Lie $n$-algebras, supersymmetry, and division algebras

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Higher Structures IV
This research began as a puzzle. Explain this pattern:

- The only normed division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. They have dimensions $k = 1, 2, 4$ and $8$.
- The classical superstring makes sense only in dimensions $k + 2 = 3, 4, 6$ and $10$.
- The classical super-2-brane makes sense only in dimensions $k + 3 = 4, 5, 7$ and $11$.
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Pulling on this thread will lead us into higher gauge theory.
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**Higher Gauge Theory**
Everything in this table can be made “super”.

A connection valued in Lie $n$-algebra is a connection on an $n$-bundle, which is like a bundle, but the fibers are “smooth $n$-categories.”

The theory of Lie $n$-algebra-valued connections was developed by Hisham Sati, Jim Stasheff and Urs Schreiber.

Let us denote the Lie 2-superalgebra for superstrings by superstring.

Let us denote the Lie 3-superalgebra for 2-branes by 2-brane.
Yet superstrings and super-2-branes are *exceptional objects*—they only make sense in certain dimensions.

The corresponding Lie 2- and Lie 3-superalgebras are similarly exceptional.

Like many exceptional objects in mathematics, they are tied to the division algebras, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$.

In this talk, I will show you how superstring and 2-brane arise from division algebras.
But why should we care about superstring and 2-brane?

- In dimensions 3, 4, 6 and 10, we will define the superstring Lie 2-superalgebra to be the chain complex:

$$ \mathfrak{siso}(V) \leftarrow \mathbb{R} $$

This is Lie 2-superalgebra extending the Poincaré Lie superalgebra, $\mathfrak{siso}(V)$.

- In dimensions 4, 5, 7 and 11, we will define the 2-brane Lie 3-superalgebra to be a chain complex:

$$ \mathfrak{siso}(V) \leftarrow 0 \leftarrow \mathbb{R} $$

This is a Lie 3-superalgebra extending the Poincaré Lie superalgebra, $\mathfrak{siso}(V)$. 
Connections valued in these Lie $n$-superalgebras describe the \textit{parallel transport} of superstrings and super-2-branes in the appropriate dimension:

\begin{center}
\begin{tabular}{c|c}
\text{superstring}(V) & \text{Connection component} \\
\hline
$\mathbb{R}$ & $\mathbb{R}$-valued 2-form, the $B$ field. \\
\downarrow & \\
$\text{siso}(V)$ & $\text{siso}(V)$-valued 1-form.
\end{tabular}
\end{center}
### 2-brane ($\mathcal{V}$) Connection component

| $\mathbb{R}$ | $\mathbb{R}$-valued 3-form, the $C$ field. |
| $\downarrow$ | $\downarrow$ |
| 0 | $\downarrow$ |
| $\text{siso}(\mathcal{V})$ | $\text{siso}(\mathcal{V})$-valued 1-form. |
The $B$ and $C$ fields are very important in physics.

- The $B$ field, or Kalb-Ramond field, is to the string what the electromagnetic $A$ field is to the particle.
- The $C$ field is to the 2-brane what the electromagnetic $A$ field is to the particle.
The $B$ and $C$ fields are very important in physics...

- The $B$ field, or Kalb-Ramond field, is to the string what the electromagnetic $A$ field is to the particle.
- The $C$ field is to the 2-brane what the electromagnetic $A$ field is to the particle.

...and geometry:

- The $A$ field is really a connection on a $U(1)$-bundle.
- The $B$ field is really a connection on a $U(1)$-gerbe, or 2-bundle.
- The $C$ field is really a connection on a $U(1)$-2-gerbe, or 3-bundle.
Using superstring and 2-brane, we neatly package these fields with the Levi–Civita connection on spacetime.

Let us see where these Lie $n$-superalgebras come from, starting with the reason superstrings and 2-branes only make sense in certain dimensions.
In the physics literature, the classical superstring and super-2-brane require certain spinor identities to hold:

**Superstring** In dimensions 3, 4, 6 and 10, we have:

\[ [\psi, \psi] \psi = 0 \]

for all spinors \( \psi \in S \).

Here, we have:
- the bracket is a symmetric map from spinors to vectors:
  \[ [,] : \text{Sym}^2 S \to V \]
- vectors can “act” on spinors via the Clifford action, since \( V \subseteq \text{Cliff}(V) \).
Recall that:

- $V$ is the vector representation of $\text{Spin}(V) = \widetilde{SO}_0(V)$.
- $S$ is a spinor representation, i.e. a representation coming from a module of $\text{Cliff}(V)$.
- $\text{Cliff}(V) = \frac{TV}{v^2 = \|v\|^2}$. 


Similarly, for the 2-brane:

**Super-2-brane** In dimensions 4, 5, 7 and 11, the 3-ψ’s rule need not hold:

\[ [\psi, \psi]\psi \neq 0 \]

Instead, we have the 4-ψ’s rule:

\[ [\psi, [\psi, \psi]\psi] = 0 \]

for all spinors \( \psi \in S \).

Again:

- \( \mathcal{V} \) and \( S \) are vectors and spinors for these dimensions.
- \( [\cdot, \cdot] : \text{Sym}^2 S \to \mathcal{V} \).
Where do the division algebras come in?
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- We can use $\mathbb{K}$ to build $V$ and $S$ in dimensions 3, 4, 6 and 10, $\mathcal{V}$ and $\mathcal{S}$ in 4, 5, 7 and 11.
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- We can use $\mathbb{K}$ to build $V$ and $S$ in dimensions 3, 4, 6 and 10, $\mathcal{V}$ and $\mathcal{S}$ in 4, 5, 7 and 11.
- The 3-$\psi$’s and 4-$\Psi$’s rules are consequences of this construction.
In superstring dimensions 3, 4, 6 and 10:

- The vectors $V$ are the $2 \times 2$ Hermitian matrices with entries in $\mathbb{K}$:

$$V = \left\{ \left( \begin{array}{cc} t + x & \bar{y} \\ y & t - x \end{array} \right) : t, x \in \mathbb{R}, \ y \in \mathbb{K} \right\}.$$

- The determinant is then the norm:

$$-\det \left( \begin{array}{cc} t + x & \bar{y} \\ y & t - x \end{array} \right) = -t^2 + x^2 + |y|^2.$$

- This uses the properties of $\mathbb{K}$:

$$|y|^2 = y\bar{y}.$$
In superstring dimensions 3, 4, 6 and 10:

- The spinors are $S = \mathbb{K}^2$. 
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$$[\psi, \psi] = 2\psi\psi^T - 2\psi^T\psi 1 \in V$$
In superstring dimensions 3, 4, 6 and 10:

- The spinors are $S = \mathbb{K}^2$.
- The Clifford action is just matrix multiplication.
- $[-,-]$ has a nice formula using matrix operations:

$$[\psi, \psi] = 2\psi\overline{\psi}^T - 2\overline{\psi}^T\psi 1 \in V$$

- Showing $[\psi, \psi]\psi = 0$ is now an easy calculation!
These constructions are originally due to Tony Sudbery, with help from Corrinne Manogue, Tevian Dray and Jorg Schray.

We have shown to generalize them to the 2-brane dimensions 4, 5, 7 and 11, taking $\mathcal{V} \subseteq \mathbb{K}[4]$ and $\mathcal{S} = \mathbb{K}^4$.

The 4-$\Psi$’s rule

$$[\Psi, [\Psi, \Psi] \Psi] = 0$$

is then also an easy calculation.
What are the 3-ψ’s and 4-Ψ’s rules?
They are cocycle conditions.

- In 3, 4, 6 and 10, there is a 3-cochain $\alpha$:

  $$\alpha(\psi, \phi, v) = \langle \psi, v\phi \rangle.$$ 

  Here, $\langle - , - \rangle$ is a Spin$(V)$-invariant pairing on spinors.

- $d\alpha = 0$ is the 3-ψ’s rule!
What are the $3$-$\psi$’s and $4$-$\Psi$’s rules?
They are *cocycle conditions*.

- In $3$, $4$, $6$ and $10$, there is a $3$-cochain $\alpha$:

  $$
  \alpha(\psi, \phi, v) = \langle \psi, v \phi \rangle.
  $$

  Here, $\langle - , - \rangle$ is a $\text{Spin}(V)$-invariant pairing on spinors.

- $d\alpha = 0$ is the $3$-$\psi$’s rule!

- In $4$, $5$, $7$ and $11$, there is a $4$-cochain $\beta$:

  $$
  \beta(\Psi, \Phi, V, W) = \langle \Psi, (VW - WV) \Phi \rangle.
  $$

  Here, $\langle - , - \rangle$ is a $\text{Spin}(V)$-invariant pairing on spinors.

- $d\beta = 0$ is the $4$-$\Psi$’s rule!
Lie (super)algebra cohomology:

- Let $g = g_0 \oplus g_1$ be a Lie superalgebra,
- which has bracket $[,] : \Lambda^2 g \to g$,
- where $\Lambda^2 g = \Lambda^2 g_0 \oplus g_0 \otimes g_1 \oplus \text{Sym}^2 g_1$ is the graded exterior square.
- We get a cochain complex:

$$\Lambda^0 g^* \to \Lambda^1 g^* \to \Lambda^2 g^* \to \cdots$$
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- We get a cochain complex:

\[
\Lambda^0 g^* \to \Lambda^1 g^* \to \Lambda^2 g^* \to \cdots
\]

- where \( d = [\,\,]^* : \Lambda^1 g^* \to \Lambda^2 g^* \), the dual of the bracket.
- \( d^2 = 0 \) is the Jacobi identity!
In 3, 4, 6 and 10:

\[ T = V \oplus S \]

is a Lie superalgebra, with bracket

\[ [,] : \text{Sym}^2 S \to V. \]

\( \alpha(\psi, \phi, \nu) = \langle \psi, \nu \phi \rangle \) is a 3-cocycle on \( T \).

In 4, 5, 7 and 11:

\[ \mathcal{T} = \mathcal{V} \oplus S \]

is a Lie superalgebra, with bracket

\[ [,] : \text{Sym}^2 S \to \mathcal{V}. \]

\( \beta(\psi, \Phi, V, W) = \langle \psi, (VW - WV)\Phi \rangle \) is a 4-cocycle on \( \mathcal{T} \).
In 3, 4, 6 and 10: we can extend $\alpha$ to a cocycle on

$$siso(V) = \text{spin}(V) \rtimes T$$

the Poincaré superalgebra.
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In 4, 5, 7 and 11: we can extend $\beta$ to a cocycle on

$$\text{siso}(V) = \text{spin}(V) \rtimes T$$

the Poincaré superalgebra.
The spinor identities were cocycle conditions for $\alpha$ and $\beta$. What are $\alpha$ and $\beta$ good for?

Building Lie $n$-superalgebras!

**Definition**

A **Lie $n$-superalgebra** is an $n$ term chain complex of $\mathbb{Z}_2$-graded vector spaces:

$$L_0 \leftarrow L_1 \leftarrow \cdots \leftarrow L_{n-1}$$

endowed with a bracket that satisfies Lie superalgebra axioms up to chain homotopy.

This is a special case of an $L_\infty$-superalgebra.
Definition
An $L_\infty$-algebra is a graded vector space $L$ equipped with a system of grade-antisymmetric linear maps

$$[-, \cdots, -]: L^\otimes k \to L$$

satisfying a generalization of the Jacobi identity.

So $L$ has:
- a boundary operator $\partial = [-]$ making it a chain complex,
- a bilinear bracket $[-, -]$, like a Lie algebra,
- but also a trilinear bracket $[-, -, -]$ and higher, all satisfying various identities.
The following theorem says we can package cocycles into Lie \( n \)-superalgebras:

**Theorem (Baez–Crans)**

If \( \omega \) is an \( n + 1 \) cocycle on the Lie superalgebra \( g \), then the \( n \) term chain complex

\[
g \leftarrow 0 \leftarrow \cdots \leftarrow 0 \leftarrow \mathbb{R}
\]

equipped with

\[
[-, -]: \Lambda^2 g \to g
\]

\[
\omega = [-, \cdots, -]: \Lambda^{n+1} g \to \mathbb{R}
\]

is a Lie \( n \)-superalgebra.
**Theorem**

*In dimensions 3, 4, 6 and 10, there exists a Lie 2-superalgebra, which we call \textit{superstring}(V), formed by extending the Poincaré superalgebra \( \mathfrak{siso}(V) \) by the 3-cocycle \( \alpha \).*

**Theorem**

*In dimensions 4, 5, 7 and 11, there exists a Lie 3-superalgebra, which we call \textit{2-brane}(V), formed by extending the Poincaré superalgebra \( \mathfrak{siso}(V) \) by the 4-cocycle \( \beta \).*