The $C_1$ Series

These are the compact simply-connected simple Lie groups:

$$Sp(n) = \{ \text{n \times n quaternionic matrices } T \text{ w/ } TT^* = 1 \}$$

where $T^*$ is the conjugate transpose of $T$:

$$(T^*)_{ij} = (T_{ji})$$

where

$$(a + bi + cj + dk)^* = a - bi - cj - dk$$

We have

$$Sp(1) = \{ \text{unit quaternions } \}^2$$

$$\cong \text{Spin}(3)$$
More generally, $\text{Sp}(n)$ has maximal torus

$$T = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} : \theta \in \mathbb{R} \right\}$$

where

$$e^{i\theta} = \sum_{n=1}^{\infty} \frac{(i\theta)^n}{n!} = \cos \theta + i \sin \theta$$

So $\dim T = n$.

This gives

$$T = \left\{ \begin{pmatrix} 0 & -i \theta \\ i \theta & 0 \end{pmatrix} : \theta \in \mathbb{R} \right\} \cong \mathbb{R}^n$$

so

$$L = \left\{ x \in T : \exp(2\pi x) = 1 \right\}$$
\[
\begin{bmatrix}
\theta & 0 \\
0 & e^{i\theta}
\end{bmatrix}
\]

So our lattice is just a "hyperspherical" lattice, just like \( B_n \).

For \( B_n \), the Weyl group was \( S_n \times Z_2 \).

How about \( C_n \)?

Find \( g \) with:

\[
g \begin{bmatrix}
e^{i\theta} & 0 \\
0 & e^{i\theta}
\end{bmatrix} g^{-1} \in \mathbb{T}
\]

If \( g \) is a permutation matrix this holds, \( W = S_n \). What else?
For example:

\[
\begin{pmatrix}
  k & 0 \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  e^{i\theta} & 0 \\
  0 & e^{i\theta_2}
\end{pmatrix}
\begin{pmatrix}
  k^{-1} & 0 \\
  0 & 1
\end{pmatrix}
\]

\[k j = -jk\]
\[kjk^{-1} = -j\]
\[ke^{i\theta}k^{-1} = e^{-j\theta}\]

So we get

\[
\begin{pmatrix}
  e^{-i\theta} & 0 \\
  0 & e^{i\theta_2}
\end{pmatrix}
\]

So, generalizing from this, we get \(W \cong \mathbb{S}_n \times \mathbb{Z}_2^n\)
In fact, that's all! \( W = S_n \times \mathbb{Z}_2 \).

So \( B_n \) and \( C_n \) have the same Weyl group.

In what way do they differ???

We'll leave this as a puzzle for now.

\[
A_0(3) \cong B_1 = \bullet = C_1 \cong A\P(1)
\]

since we've seen \( \text{Sp}(1) \) is the double cover of \( \text{SO}(3) \).

\[
A_0(5) \cong B_2 \Rightarrow C_2 \cong A\P(2)
\]

Indeed, \( \text{Sp}(2) \) is the double cover of \( \text{SO}(5) \).
But:

\[ A_0(7) \cong B_3 \not\cong C_2 \cong A_1(3) \]

Let's digress and consider all the coincidences between \( A_n, B_n, C_n, D_n \)’s:

- \( A_1 \cong B_1 \cong C_1 \) (there's no \( D_1 \))

\[ A(2) \cong A_0(3) \cong A_1(3) \]

We know \( \text{Sp}(1) \) double covers \( \text{SO}(3) \), but also

\[ \text{IH} \hookrightarrow \mathbb{C} \otimes \mathbb{R} \text{IH} \cong \mathbb{M}_2(\mathbb{C}) \]

\[ \text{Sp}(1) \sim \text{SU}(2) \]

Gives \( \text{Sp}(1) \cong \text{SU}(2) \), so \( A(2) \cong A_0(3) \cong A_1(3) \).

\[ A_1 \times A_1 \cong D_2 \]

\[ A(2) \otimes A(2) \cong A_0(4) \]
\( SU(2) \times SU(2) \cong Spin(4) \)

Start with \( \mathbb{R}^4 \) w/ usual inner product & orientation. It's symmetry group is \( SO(4) \). This acts on the exterior algebra \( \Lambda \mathbb{R}^4 \), preserving the Hodge star operator:

\[ * : \Lambda^p \mathbb{R}^4 \rightarrow \Lambda^{4-p} \mathbb{R}^4 \]

defined by

\[ w \wedge^* v = \langle w, v \rangle \text{ vol}, \quad w, v \in \Lambda^p \mathbb{R}^4 \]

where \( \langle \cdot, \cdot \rangle \) on \( \Lambda \mathbb{R}^4 \) is induced from the inner product on \( \mathbb{R}^4 \) & \( \text{ vol} = e_1 \wedge e_2 \wedge e_3 \wedge e_4 \in \Lambda^4 \mathbb{R}^4 \)

where \( e_1, \ldots, e_4 \) is any oriented basis.

We have

\[ * : \Lambda^2 \mathbb{R}^4 \rightarrow \Lambda^2 \mathbb{R}^4 \]
\( * (e_i \wedge e_j) = e_k \wedge e_l \) where \( i, j, k, l \) are in an even permutation of \( 1, 2, 3, 4 \).

\( \ast \ast (e_i \wedge e_n) = \ast (e_3 \wedge e_4) = e_i \wedge e_n \)

In general \( \ast^2 = 1 \) on \( \Lambda^2 \mathbb{R}^4 \). So

\[
\Lambda^2 \mathbb{R}^4 = \Lambda^+ \mathbb{R}^4 \oplus \Lambda^- \mathbb{R}^4
\]

Where \( \ast \mu = \pm \mu \) if \( \mu \in \Lambda^\pm \mathbb{R}^4 \).

So: \( \text{SO}(4) \) acts on \( \Lambda^2 \mathbb{R}^4 \) preserving this splitting & inner product. So we get a homomorphism:

\[
\text{SO}(4) \rightarrow \text{SO}(\Lambda^+ \mathbb{R}^4) \times \text{SO}(\Lambda^- \mathbb{R}^4)
\]
\[ \dim \Lambda^2 \mathbb{R}^4 = \binom{4}{2} = 6 \]

\[ \Lambda^2 \mathbb{R}^4 = \langle e_1^* e_2 + e_3^* e_4, e_1^* e_3 + e_4^* e_2, e_1^* e_4 + e_2^* e_3 \rangle \]

has \( \dim = 3 \), \( \& \) similarly for \( \Lambda^2 \mathbb{R}^3 \).

So we get:

\[ \text{SO}(4) \rightarrow \text{SO}(3) \times \text{SO}(3) \]

This is 2-1, so

\[ \text{A}_0(4) \rightarrow \text{A}_0(3) \oplus \text{A}_0(3) \]

is 1-1, since both sides have dimension 6, it's onto. Thus

\[ \text{A}_0(4) \cong \text{A}_0(3) \oplus \text{A}_0(3) \]
\[ \text{Spin}(4) \cong \text{Spin}(3) \times \text{Spin}(3) \cong \text{SU}(2) \times \text{SU}(2) \]

Next:

\[ A_3 \cong D_3 \]

\[ A_4(4) \cong A_0(6) \]

\[ SU(4) \cong \text{Spin}(6) \]

\[ SU(4) \text{ is the symmetries of } \mathbb{C}^4 \text{ w/ its usual inner product & volume form } e_1 e_2 e_3 e_4 e_5 e_6. \]

We can define a Hodge star operator

\[ \star : \Lambda^2 \mathbb{C}^4 \rightarrow \Lambda^2 \mathbb{C}^4 \]
\[ w / * = 1 \text{ as before, but now } * \text{ is conjugate linear}, \quad \text{since} \]

\[ w ^ * \nu = \langle w, \nu \rangle \text{ Vol} \]

Now

\[ \Lambda^+ \mathbb{C}^4 = \{ * w = \pm w \} \]

are real subspaces of \( \Lambda^4 \mathbb{C}^4 \), w/ real dimension 6.

So we get 2-1 map

\[ \text{SU}(4) \to \text{SO}(\Lambda^+ \mathbb{C}^4) \cong \text{SO}(6) \]

So

\[ \Delta \mu(4) \to \Delta O(6) \]

is 1-1, i.e. thus onto since
\[ \dim SO(4) = 4^2 - 1 = 15 \]
\[ \dim SO(6) = \frac{6(6-1)}{2} = 15 \]

Thus, the dimensions are equal.

Thus

\[ A_4(4) \cong A_6(6) \]

\[ SU(4) \cong Spin(6). \]

The last coincidence will be

\[ B_2 \cong C_2 \]

\[ A_8(5) \cong A_7(2) \]

\[ Spin(5) \cong Sp(2) \]

to be discussed next time!