \alpha(n)\chi(n)1_{(n,e)=1}\right| \ll M\mathcal{L}^{O(1)}, \tag{202}$$

and using the fact that $MN \ll x$ we get

$$\begin{aligned}\left|\sum_n \beta(n)\chi(n)1_{(n,e)=1}\right| &\ll x\mathcal{L}^{-A'}\sup_{\substack{e\le Q\\ D\le\mathcal{L}^C}}\sum_{D<d\le 2D}\frac{1}{\varphi(d)}\sideset{}{^*}\sum_{\chi\ne\chi_0 \ (\mathrm{mod}\ d)}\tau(de)^{O(1)} \\ &\ll x\mathcal{L}^{-A'}\sup_{\substack{e\le Q\\ D\le\mathcal{L}^C}}\sum_{D<d\le 2D}\tau(de)^{O(1)} \\ &\ll x\mathcal{L}^{-A'} \end{aligned} \tag{203}$$

for all $A' \in \mathbb{R}^+$, by multiplicative function means. This is acceptable so we deal with the case where $D > \mathcal{L}^C$. Then, again by Lemma 37

$$\sum_m |\alpha(m)1_{(m,e)=1}|^2 \ll M\mathcal{L}^{O(1)} \tag{204}$$

$$\sum_n |\beta(n)1_{(n,e)=1}|^2 \ll N\mathcal{L}^{O(1)}. \tag{205}$$

Using the large sieve inequality and the Cauchy-Schwarz inequality gives us

$$\begin{aligned} T &:= \sum_{D<d\le 2D}\frac{1}{\varphi(d)}\sideset{}{^*}\sum_{\chi \ (\mathrm{mod}\ ()d)}\left|\left(\sum_m \alpha(m)\chi(m)1_{(m,e)=1}\right)\left(\sum_n \beta(n)\chi(n)1_{(n,e)=1}\right)\right| \\ &\le \sum_{D<d\le 2D}\frac{1}{\varphi(d)}\left(\sideset{}{^*}\sum_{\chi \ (\mathrm{mod}\ d)}\left|\sum_m \alpha(m)\chi(m)1_{(m,e)=1}\right|^2\right)^{1/2}\left(\sideset{}{^*}\sum_{\chi \ (\mathrm{mod}\ d)}\left|\sum_n \beta(n)\chi(n)1_{(n,e)=1}\right|^2\right)^{1/2} \\ &\le \left(\sum_{D<d\le 2D}\frac{1}{\varphi(d)}\sideset{}{^*}\sum_{\chi \ (\mathrm{mod}\ d)}\left|\sum_m \alpha(m)\chi(m)1_{(m,e)=1}\right|^2\right)^{1/2}\left(\sum_{D<d\le 2D}\frac{1}{\varphi(d)}\sideset{}{^*}\sum_{\chi \ (\mathrm{mod}\ d)}\left|\sum_n \beta(n)\chi(n)1_{(n,e)=1}\right|^2\right)^{1/2} \\ &\ll \left(\frac{M+D^2}{D}\right)^{1/2}\left(\frac{N+D^2}{D}\right)^{1/2}\left(\sum_m |\alpha(m)1_{(m,e)=1}|^2\right)^{1/2}\left(\sum_n |\beta(n)1_{(n,e)=1}|^2\right)^{1/2} \end{aligned}$$

$$\ll \frac{\mathcal{L}^{O(1)}}{D}\left(MN(M+D^2)(N+D^2)\right)^{1/2}. \tag{206}$$

Again, using the fact that $MN \asymp x$ we get

$$\begin{aligned} T &\ll \frac{x^{1/2}\mathcal{L}^{O(1)}}{D}(MN+MD^2+ND^2+D^4)^{1/2} \\ &\leq \frac{x^{1/2}\mathcal{L}^{O(1)}}{D}((MN)^{1/2}+M^{1/2}D+N^{1/2}D+D^2) \\ &\ll x\mathcal{L}^{O(1)}\left(\frac{1}{D}+N^{-1/2}+M^{-1/2}+x^{-1/2}D\right). \end{aligned} \tag{207}$$

But $\mathcal{L}^C \leq D \leq Q = x^{1/2-\epsilon}$ so

$$\begin{aligned} T &\ll x\mathcal{L}^{O(1)}\left(\mathcal{L}^{-C}+x^{-\epsilon}+x^{-\epsilon}+x^{-\epsilon}\right) \\ &\ll x\mathcal{L}^{O(1)-C} \end{aligned} \tag{208}$$

which, for large enough C, is $\ll x\mathcal{L}^{-A}$ for all A. Substituting both these values back into (199) will only multiply by a bounded power of $\mathcal{L}$ which we can safely ignore. This proves the result. □

C. Assorted Lemmata

Lemma 40. *Let $\kappa, A_1, A_2, L \in \mathbb{R}^+$ and let γ be a multiplicative function satisfying*

$$0 \leq \frac{\gamma(p)}{p} \leq 1 - A_1 \tag{209}$$

and

$$-L \leq \sum_{w \leq p \leq z} \frac{\gamma(p)\log p}{p} - \kappa \log\frac{z}{w} \leq A_2 \tag{210}$$

for any $2 \leq w \leq z$. Also let h be the totally multiplicative function defined on primes by

$$h(p) = \frac{\gamma(p)}{p-\gamma(p)} \tag{211}$$

and, finally, let $G : [0,1] \to \mathbb{R}$ be a piecewise-differentiable function with

$$G_{\max} = \sup_{t\in[0,1]}(|G(t)| - |G'(t)|). \tag{212}$$

Then

$$\sum_{d<z}\mu(d)^2h(d)G\left(\frac{\log d}{\log z}\right) = \mathfrak{S}\frac{(\log z)^\kappa}{\Gamma(\kappa)}\int_0^1 G(x)x^{\kappa-1}dx + O_{A_1,A_2,\kappa}(\mathfrak{S}LG_{\max}(\log z)^{\kappa-1}) \tag{213}$$

where

$$\mathfrak{S} = \prod_p\left(1-\frac{\gamma(p)}{p}\right)^{-1}\left(1-\frac{1}{p}\right)^\kappa \tag{214}$$

is the singular series and the 'O' term is independent of G and L.

Proof. Found in [7] with slight notational changes. □

Lemma 41. *Let $k, m \in \mathbb{N}$ and suppose there exists some constant $C \in \mathbb{R}^+$, independent of m and k, such that $k \geq Cm^2e^{4m}$. Then*

$$\left(\frac{1}{4}-\frac{1}{2k}\right)(\log k - 2\log\log k - 2) > m. \tag{215}$$

Proof. Suppose that the lemma does not hold. Then there is some $k \geq Cm^2e^{4m}$ with

$$\begin{aligned}\log m &\geq \log\left(\frac{1}{4}-\frac{1}{2k}\right)+\log(\log k-2\log\log k-2)\\ &=\log\left(\frac{1}{4}-\frac{1}{2k}\right)+\log\log k+\log\left(1-2\frac{\log\log k}{\log k}-\frac{2}{\log k}\right). \qquad (216)\end{aligned}$$

Now $\log k \geq \log C + 2\log m + 4m$ so

$$\begin{aligned}0 \geq{}& \log C + 2\left(\log\left(\frac{1}{4}-\frac{1}{2k}\right)+\log\log k+\log\left(1-2\frac{\log\log k}{\log k}-\frac{2}{\log k}\right)\right)\\ &+\left(1-\frac{2}{k}\right)(\log k+2\log\log k-2)-\log k\\ &=\log C+2\left(\log\left(\frac{1}{4}-\frac{1}{2k}\right)+\log\left(1-2\frac{\log\log k}{\log k}-\frac{2}{\log k}\right)-1\right)\\ &-\frac{2}{k}(\log k+2\log\log k-2). \qquad (217)\end{aligned}$$

This is continuous and tends to a finite limit and so is bounded. Thus we can increase C sufficiently to violate this inequality for all k, yielding a contradiction. This implies that the lemma must hold. □

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