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SYMMETRY AND ASYMMETRY: THE METHOD OF MOVING SPHERES

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Abstract. This paper consists of two parts. The first part concerns a question raised by Véron on the symmetry property of positive solutions of the semilinear elliptic equation

$$\Delta u + \frac{c}{|x|^2} u + u^{(n+2)/(n-2)} = 0 \quad \text{in } \mathbb{R}^n \setminus \{0\}.$$

The second part concerns some nonlinear elliptic equations on the unit sphere $\mathbb{S}^n$. By the method of moving spheres and the global bifurcation theory, we obtain various symmetry, asymmetry, and non-existence results.

1. Introduction

In this paper, we will consider some non-linear elliptic equations on $\mathbb{R}^n$ and $\mathbb{S}^n$. We first consider

$$\Delta u + \frac{c}{|x|^2} u + u^{(n+2)/(n-2)} = 0 \quad \text{and} \quad u > 0 \text{ in } \mathbb{R}^n \backslash \{0\}. \tag{1.1}$$

Through the work of Obata [25], Gidas-Ni-Nirenberg [10] and Caffarelli-Gidas-Spruck [6], the asymptotic behavior of solutions of (1.1) as well as the classification of global solutions are well understood in the case when $c = 0$. In [28], Véron raised the following question: *For $c \in \mathbb{R}$, $c \neq 0$ and $n \geq 3$, let $u \in C^\infty(\mathbb{R}^n \backslash \{0\})$ satisfy* (1.1). *Is it true that u must be radially symmetric about the origin?* He pointed out that there might be non-radial solutions of

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a certain form as suggested in section 4 of [3]. The following result partially answers this question.

Theorem 1.1.

(i) *If* $c \geq (n-2)^2/4$, *then* (1.1) *has no smooth solution.*
(ii) *If* $c > 0$, *then any* $u \in C^\infty(\mathbb{R}^n \backslash \{0\})$ *satisfying* (1.1) *must be radially symmetric about the origin,* $u'(r) < 0$ *for* $0 < r < \infty$, *and there exists a positive constant* C *such that* $u(x) \leq C|x|^{-(n-2)/2}$ *for* $x \in \mathbb{R}^n \backslash \{0\}$.
(iii) *If* $c < (n-2)^2/4$, *then* (1.1) *has infinitely many smooth radial solutions.*
(iv) *For any* $c < -(n-2)/4$, (1.1) *has non-radial solutions. Moreover, the number of non-radial solutions goes to* ∞ *when* $c \to -\infty$.

We remark that the non-radial solutions we produced for (iv) are not of the form suggested in [3] and it will be an interesting question to study the existence of solutions of the suggested form. We also remark that the above question of Véron remains open for $-(n-2)/4 \leq c < 0$. Assertion (iii) was a known result; a proof can be found in [9]. It is interesting to note that the number $(n-2)^2/4$ appearing in Theorem 1.1 is exactly the best constant in the classical Hardy inequality which states that

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \Big(\frac{n-2}{2}\Big)^2 \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx,$$

for all $u \in C_0^\infty(\mathbb{R}^n)$ with $n \geq 3$.

In order to prove the radial symmetry of a solution u of (1.1) with $c > 0$, we will use the method of moving spheres, a variant of the method of moving planes [10], to compare u with its Kelvin transforms $u_{\bar{x},\lambda}$:

$$u_{\bar{x},\lambda}(x) := \Big(\frac{\lambda}{|x-\bar{x}|}\Big)^{n-2} u\Big(\bar{x} + \frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}\Big), \quad x \in \mathbb{R}^n \setminus \{\bar{x}\},$$

where $\lambda > 0$ and $\bar{x} \in \mathbb{R}^n$.

In order to find non-radial solutions of (1.1) for $c < -(n-2)/4$, let $v(t,\theta) := e^{-\frac{n-2}{2}t} u(r,\theta)$, where (r,θ), $0 < r < \infty$, $\theta \in \mathbb{S}^{n-1}$, are the polar coordinates of $\mathbb{R}^n$ and $t = -\log r$. Then u is a solution of (1.1) if and only if v satisfies the equation

$$v_{tt} + \Delta_{\mathbb{S}^{n-1}} v + \Big(c - \frac{(n-2)^2}{4}\Big) v + v^{(n+2)/(n-2)} = 0 \quad \text{on } \mathbb{S}^{n-1},$$

where $\mathbb{S}^{n-1}$ is the unit sphere with the canonical metric g_0 induced from $\mathbb{R}^n$, and $\Delta_{\mathbb{S}^{n-1}}$ is the corresponding Laplace-Beltrami operator on $\mathbb{S}^{n-1}$. If

v depends only on $\theta \in \mathbb{S}^{n-1}$, then

$$\Delta_{\mathbb{S}^{n-1}} v + \Big(c - \frac{(n-2)^2}{4}\Big) v + v^{(n+2)/(n-2)} = 0 \quad \text{and} \quad v > 0 \quad \text{on } \mathbb{S}^{n-1}. \tag{1.2}$$

The way we prove Theorem 1.1 (iv) is to show the existence of non-constant solutions of (1.2).

Setting $N = n-1$, we will consider the existence of non-constant solutions of the equation

$$-\Delta_{\mathbb{S}^N} v = v^p - \lambda v \quad \text{and} \quad v > 0 \quad \text{on } \mathbb{S}^N, \tag{1.3}$$

where $\lambda \in \mathbb{R}$, $N \geq 2$ and $1 < p < N^*$; here N^* denotes $(N+2)/(N-2)$ if $N \geq 3$ and ∞ if $N = 2$ respectively. It is clear that (1.3) has no solution if $\lambda \leq 0$. It was proved in [3], which sharpened an earlier result in [12], that if $0 < \lambda \leq N/(p-1)$, then the only solution of (1.3) is the constant $v = \lambda^{1/(p-1)}$. Since we will use this result, let us state it in the following form.

Theorem 1.2. ([3, 12]) *If $1 < p < N^*$, then the only solution of* (1.3) *is the constant $v = \lambda^{1/(p-1)}$ for every $0 < \lambda \leq N/(p-1)$.*

By using bifurcation theories and a priori estimates of solutions, we will show that (1.3) has non-constant solutions for every $\lambda > N/(p-1)$. In [3] Bidaut-Véron and Véron showed that for $1 < p < N^*$ and $\lambda > N/(p-1)$ but close to $N/(p-1)$ there exists non-constant solution of (1.3) due to the local bifurcation theory. In [4] Brezis and Li gave a somewhat different proof which also implies that for $p > N^*$ and $\lambda < N/(p-1)$ with $|\lambda - N/(p-1)|$ small, (1.3) has non-constant solutions. See [5] for some related works. Our result will employ the global bifurcation theorem of Rabinowitz [26].

In order to state our result more precisely, let us introduce some notation and terminology. Let $O(N+1)$ be the group consisting of $(N+1)\times(N+1)$ orthogonal matrices, and let G be the subgroup of $O(N+1)$ consisting of those elements which fix $\mathbf{e}_{N+1} = (0, \cdots, 0, 1)$. We say a function v defined on $\mathbb{S}^N$ is G-invariant if $v(O\theta) = v(\theta)$ for $\theta \in \mathbb{S}^N$ and $O \in G$. It is clear that a G-invariant function v on $\mathbb{S}^N$ can be written as $v(\theta) = \tilde{v}(\theta_{N+1})$ for some function $\tilde{v}$ defined on $[-1, 1]$, where θ_{N+1} denotes the $(N+1)$-th coordinate of θ. In the following, given a G-invariant function v on $\mathbb{S}^N$, we will always use $\tilde{v}$ to denote the corresponding function defined on $[-1, 1]$ such that $v(\theta) = \tilde{v}(\theta_{N+1})$. For each integer $l \geq 0$ and $0 < \alpha < 1$ we introduce the Banach space

$$C^{l,\alpha}_G(\mathbb{S}^N) = \{v \in C^{l,\alpha}(\mathbb{S}^N) : v \text{ is } G\text{-invariant}\},$$

where $C^{l,\alpha}(\mathbb{S}^N)$ denotes the usual Hölder spaces.

We will show that (1.3) has non-constant solutions in $C_G^{2,\alpha}(\mathbb{S}^N)$ when $\lambda > N/(p-1)$. For each k we set

$$\mathcal{S}_k := \big\{v \in C_G^{2,\alpha}(\mathbb{S}^N) : \tilde{v} \text{ has exactly } k \text{ zeroes, all of them in } (-1,1) \text{ and simple}\big\}. \tag{1.4}$$

Clearly, the S_js are mutually disjoint. Our bifurcation result reads as follows.

Theorem 1.3. *Assume that $N \geq 2$ and $1 < p < N^*$. Let $\nu_k = k(k+N-1)$ and $\lambda_k := \nu_k/(p-1)$ for $k \geq 1$. Then for each $k \geq 1$ and $\lambda_k < \lambda \leq \lambda_{k+1}$,* (1.3) *has k distinct non-constant solutions $v_1, \cdots, v_k$ such that $v_j - \lambda^{1/(p-1)} \in \mathcal{S}_j$ for $1 \leq j \leq k$.*

Next, we will consider symmetry properties of solutions of some elliptic equations on $\mathbb{S}^n$. A point $\theta \in \mathbb{S}^n$ is represented as $\theta = (\theta_1, \cdots, \theta_{n+1}) \in \mathbb{R}^{n+1}$ with $\sum \theta_i^2 = 1$. In the following, we will always use $\mathbf{n}$ and $\mathbf{s}$ to denote the north pole and south pole respectively; i.e., $\mathbf{n} = (0, \cdots, 0, 1)$ and $\mathbf{s} = (0, \cdots, 0, -1)$. When $n \geq 3$, the conformal Laplacian on $\mathbb{S}^n$ is defined as

$$\mathcal{L}_{\mathbb{S}^n} := \Delta_{\mathbb{S}^n} - \frac{n(n-2)}{4}.$$

In the following, g denotes a given function in $C^0(\Omega \times (0,\infty))$, where Ω is one of the sets $\mathbb{S}^n$, $\mathbb{S}^n \setminus \{\mathbf{n}\}$ or $\mathbb{S}^n \setminus \{\mathbf{n}, \mathbf{s}\}$, which should be clear from the context.

We first consider the equation

$$-\mathcal{L}_{\mathbb{S}^n} v = g(\theta, v) \quad \text{and} \quad v > 0 \text{ on } \mathbb{S}^n \backslash \{\mathbf{n}\}. \tag{1.5}$$

We will give the symmetry property of solutions of (1.5) under various conditions on g. The following conditions are used in the first result.

- (g1) For each $s > 0$ the function $\theta \to g(\theta, s)$ is rotationally symmetric about the line through $\mathbf{n}$ and $\mathbf{s}$.
- (g2) For each $s > 0$ and any θ, θ' on the same geodesic passing through $\mathbf{n}$ and $\mathbf{s}$, the function $\theta \to g(\theta, s)$ satisfies $g(\theta, s) > g(\theta', s)$ if $\theta_{n+1} > \theta'_{n+1}$.
- (g3) For each $\theta \in \mathbb{S}^n \setminus \{\mathbf{n}, \mathbf{s}\}$ the function $s \to s^{-(n+2)/(n-2)} g(\theta, s)$ is non-increasing on $(0, \infty)$.
- (g4) For each $\theta \in \mathbb{S}^n \setminus \{\mathbf{n}, \mathbf{s}\}$ the function $s \to g(\theta, s)$ is non-decreasing on $(0, \infty)$.

Theorem 1.4. *For $n \geq 3$, assume that g is continuous on $(\mathbb{S}^n \setminus \{\mathbf{n}\}) \times (0, \infty)$ with $g(\cdot, s)$ bounded from below in $\mathbb{S}^n \setminus \{\mathbf{n}\}$ for each $s \in (0, \infty)$ and satisfies* (g1)-(g4). *If $v \in C^2(\mathbb{S}^n \setminus \{\mathbf{n}\})$ is a solution of* (1.5) *satisfying*

$$\liminf_{\theta \to \mathbf{n}} v(\theta) > 0, \tag{1.6}$$

then v is rotationally symmetric about the line through $\mathbf{n}$ and $\mathbf{s}$.

Remark 1.1. Condition (1.6) ensures that our moving sphere procedure can start; the proof can be found in Lemma 3.1 in section 3. If we assume that $g(\theta, v) \geq 0$ on $\mathbb{S}^n \setminus \{\mathbf{n}\}$, then (1.6) is satisfied automatically (see Lemma 3.2).

The proof of Theorem 1.4 is based on a moving sphere procedure on $\mathbb{S}^n$, with a feature of varying both the radius and the center of the moving sphere, which we will introduce in the following. Given a function v on $\mathbb{S}^n$, let us first define its Kelvin transforms. Fix $\mathbf{p} \in \mathbb{S}^n$ and $0 < \lambda < \pi$, and let $B_\lambda(\mathbf{p})$ be the geodesic ball on $\mathbb{S}^n$ with center $\mathbf{p}$ and radius λ. Set $\Sigma_{\mathbf{p},\lambda} := \mathbb{S}^n \setminus \overline{B_\lambda(\mathbf{p})}$. Let $\varphi_{\mathbf{p},\lambda} : \mathbb{S}^n \to \mathbb{S}^n$ be the uniquely determined conformal diffeomorphism such that $\varphi_{\mathbf{p},\lambda}(B_\lambda(\mathbf{p})) = \Sigma_{\mathbf{p},\lambda}$, $\varphi(\Sigma_{\mathbf{p},\lambda}) = B_\lambda(\mathbf{p})$ and $\varphi_{\mathbf{p},\lambda}$ fixes every point on $\partial B_\lambda(\mathbf{p})$. Then the Kelvin transforms of v are defined by

$$v_{\mathbf{p},\lambda} := |J_{\varphi_{\mathbf{p},\lambda}}|^{\frac{n-2}{2n}} (v \circ \varphi_{\mathbf{p},\lambda}), \tag{1.7}$$

where $J_{\varphi_{\mathbf{p},\lambda}}$ denotes the Jacobian of $\varphi_{\mathbf{p},\lambda}$. By the conformal invariance we have

$$-\mathcal{L}_{\mathbb{S}^n} v_{\mathbf{p},\lambda} = |J_{\varphi_{\mathbf{p},\lambda}}|^{\frac{n+2}{2n}} (-\mathcal{L}_{\mathbb{S}^n} v) \circ \varphi_{\mathbf{p},\lambda}. \tag{1.8}$$

If we use (r, ω), $0 < r < \pi$, $\omega \in \mathbb{S}^{n-1}$, to denote the geodesic polar coordinates on $\mathbb{S}^n$ with respect to $\mathbf{p}$, then

$$\varphi_{\mathbf{p},\lambda}(r, \omega) = (h_\lambda(r), \omega),$$

where $h_\lambda(r) \in (0, \pi)$ is determined by the equation

$$\cos h_\lambda(r) = \frac{2\cos\lambda - (1 + \cos^2\lambda)\cos r}{1 + \cos^2\lambda - 2\cos\lambda\cos r}.$$

Some straightforward calculation then gives

$$|J_{\varphi_{\mathbf{p},\lambda}}|(r, \omega) = \Big(\frac{\sin^2\lambda}{1 + \cos^2\lambda - 2\cos\lambda\cos r}\Big)^n.$$

From this we can see that $|J_{\varphi_{\mathbf{p},\lambda}}| < 1$ on $\Sigma_{\mathbf{p},\lambda}$ if $\lambda < \pi/2$ and $|J_{\varphi_{\mathbf{p},\lambda}}| > 1$ on $\Sigma_{\mathbf{p},\lambda}$ if $\lambda > \pi/2$.

For a solution v of (1.5), the proof of its rotational symmetry is reduced to showing that $v = v_{\mathbf{p},\pi/2}$ on $\Sigma_{\mathbf{p},\pi/2} \setminus \{\mathbf{n}\}$ for every $\mathbf{p} \in \partial B_{\mathbf{p},\pi/2}(\mathbf{n})$. The comparison of v with $v_{\mathbf{p},\lambda}$ is always possible for small $\lambda > 0$ if v is regular at $\mathbf{p}$; i.e., there exists $\lambda_0 > 0$ such that

$$v_{\mathbf{p},\lambda} \le v \quad \text{on } \Sigma_{\mathbf{p},\lambda} \setminus \{\mathbf{n}\} \text{ for each } 0 < \lambda < \lambda_0.$$

The number λ_0 in general depends on $\mathbf{p}$. Under the conditions in Theorem 1.4 we can show that λ_0 can be taken as $\pi/2$ if $\mathbf{p} = \mathbf{s}$. We define

$$\Sigma := \{\mathbf{p} \in \mathbb{S}^n : v \ge v_{\mathbf{p},\pi/2} \text{ in } \Sigma_{\mathbf{p},\pi/2} \setminus \{\mathbf{n}\}\}.$$

By using the strong maximum principle and the Hopf lemma we are able to show $\Sigma \supset \overline{B_{\pi/2}(\mathbf{s})}$. This is enough for our purpose. The way we prove $\Sigma \supset \overline{B_{\pi/2}(\mathbf{s})}$ is of some independent interest: For any point $\mathbf{p} \in \partial B_{\pi/2}(\mathbf{s})$, we construct $x \in C^1([0,1], \mathbb{S}^n)$, $\lambda \in C^0([0,1],[0,\pi/2])$, satisfying

$$x(0) = \mathbf{s},\ x(1) = \mathbf{p},\ \lambda(0) = 0,\ \lambda(1) = \pi/2,$$

and prove

$$v_{x(t),\lambda(t)} \le v \quad \text{on } \Sigma_{x(t),\lambda(t)} \setminus \{\mathbf{n}\} \text{ for all } 0 \le t \le 1.$$

In fact, we take

$$x(t) = \mathbf{s} \ \text{ and } \ \lambda(t) = t\pi \ \text{ for } 0 \le t \le \tfrac{1}{2},$$

and

$$\lambda(t) = \tfrac{\pi}{2} \quad \text{for} \quad \tfrac{1}{2} \le t \le 1,$$

while for $\frac{1}{2} \le t \le 1$, $x(t)$ goes from $\mathbf{s}$ to $\mathbf{p}$ along the shortest geodesic (the largest circle).

We next give a symmetry result on the equation

$$-\mathcal{L}_{\mathbb{S}^n} v = g(\theta, v) \quad \text{and} \quad v > 0 \quad \text{on } \mathbb{S}^n \setminus \{\mathbf{n}, \mathbf{s}\}, \tag{1.9}$$

where a solution v is allowed to have two singularities. For the function g, in addition to (g1) and (g3), we will assume the following two conditions.

- (g5) For each $s > 0$ and any θ, θ' on the same geodesic passing through $\mathbf{n}$ and $\mathbf{s}$, there holds $g(\theta, s) \ge g(\theta', s)$ if $\theta_{n+1} > \theta'_{n+1} > 0$ and $g(\theta, s) \ge g(\theta', s)$ if $\theta_{n+1} < \theta'_{n+1} < 0$.
- (g6) Either the inequalities in (g5) are strict or the function in (g3) is strictly decreasing.

Theorem 1.5. *For $n \ge 3$, assume that g is continuous on* $(\mathbb{S}^n \setminus \{\mathbf{n}, \mathbf{s}\}) \times (0, \infty)$ *with $g(\cdot, s)$ bounded from below in $\mathbb{S}^n \setminus \{\mathbf{n}, \mathbf{s}\}$ for each $s \in (0, \infty)$ and satisfies* (g1), (g3), (g5) *and* (g6). *If v is a solution of* (1.9) *on $\mathbb{S}^n \setminus \{\mathbf{n}, \mathbf{s}\}$ satisfying*

$$\liminf_{\theta \to \mathbf{n}} v(\theta) > 0 \quad \textit{and} \quad \liminf_{\theta \to \mathbf{s}} v(\theta) > 0, \tag{1.10}$$

then v is rotationally symmetric about the line through $\mathbf{n}$ *and* $\mathbf{s}$*.*

Remark 1.2. Similar to Remark 1.1, (1.10) is sufficient for the moving sphere procedure to start on the equator. If we assume that $g(\theta, v) \geq 0$ on $\mathbb{S}^n \setminus \{\mathbf{n}, \mathbf{s}\}$, then (1.10) is automatically satisfied.

As the first application of Theorem 1.4 and Theorem 1.5, we consider the Matukuma equation

$$-\Delta u = \frac{1}{1+|x|^2} u^p \quad \text{and} \quad u > 0 \quad \text{in } \mathbb{R}^n, \tag{1.11}$$

where $n \geq 3$ and $p \geq 0$. Let $\pi_{\mathbf{n}} : \mathbb{S}^n \setminus \{\mathbf{n}\} \to \mathbb{R}^n$ be the stereographic projection which sends $\mathbf{n}$ to ∞. Let g_0 be the standard metric on $\mathbb{S}^n$. It is well known that

$$(\pi_{\mathbf{n}}^{-1})^*(g_0) = \xi(x)^{4/(n-2)} \sum_{i=1}^{n} (dx_i)^2,$$

where

$$\xi(x) = \Big(\frac{2}{1+|x|^2}\Big)^{(n-2)/2}, \quad x \in \mathbb{R}^n.$$

For a solution u of (1.11), we define a function v on $\mathbb{S}^n \setminus \{\mathbf{n}\}$ by

$$v(\theta) = (\xi^{-1} u)(\pi_{\mathbf{n}}(\theta)), \quad \theta \in \mathbb{S}^n \setminus \{\mathbf{n}\}.$$

By using the conformal invariance one can check that v satisfies (1.5) with

$$g(\theta, s) := \frac{1}{2}(1-\theta_{n+1})^{\frac{n-2}{2}(p-\frac{n}{n-2})} s^p.$$

Thus, g satisfies (g1)–(g4) if $0 \leq p < \frac{n}{n-2}$ and g satisfies (g1), (g3), (g5) and (g6) if $p = \frac{n}{n-2}$. Theorem 1.4 and Theorem 1.5 then imply that v is rotationally symmetric about the line through $\mathbf{n}$ and $\mathbf{s}$, which in turn implies that u is radially symmetric about the origin. On the other hand, it is easy to see that there is no smooth positive radially symmetric solution to $-\Delta u \geq \frac{1}{1+|x|^2} u^p$ in $\mathbb{R}^n$ for $0 \leq p < 1$. We, thus, obtain the following.

Corollary 1.1. *If* $1 \leq p \leq n/(n-2)$*, then any smooth solution of the Matukuma equation* (1.11) *must be radially symmetric about the origin. If* $0 \leq p < 1$*,* (1.11) *has no smooth solutions.*

Remark 1.3. The result, as far as we know, is new for the case $n \geq 4$ and for the case $n = 3$ and $0 \leq p \leq 1$. When $n = 3$ and $1 < p < 5$, the result was proved by Li in [16]. For $n \geq 3$ and $1 < p < \frac{n+2}{n-2}$, the result was proved earlier by Li and Ni in [17, 18, 19] under an additional finite total mass condition:

$$\int_{\mathbb{R}^n} \frac{1}{1+|x|^2} u^p dx < \infty.$$

Our method is different from theirs. In [17, 18, 19] under the finite total mass condition, they analyzed the asymptotic behavior of solutions at ∞ to ensure that the moving plane method can start at ∞. In [16], Li obtained the asymptotic behavior of the solution in dimension $n = 3$ without the finite total mass condition. Our proof, via the method of moving spheres, does not need to analyze the asymptotic behavior of solutions at ∞.

We now consider a special form of equation (1.9) as follows

$$-\mathcal{L}_{\mathbb{S}^n} v = K(\theta) v^{(n+2)/(n-2)} \quad \text{and} \quad v > 0 \quad \text{on } \mathbb{S}^n \setminus \{\mathbf{n}, \mathbf{s}\}; \tag{1.12}$$

i.e., $g(\theta, s) = K(\theta) s^{(n+2)/(n-2)}$, where $K(\theta)$ is a function defined on $\mathbb{S}^n \setminus \{\mathbf{n}, \mathbf{s}\}$. As a consequence of Theorem 1.5 we have the following.

Corollary 1.2. *For $n \geq 3$, assume that K is continuous on $\mathbb{S}^n \setminus \{\mathbf{n}, \mathbf{s}\}$ and that K is rotationally symmetric about the line through $\mathbf{n}$ and $\mathbf{s}$. Assume further that there exists $-1 < c < 1$ such that $K(\theta) > K(\theta')$ if $\theta_{n+1} > \theta'_{n+1} \geq c$ and $K(\theta) < K(\theta')$ if $\theta_{n+1} < \theta'_{n+1} \leq c$. Then any solution $v \in C^2(\mathbb{S}^n \setminus \{\mathbf{n}, \mathbf{s}\})$ of (1.12) satisfying (1.10) is rotationally symmetric about the line through $\mathbf{n}$ and $\mathbf{s}$.*

To see this, let $\varphi_c : \mathbb{S}^n \to \mathbb{S}^n$ be the conformal diffeomorphism such that $\varphi_c(B_{\pi/2}(\mathbf{s})) = \Gamma_c$ and $\varphi_c(\Sigma_{\mathbf{s},\pi/\mathbf{2}}) = \mathbb{S}^n \setminus \Gamma_c$, where $\Gamma_c := \{\theta \in \mathbb{S}^n : \theta_{n+1} < c\}$. For a solution v of (1.12), we define

$$\hat{v} := |J_{\varphi_c}|^{2n/(n-2)} (v \circ \varphi_c),$$

where J_{φ_c} is the Jacobian of φ_c. Then

$$-\mathcal{L}_{\mathbb{S}^n} \hat{v} = (K \circ \varphi_c) \hat{v}^{(n+2)/(n-2)} \quad \text{and} \quad \hat{v} > 0 \quad \text{on } \mathbb{S}^n \setminus \{\mathbf{n}, \mathbf{s}\}.$$

We can use Theorem 1.5 to conclude that $\hat{v}$ is rotationally symmetric about the line through $\mathbf{n}$ and $\mathbf{s}$ and so is v.

Theorem 1.5 can be used to classify C^2 solutions of the equation

$$-\mathcal{L}_{\mathbb{S}^n} v = f(v) \quad \text{and} \quad v > 0 \quad \text{on } \mathbb{S}^n, \tag{1.13}$$

where $f : [0, \infty) \to \mathbb{R}$ is a continuous function.

Corollary 1.3. *Assume that $n \geq 3$ and that $s^{-(n+2)/(n-2)} f(s)$ is strictly decreasing on $(0, \infty)$. Then any solution $v \in C^2(\mathbb{S}^n)$ of (1.13) must be a constant $v \equiv c$ on $\mathbb{S}^n$ satisfying $f(c) = \frac{n(n-2)}{4} c$.*

When f satisfies some differentiability conditions, Gidas and Spruck obtained this result in [12] by using an Obata type argument. On the other hand, Brezis and Li obtained it in [4] by transforming the equation into an equation in $\mathbb{R}^n$ and then using the result established in [11] by moving plane methods.

If we take

$$g(\theta, s) = s^{\frac{n+2}{n-2}} + \Big(\frac{n(n-2)}{4} - \beta\Big)s, \tag{1.14}$$

for some number $\beta \in \mathbb{R}^n$, then (1.9) reduces to the form

$$-\Delta_{\mathbb{S}^n} v + \beta v = v^{\frac{n+2}{n-2}} \quad \text{and} \quad v > 0 \quad \text{on } \mathbb{S}^n \backslash \{\mathbf{n}, \mathbf{s}\}. \tag{1.15}$$

By using Theorem 1.5 we can analyze the solutions of (1.15) in some detail.

Corollary 1.4.

(i) *If $\beta < n(n-2)/4$, then any solution $v \in C^2(\mathbb{S}^n \setminus \{\mathbf{n}, \mathbf{s}\})$ of* (1.15) *is rotationally symmetric about the line through* $\mathbf{n}$ *and* $\mathbf{s}$.

(ii) *If $\beta \le 0$, then* (1.15) *has no solution $v \in C^2(\mathbb{S}^n \setminus \{\mathbf{n}, \mathbf{s}\})$.*

(iii) *If $(n-2)/2 < \beta < n(n-2)/4$, then the equation* (1.15) *has infinitely many solutions in $C^2(\mathbb{S}^n \setminus \{\mathbf{n}, \mathbf{s}\})$, which have exactly two singularities.*

It is well known that if $v \in C^\infty(\mathbb{S}^n \backslash \{\mathbf{n}, \mathbf{s}\})$ is a solution of (1.15) satisfying $v \in L^{\frac{2n}{n-2}}(\mathbb{S}^n)$, then $v \in C^\infty(\mathbb{S}^n)$ and thus v must be constant if $0 < \beta < n(n-2)/4$. Corollary 1.4 (iii) indicates that (1.15) has non-constant solutions at least for $(n-2)/2 < \beta < n(n-2)/4$ if one drops the condition $v \in L^{\frac{2n}{n-2}}(\mathbb{S}^n)$. It is interesting to point out that, for $\beta = \beta_0 := \frac{1}{16}(n-2)(3n-2)$,

$$v(\theta) := \Big(\frac{n-2}{2}\Big)^{\frac{n-2}{2}} (1 - \theta_{n+1})^{-\frac{n-2}{4}}$$

is also a solution of (1.15) which has only one singularity at $\mathbf{n}$. Note that $\beta_0 \le (n-2)/2$ for $n = 3, 4$, therefore it is probably true that (1.15) has many solutions even if $0 < \beta \le (n-2)/2$.

Similar problems can be considered on $\mathbb{S}^2$. We first consider the equation of the form

$$-\Delta_{\mathbb{S}^2} v + 1 = K(\theta)e^{2v} + f(\theta) \quad \text{on } \mathbb{S}^2 \setminus \{\mathbf{n}, \mathbf{s}\}, \tag{1.16}$$

where K and f are continuous functions on $\mathbb{S}^2$ satisfying the following conditions.

(K1) For any θ, θ' on the same geodesic passing through $\mathbf{n}$ and $\mathbf{s}$, $K(\theta) \ge K(\theta')$ if $\theta_{n+1} > \theta'_{n+1} \ge 0$ and $K(\theta) \le K(\theta')$ if $\theta_{n+1} < \theta'_{n+1} \le 0$.

(f1) For any θ, θ' on the same geodesic passing through $\mathbf{n}$ and $\mathbf{s}$, $f(\theta) \geq f(\theta')$ if $\theta_{n+1} > \theta'_{n+1} \geq 0$ and $f(\theta) \leq f(\theta')$ if $\theta_{n+1} < \theta'_{n+1} \leq 0$.

(Kf1) $f^2 + (K - ⨍_{\mathbb{S}^2} K)^2$ is not identically zero on $\mathbb{S}^2$.

Similarly to Theorem 1.5 we have the following.

Theorem 1.6. *Assume that K and f are continuous non-negative functions defined on $\mathbb{S}^2 \setminus \{\mathbf{n}, \mathbf{s}\}$ satisfying* (K1), (f1), *and* (Kf1). *If both K and f are rotationally symmetric about the line through $\mathbf{n}$ and $\mathbf{s}$, then any solution $v \in C^2(\mathbb{S}^2 \setminus \{\mathbf{n}, \mathbf{s}\})$ of (1.16) satisfying*

$$\limsup_{\theta \to \mathbf{n}} \frac{v(\theta)}{\log d(\theta, \mathbf{n})} \leq 0 \quad \textit{and} \qquad \limsup_{\theta \to \mathbf{s}} \frac{v(\theta)}{\log d(\theta, \mathbf{s})} \leq 0 \tag{1.17}$$

must be rotationally symmetric about the line through $\mathbf{n}$ and $\mathbf{s}$.

Remark 1.4. It is well known that, if $f^2 + (K - ⨍_{\mathbb{S}^2} K)^2 \equiv 0$ on $\mathbb{S}^2$, the conclusion of Theorem 1.6 is not true.

As an immediate consequence of Theorem 1.6 we have the following.

Corollary 1.5. *Suppose that $v \in C^2(\mathbb{S}^2)$ satisfies the equation*

$$-\Delta_{\mathbb{S}^2} v + \beta = e^{2v} \quad \textit{on } \mathbb{S}^2, \tag{1.18}$$

with $0 < \beta < 1$. Then v must be constant.

Next, we consider the equation

$$-\Delta_{\mathbb{S}^2} v + 1 = K(\theta) e^{2v} + f(\theta) \quad \text{on } \mathbb{S}^2 \setminus \{\mathbf{n}\}, \tag{1.19}$$

where K and f are non-negative continuous functions on $\mathbb{S}^2$ satisfying the following conditions.

(K2) For any θ, θ' on the same geodesic passing through $\mathbf{n}$ and $\mathbf{s}$, $K(\theta) \geq K(\theta')$ if $\theta_{n+1} > \theta'_{n+1}$.

(f2) For any θ, θ' on the same geodesic passing through $\mathbf{n}$ and $\mathbf{s}$, $f(\theta) \geq f(\theta')$ if $\theta_{n+1} > \theta'_{n+1}$.

(Kf2) $(f - ⨍_{\mathbb{S}^2} f)^2 + (K - ⨍_{\mathbb{S}^2} K)^2$ is not identically zero on $\mathbb{S}^2$.

Similarly to Theorem 1.4 we have the following.

Theorem 1.7. *Assume that K and f are continuous non-negative functions defined on $\mathbb{S}^2 \setminus \{\mathbf{n}\}$ satisfying* (K2), (f2), *and* (Kf2). *If both K and f are rotationally symmetric about the line through $\mathbf{n}$ and $\mathbf{s}$, then any solution $v \in C^2(\mathbb{S}^2 \setminus \{\mathbf{n}\})$ of (1.19) satisfying*

$$\limsup_{\theta \to \mathbf{n}} \frac{v(\theta)}{\log d(\theta, \mathbf{n})} \leq 0 \tag{1.20}$$

must be rotationally symmetric about the line through $\mathbf{n}$ *and* $\mathbf{s}$*.*

We give an application of Theorem 1.7 to the mean field equation

$$\Delta_{\mathbb{S}^2}\varphi + \frac{\exp(\alpha\varphi(\theta) - \gamma\langle \mathbf{s}, \theta\rangle)}{\int_{\mathbb{S}^2}\exp(\alpha\varphi(\sigma) - \gamma\langle \mathbf{s}, \sigma\rangle)d\sigma} - \frac{1}{4\pi} = 0 \quad \text{on } \mathbb{S}^2, \tag{1.21}$$

where α and γ are non-negative numbers. It is easy to see that the new function

$$v = \frac{\alpha}{2}\varphi(\theta) - \frac{1}{2}\log\int_{\mathbb{S}^2}\exp(\alpha\varphi(\sigma) - \gamma\langle \mathbf{s}, \sigma\rangle)d\sigma + \frac{1}{2}\log\frac{\alpha}{2}$$

satisfies the equation

$$-\Delta_{\mathbb{S}^2}v + \frac{\alpha}{8\pi} = \exp(2v - \gamma\langle \mathbf{s}, \theta\rangle) \quad \text{on } \mathbb{S}^2.$$

This is exactly the equation (1.20) with $K(\theta) = \exp(-\gamma\langle \mathbf{s}, \theta\rangle)$ and $f(\theta) = 1 - \frac{\alpha}{8\pi}$. By using Theorem 1.7 we thus conclude the following result which was proved by Lin [23].

Corollary 1.6. *If* $0 < \alpha < 8\pi$ *and* $\gamma \geq 0$*, then any solution of* (1.21) *must be rotationally symmetric about the line through* $\mathbf{n}$ *and* $\mathbf{s}$*.*

Using the moving sphere procedure, we can also show the following result. We consider the equation

$$-\mathcal{L}_{\mathbb{S}^n}v = K(\theta)v^{(n+2)/(n-2)} \quad \text{and} \quad v > 0 \quad \text{on } \mathbb{S}^n \setminus \{\mathbf{n}\}. \tag{1.22}$$

We will use ∇_X to denote the covariant differentiation on $\mathbb{S}^n$ with respect to the vector field X.

Theorem 1.8. *Assume that* $n \geq 3$ *and that* K *is a* C^1 *function on* $\mathbb{S}^n$ *such that* $\nabla_{\frac{\partial}{\partial\theta_{n+1}}}K \geq 0$ *and is not identically zero on* $\mathbb{S}^n \setminus \{\mathbf{n}, \mathbf{s}\}$*. If* $v \in C^2(\mathbb{S}^n \setminus \{\mathbf{n}\})$ *is a solution of* (1.22) *satisfying*

$$\liminf_{\theta\to\mathbf{n}} v(\theta) > 0, \tag{1.23}$$

then

$$v(\theta) \geq C_0 d(\theta, \mathbf{n})^{-(n-2)/2} \quad \textit{for all } \theta \in \mathbb{S}^n \setminus \{\mathbf{n}\}, \tag{1.24}$$

for some positive constant C_0*, where* $d(\cdot,\cdot)$ *denotes the distance on* $\mathbb{S}^n$*. In particular,* (1.22) *has no* $C^2(\mathbb{S}^n)$ *solutions.*

Remark 1.5. If $K \geq 0$, then (1.23) is automatically satisfied.

Remark 1.6. The non-existence of a $C^2(\mathbb{S}^n)$ solution to (1.22) is due to Kazdan-Warner [14].

Our moving sphere procedure can also be used to obtain a Kazdan-Warner type obstruction for some fully non-linear elliptic equations on $\mathbb{S}^n$ for $n \geq 3$.

Let g_0 be the standard metric on $\mathbb{S}^n$. For $n \geq 3$, let A_g denote the Schouten tensor of a metric g

$$A_g = \frac{1}{n-2}\Big(Ric_g - \frac{R_g}{2(n-1)}g\Big), \tag{1.25}$$

where Ric_g and R_g denote the Ricci tensor and the scalar curvature of g respectively.

For $0 < v \in C^2(\mathbb{S}^n)$, we consider the conformal change of metric $g_1 = v^{\frac{4}{n-2}}g_0$. Then

$$\begin{aligned} A_{g_1} =& -\frac{2}{n-2}v^{-1}\nabla^2_{g_0}v + \frac{2n}{(n-2)^2}v^{-2}\nabla_{g_0}v\otimes\nabla_{g_0}v \\ & - \frac{2}{(n-2)^2}v^{-2}|\nabla_{g_0}v|^2 g_0 + A_{g_0}. \end{aligned} \tag{1.26}$$

We assume that

$$\Gamma \subset \mathbb{R}^n \text{ is an open convex symmetric cone with vertex at the origin} \tag{1.27}$$

such that

$$\Gamma_n \subset \Gamma \subset \Gamma_1, \tag{1.28}$$

with $\Gamma_1 := \{\lambda \in \mathbb{R}^n : \sum_i \lambda_i > 0\}$ and $\Gamma_n := \{\lambda \in \mathbb{R}^n : \lambda_i > 0 \text{ for all } i\}$, where Γ being symmetric means $(\lambda_1, ..., \lambda_n) \in \Gamma$ implies $(\lambda_{i_1}, ..., \lambda_{i_n}) \in \Gamma$ for any permutation $(i_1, ..., i_n)$ of $(1, 2, ..., n)$.

We also assume that f is a function defined on Γ such that

$$f \in C^1(\Gamma) \text{ is symmetric in } \Gamma, \tag{1.29}$$

and

$$f > 0 \text{ and } f_{\lambda_i} > 0 \text{ in } \Gamma. \tag{1.30}$$

Given a positive C^1 function K on $\mathbb{S}^n$, we consider the equation

$$f(\lambda(A_{g_1})) = K, \quad \lambda(A_{g_1}) \in \Gamma \quad \text{and} \quad v > 0 \quad \text{on} \quad \mathbb{S}^n, \tag{1.31}$$

where $g_1 = v^{\frac{4}{n-2}}g_0$ and $\lambda(A_{g_1})$ denotes the eigenvalues of A_{g_1} with respect to g_1.

Theorem 1.9. *For $n \geq 3$, assume that (f, Γ) satisfies conditions* (1.27)-(1.30), *and that K is a positive C^1 function on $\mathbb{S}^n$ such that $\nabla_{\frac{\partial}{\partial\theta_{n+1}}}K \geq 0$ and is not identically zero on $\mathbb{S}^n \setminus \{\mathbf{n}, \mathbf{s}\}$. Then* (1.31) *has no $C^2(\mathbb{S}^n)$ solutions.*

Remark 1.7. The non-existence of $C^2(\mathbb{S}^n)$ solution of (1.31) under the assumption on K, or some similar ones, was known for $(f,\Gamma) = (\sigma_k^{\frac{1}{k}}, \Gamma_k)$, where

$$\sigma_k(\lambda) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k},$$

and $\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_1(\lambda) > 0, \cdots, \sigma_k(\lambda) > 0\}$. For details, see the work of Viaclovsky [29], Han [13] and Delanoë [8]. Our method of proof is completely different from theirs.

We can also consider the following equation on the half sphere $\mathbb{S}^n_+ := \{\theta \in \mathbb{S}^n : \theta_1 \ge 0\}$ for $n \ge 3$:

$$\begin{cases} f\Big(\lambda(A_{v^{\frac{4}{n-2}}g_0})\Big) = K(\theta), \ \ \lambda(A_{v^{\frac{4}{n-2}}g_0}) \in \Gamma \ \text{ and } \ v > 0 \ \text{ on } \mathbb{S}^n_+ \\ \frac{\partial v}{\partial \nu} = H(\theta) v^{\frac{n}{n-2}} \quad \text{on } \partial\mathbb{S}^n_+, \end{cases} \tag{1.32}$$

where ν denotes the unit outer normal of $\partial\mathbb{S}^n_+$.

Theorem 1.10. *For $n \ge 3$, assume that (f,Γ) satisfies (1.27)-(1.30). Assume also that $K > 0$, $K \in C^1(\mathbb{S}^n_+)$ and $H \in C^1(\partial\mathbb{S}^n_+)$ such that $\nabla_{\frac{\partial}{\partial\theta_{n+1}}} K \ge 0$ on $\mathbb{S}^n_+$, $\nabla_{\frac{\partial}{\partial\theta_{n+1}}} H \ge 0$ on $\partial\mathbb{S}^n_+$ and at least one of these two inequalities is strict somewhere. Then (1.32) has no positive $C^2(\mathbb{S}^n_+)$ solutions.*

Remark 1.8. In the case that $f(\lambda) = \lambda_1 + \cdots + \lambda_n$, the non-existence of $C^2(\mathbb{S}^n_+)$ solution was known, see Bianchi and Pan [2]. Our proof, similar to that of Theorems 1.8–1.9, is completely different.

The paper is organized as follows. In section 2, we prove Theorem 1.1 and Theorem 1.3. In section 3, we use moving sphere procedures on the sphere to show the other theorems and their corollaries.

2. Some results on Véron's question

In this section, we give the proof of Theorem 1.1. The radial symmetry of solutions of (1.1) will be proved in subsection 2.1 by using the method of moving spheres. The existence of non-radial solutions of (1.1) follows from Theorem 1.3 whose proof is based on a global bifurcation analysis and will be provided in subsection 2.2.

2.1. **Proof of Theorem 1.1.** We first state a calculus lemma due to [21], which gives the symmetric property of a function through the investigation of its Kelvin transforms.

Lemma 2.1. *If $u \in C^1(\mathbb{R}^n \backslash \{0\})$ is a function such that for each $y \neq 0$ there holds*

$$u_{y,\lambda}(x) \le u(x) \quad \forall 0 < \lambda < |y| \text{ and } |x-y| \ge \lambda \text{ with } x \neq 0, \tag{2.1}$$

then u must be radially symmetric about the origin, and $u'(r) \le 0$ for $0 < r < \infty$.

Proof. We include here the proof for completeness. For any $x \in \mathbb{R}^n \backslash \{0\}$ and any number $a > 0$, let $\mathbf{e}$ be any unit vector in $\mathbb{R}^n$ such that $\langle x - a\mathbf{e}, \mathbf{e}\rangle < 0$. For any number $\tau > a$, if we set $\lambda = \tau - a$ and $y = \tau\mathbf{e}$, then $0 < \lambda < |y|$ and $|x - y| > \lambda$. So we may apply (2.1) to get

$$u(x) \ge \Big(\frac{\tau - a}{|x - \tau\mathbf{e}|}\Big)^{n-2} u\Big(\tau\mathbf{e} + \frac{(\tau-a)^2(x-\tau\mathbf{e})}{|x-\tau\mathbf{e}|^2}\Big). \tag{2.2}$$

It is easy to check that

$$\tau\mathbf{e} + \frac{(\tau-a)^2(x-\tau\mathbf{e})}{|x-\tau\mathbf{e}|^2} \to x - 2\Big(\langle x, \mathbf{e}\rangle - a\Big)\mathbf{e} \quad \text{as } \tau \to \infty.$$

Since $\langle x - a\mathbf{e}, \mathbf{e}\rangle < 0$, we must have $x - 2\big(\langle x, \mathbf{e}\rangle - a\big)\mathbf{e} \neq 0$. Therefore, by sending $\tau \to \infty$ in (2.2) and using the continuity of u in $\mathbb{R}^n \backslash \{0\}$, we obtain

$$u(x) \ge u\Big(x - 2\big(\langle x, \mathbf{e}\rangle - a\big)\mathbf{e}\Big).$$

This immediately implies that u is radially symmetric about the origin and $u'(r) \le 0$ for $0 < r < \infty$. $\square$

Instead of proving the symmetric property about solutions of (1.1) directly, we consider the following more general equation

$$\Delta u + f(x, u) = 0 \quad \text{and} \quad u > 0 \text{ in } \mathbb{R}^n \backslash \{0\}, \tag{2.3}$$

where $n \ge 3$, Δ denotes the Laplace operator on $\mathbb{R}^n$, and $f(x,t) : \mathbb{R}^n \backslash \{0\} \times [0, \infty) \to [0, \infty)$ is a continuous function satisfying the property:

(F) for any $x \neq 0$, $0 < \lambda < |x|$, $|z| > \lambda$ and $a \le b$, there holds

$$\Big(\frac{\lambda}{|z|}\Big)^{n+2} f\Big(x + \frac{\lambda^2 z}{|z|^2}, \Big(\frac{|z|}{\lambda}\Big)^{n-2} a\Big) < f(x+z, b).$$

We have the following symmetry result for solutions of (2.3).

Proposition 2.1. *Let $f(x,t) : \mathbb{R}^n\backslash\{0\} \times [0,\infty) \to [0,\infty)$ be a continuous function satisfying (F). If $u \in C^2(\mathbb{R}^n\backslash\{0\})$ is a solution of (2.3), then u must be radially symmetric about the origin and $u'(r) < 0$ for all $0 < r < \infty$.*

Proof. The proof is based on the method of moving spheres. From (2.3) it follows that $\Delta u \le 0$ and $u > 0$ in $\mathbb{R}^n\backslash\{0\}$. So, by the maximum principle, we have

$$\liminf_{|x|\to 0} u(x) > 0 \quad \text{and} \quad \liminf_{|x|\to\infty} |x|^{n-2}u(x) > 0. \tag{2.4}$$

One can follow the proof of [22, Lemma 2.1] to conclude that for each $y \neq 0$ there exists $\lambda(y) > 0$ such that

$$u_{y,\lambda}(x) \le u(x), \quad \forall 0 < \lambda < \lambda(y) \text{ and } |x-y| \ge \lambda \text{ with } x \neq 0.$$

Define

$$\bar\lambda(y) := \{0 < \mu \le |y| : u_{y,\lambda}(x) \le u(x), \forall 0 < \lambda < \mu,\, |x-y| > \lambda \text{ with } x \neq 0\}.$$

Then $\bar\lambda(y) > 0$. By using Lemma 2.1, it suffices to show that

$$\bar\lambda(y) = |y|, \quad \forall y \in \mathbb{R}^n\backslash\{0\}. \tag{2.5}$$

Suppose that (2.5) is not true, then there exists $y_0 \neq 0$ such that $\bar\lambda(y_0) < |y_0|$. Let $\lambda_0 := \bar\lambda(y_0)$, then from the definition of λ_0 we have

$$u_{y_0,\lambda_0}(x) \le u(x) \qquad \forall |x-y_0| > \lambda_0 \text{ with } x \neq 0. \tag{2.6}$$

A straightforward calculation shows that

$$\Delta u_{y_0,\lambda_0}(x) = -\Big(\frac{\lambda_0}{|x-y_0|}\Big)^{n+2} f\Big(y_0 + \frac{\lambda_0^2(x-y_0)}{|x-y_0|^2}, u(y_0 + \frac{\lambda_0^2(x-y_0)}{|x-y_0|^2})\Big).$$

Therefore, by using (F) and (2.6), we have for $|x-y_0| > \lambda_0$ with $x \neq 0$ that

$$\begin{aligned} -\Delta u_{y_0,\lambda_0}(x) &= \Big(\frac{\lambda_0}{|x-y_0|}\Big)^{n+2} f\Big(y_0 + \frac{\lambda_0^2(x-y_0)}{|x-y_0|^2}, \Big(\frac{|x-y_0|}{\lambda_0}\Big)^{n-2} u_{y_0,\lambda_0}(x)\Big) \\ &< f(x,u(x)) = -\Delta u(x). \end{aligned}$$

This, together with the strong maximum principle and the Hopf lemma, gives

$$u(x) - u_{y_0,\lambda_0}(x) > 0, \quad \forall |x-y_0| > \lambda_0 \text{ with } x \neq 0, \tag{2.7}$$

$$\liminf_{|x|\to 0}(u(x) - u_{y_0,\lambda_0}(x)) > 0 \quad \text{and} \quad \liminf_{|x|\to\infty} |x|^{2-n}(u(x) - u_{y_0,\lambda_0}(x)) > 0, \tag{2.8}$$

and

$$\frac{d}{dr}(u - u_{y_0,\lambda_0})\Big|_{\partial B_{\lambda_0}(y_0)} > 0. \tag{2.9}$$

Properties (2.7)-(2.9) lead to, as in section 2 of [22], a contradiction to the definition of $\bar{\lambda}(y_0)$. For the reader's convenience, we include a proof. From (2.9) it follows that there exists R_0 satisfying $\lambda_0 < R_0 < |y_0|$ such that

$$\frac{d}{dr}(u - u_{y_0,\lambda})(x) > 0 \quad \text{for } \lambda_0 \le \lambda \le R_0 \text{ and } \lambda \le |x - y_0| \le R_0.$$

Since $u - u_{y_0,\lambda} = 0$ on $\partial B_\lambda(y_0)$, we have

$$u(x) - u_{y_0,\lambda}(x) > 0 \quad \text{for } \lambda_0 \le \lambda < R_0 \text{ and } \lambda < |x - y_0| \le R_0. \tag{2.10}$$

From (2.8) one can find $c > 0$, $R_1 > |y_0| > R_0$ and $\eta > 0$ such that

$$u(x) - u_{y_0,\lambda_0}(x) \ge \begin{cases} c|x - y_0|^{2-n} & \text{if } |x - y_0| \ge R_1 \\ c & \text{if } 0 < |x| < \eta. \end{cases}$$

But it is easy to see that there exists $\varepsilon_1 > 0$ such that, if $\lambda_0 \le \lambda \le \lambda_0 + \varepsilon_1$, then

$$|u_{y_0,\lambda}(x) - u_{y_0,\lambda_0}(x)| \le \begin{cases} \frac{1}{2}c|x - y_0|^{2-n} & \text{if } |x - y_0| \ge R_1 \\ \frac{1}{2}c & \text{if } 0 < |x| \le \eta. \end{cases}$$

Therefore, for $\lambda_0 \le \lambda \le \lambda_0 + \varepsilon_1$ there holds

$$u(x) - u_{y_0,\lambda}(x) > 0 \quad \text{if } |x - y_0| \ge R_1 \text{ or } 0 < |x| \le \eta. \tag{2.11}$$

Finally, by continuity, (2.7) implies that there exists $\varepsilon_2 > 0$ such that

$$u(x) - u_{y_0,\lambda}(x) > 0 \quad \text{if } \lambda_0 \le \lambda \le \lambda_0 + \varepsilon_2,\ R_0 \le |x - y_0| \le R_1 \text{ and } |x| \ge \eta. \tag{2.12}$$

Combining (2.10), (2.11) and (2.12) we have for some $\varepsilon > 0$ that

$$u(x) - u_{y_0,\lambda}(x) > 0 \quad \text{if } \lambda_0 \le \lambda \le \lambda_0 + \varepsilon \text{ and } |x - y_0| \ge \lambda \text{ with } x \ne 0.$$

This gives a contradiction to the definition of λ_0. We thus obtain (2.5). □

Remark 2.1. The condition (F) imposed on $f(x,u)$ arises mainly because our argument involves the Kelvin transform. When $f(x,u) = a(x)u + u^{\frac{n+2}{n-2}}$ for some continuous function $a : \mathbb{R}^n \setminus \{0\} \to [0,\infty)$, the condition (F) can be verified easily if a satisfies the property:

(A) for each $x \ne 0$ there holds

$$\Big(\frac{\lambda}{|z|}\Big)^4 a\Big(x + \frac{\lambda^2 z}{|z|^2}\Big) < a(x+z), \quad \forall 0 < \lambda < |x| \text{ and } |z| > \lambda.$$

When $f(x,u) = b(x)u^p$ for some continuous function $b : \mathbb{R}^n\backslash\{0\} \to [0,\infty)$ and some number $p > 0$, the condition (F) can be verified easily if b has the property:

(B) for each $x \neq 0$ there holds

$$\Big(\frac{\lambda}{|z|}\Big)^{n+2-p(n-2)} b\Big(x + \frac{\lambda^2 z}{|z|^2}\Big) < b(x+z), \quad \forall 0 < \lambda < |x| \text{ and } |z| > \lambda.$$

Remark 2.2. Note that the Matukuma equation (1.11) takes the form (2.3) with $f(x,u) = b(x)u^p$ and $b(x) = \frac{1}{1+|x|^2}$. It is easy to check that b satisfies the condition (B) for any $p \leq \frac{n}{n-2}$. Therefore, Proposition 2.1 applies to conclude that the Matukuma equation (1.11) only has radially symmetric positive smooth solution in $\mathbb{R}^n$ for $n \geq 3$ when $p \leq \frac{n}{n-2}$. Moreover, we even allow the solution to have singularities at the origin.

Now, we are ready to give the proof of Theorem 1.1 by assuming Theorem 1.3.

Proof of Theorem 1.1. We first show part (ii) by using Proposition 2.1. Let $\varphi(x) = c/|x|^2$, then it suffices to verify (A) for φ. This is equivalent to showing that

$$\lambda^4 |x+z|^2 < |z|^4 \Big| x + \frac{\lambda^2 z}{|z|^2}\Big|^2, \quad \forall 0 < \lambda < |x| \text{ and } |z| > \lambda.$$

It can be confirmed by the following computation:

$$\begin{aligned}
\lambda^4 |x+z|^2 - |z|^4 \Big| x + \frac{\lambda^2 z}{|z|^2}\Big|^2 &= \Big(\lambda^2 - |z|^2\Big)\left\{(\lambda^2 + |z|^2)|x|^2 + 2\lambda^2 \langle x, z\rangle\right\} \\
&\leq (\lambda^2 - |z|^2)\left\{(\lambda^2 + |z|^2)|x|^2 - \lambda^2(|x|^2 + |z|^2)\right\} \\
&= (\lambda^2 - |z|^2)(|x|^2 - \lambda^2)|z|^2 < 0.
\end{aligned}$$

Thus, Proposition 2.1 applies to conclude that u must be radially symmetric about the origin, and consequently u satisfies the ordinary differential equation

$$u'' + \frac{n-1}{r}u' + \frac{c}{r^2}u + u^{(n+2)/(n-2)} = 0, \quad u > 0 \quad \text{and} \quad u' < 0 \text{ for } r \in (0,\infty).$$

Then

$$(r^{n-1}u')' < -r^{n-1}u^{(n+2)/(n-2)}.$$

Therefore, for any $r > \varepsilon > 0$ we have

$$r^{n-1}u'(r) - \varepsilon^{n-1}u'(\varepsilon) < -\int_\varepsilon^r s^{n-1}u(s)^{(n+2)/(n-2)} ds$$

$$\leq -\frac{1}{n}(r^n - \varepsilon^n)u(r)^{(n+2)/(n-2)}.$$

Since $u'(\varepsilon) \leq 0$ we may drop the second term on the left-hand side and then take $\varepsilon \to 0$ to get

$$r^{n-1}u'(r) \leq -\frac{1}{n}r^n u(r)^{(n+2)/(n-2)} \quad \text{for all } r > 0.$$

This is equivalent to $\frac{d}{dr}\big(u(r)^{-4/(n-2)}\big) \geq \frac{4}{n(n-2)}r$. Hence, for $r \geq \varepsilon > 0$ there holds

$$u(r)^{-4/(n-2)} \geq u(\varepsilon)^{-4/(n-2)} + \frac{2}{n(n-2)}\big(r^2 - \varepsilon^2\big) \geq \frac{2}{n(n-2)}\big(r^2 - \varepsilon^2\big).$$

Letting $\varepsilon \to 0$ we then obtain $u(r) \leq Cr^{-(n-2)/2}$ for all $r > 0$. This gives the desired estimate.

We now use (ii) to show (i). Suppose (1.11) has a solution u for some $c \geq (n-2)^2/4$. We define

$$w(t) := e^{-\frac{n-2}{2}t}u(e^{-t}), \quad t \in (-\infty, \infty).$$

One can verify that w satisfies the ordinary differential equation

$$w'' + \Big(c - \frac{(n-2)^2}{4}\Big)w + w^{(n+2)/(n-2)} = 0 \quad \text{and} \quad w > 0 \text{ on } (-\infty, \infty). \tag{2.13}$$

Thus, w is a positive strictly concave function defined on $(-\infty, \infty)$. However, such a function does not exist.

(iv) From Theorem 1.3 it follows that (1.2) has non-constant solutions when $c < -(n-2)/4$, which give non-radial solutions of (1.1). □

2.2. **Proof of Theorem 1.3: A bifurcation analysis.** We first give a fact concerning solutions of (1.3).

Lemma 2.2. *Suppose* $1 < p < N^*$. *Then for any* $\Lambda > 0$ *there exists a positive constant* $C(N,p,\Lambda)$ *depending only on* N, p *and* Λ,*such that any non-constant solution* v *of* (1.3) *with* $\lambda \leq \Lambda$ *satisfies*

$$1/C(N,p,\Lambda) \leq v \leq C(N,p,\Lambda) \quad on\ \mathbb{S}^n.$$

Proof. Note that (1.3) has no solution for $\lambda \leq 0$. Considering Theorem 1.2, we may assume $N/(p-1) \leq \lambda < \Lambda$. The upper bound can be obtained by using the blow-up technique together with the fact that the equation $-\Delta u = u^p$ in $\mathbb{R}^N$ has no positive solution if $1 < p < N^*$ (see e.g. [27]). In order to get the lower bound, we first use the Harnack inequality to get

$$\max_{\mathbb{S}^N} v \leq C_1(N,p,\Lambda) \min_{\mathbb{S}^n} v,$$

for some positive constant $C_1(N, p, \Lambda)$ depending only on N, p and Λ. So it suffices to derive a lower bound of $\max_{\mathbb{S}^n} v$. Suppose $\max_{\mathbb{S}^n} v = v(x_0)$ for some $x_0 \in \mathbb{S}^N$. Then $-\Delta_{\mathbb{S}^N} v(x_0) \geq 0$ and hence $v(x)^p - \lambda v(x_0) \geq 0$. This implies $v(x_0) \geq \lambda^{1/(p-1)} \geq (N/(p-1))^{1/(p-1)}$. □

Lemma 2.3. *The eigenvalues of $-\Delta_{\mathbb{S}^N}$ restricted to $C_G^{4,\alpha}(\mathbb{S}^N)$ are $\nu_k = k(N+k-1)$ for $k = 0, 1, \cdots$; they are all simple and the eigenspace of ν_k is spanned by a function p_k which can be written as $p_k(\theta) = \tilde{p}_k(\theta_{n+1})$, where $\tilde{p}_k(t)$ is a polynomial of degree k. Moreover, all the zeroes of $\tilde{p}_k(t)$ are simple and in $(-1, 1)$.*

Proof. All the assertions can be found in [1] except the last part. In the following, we will show by induction on k that $\tilde{p}_k$ has exactly k simple zeroes in $(-1, 1)$. This is clear for $k = 0$ since $\tilde{p}_0 = 1$. Now, we assume that $\tilde{p}_k$ has k simple zeroes in $(-1, 1)$, say $-1 < t_1 < t_2 < ... < t_k < 1$. Set $t_0 = -1$ and $t_{k+1} = 1$. It suffices to show that $\tilde{p}_{k+1}$ has a zero in each interval (t_i, t_{i+1}) for $i = 0, \cdots, k$. Suppose for some $0 \leq i \leq k$ the polynomial $\tilde{p}_{k+1}$ has no zeroes in (t_i, t_{i+1}), then both $\tilde{p}_k$ and $\tilde{p}_{k+1}$ do not change sign in this interval. Without loss of generality, we may assume both $\tilde{p}_k$ and $\tilde{p}_{k+1}$ are positive in (t_i, t_{i+1}). Let

$$\Sigma_i := \begin{cases} \{\theta \in \mathbb{S}^N : \theta_{N+1} < t_1\}, & \text{if } i = 0, \\ \{\theta \in \mathbb{S}^N : t_i < \theta_{N+1} < t_{i+1}\}, & \text{if } 1 \leq i \leq k-1, \\ \{\theta \in \mathbb{S}^N : \theta_{N+1} > t_k\}, & \text{if } i = k. \end{cases}$$

Then, both p_k and p_{k+1} are non-negative on the domain Σ_i, $p_k = 0$ and $\frac{\partial p_k}{\partial \nu} \leq 0$ on $\partial \Sigma_i$. Recall that

$$-\Delta_{\mathbb{S}^N} p_k = \nu_k p_k \quad \text{and} \quad -\Delta_{\mathbb{S}^N} p_{k+1} = \nu_{k+1} p_{k+1} \quad \text{on } \mathbb{S}^n.$$

We have

$$\begin{aligned}(\nu_{k+1} - \nu_k) \int_{\Sigma_i} p_k p_{k+1} &= -\int_{\Sigma_i} p_k \Delta_{\mathbb{S}^N} p_{k+1} + \int_{\Sigma_i} p_{k+1} \Delta_{\mathbb{S}^N} p_k \\ &= \int_{\partial \Sigma_i} \Big(p_{k+1} \frac{\partial p_k}{\partial \nu} - p_k \frac{\partial p_{k+1}}{\partial \nu} \Big) \leq 0,\end{aligned}$$

which is a contradiction. □

In order to study the solutions of (1.3), let $v = \lambda^{1/(p-1)}(w+1)$, then w satisfies the equation

$$-\Delta_{\mathbb{S}^N} w = \lambda\big((w+1)^p - w - 1\big) \quad \text{and} \quad w > -1 \quad \text{on } \mathbb{S}^N. \tag{2.14}$$

We are going to find G-invariant non-zero solutions of (2.14) for each $\lambda > N/(p-1)$.

Lemma 2.4. *(a) For any $0 < \lambda \le N/(p-1)$ the only solution of (2.14) is $w = 0$.*

(b) For any $\Lambda > 0$, there exist positive constants $C(N,p,\Lambda)$ and $\varepsilon(N,p,\Lambda)$ depending only on N, p and Λ such that

$$-1 + \varepsilon(N,p,\Lambda) \le w \le C(N,p,\Lambda) \quad \text{on } \mathbb{S}^N,$$

for any solution w of (2.14) with $\lambda \le \Lambda$.

(c) Any non-zero G-invariant solution w of (2.14) neither vanishes at the north pole nor at the south pole on $\mathbb{S}^n$; moreover, by writing $w(\theta) = \tilde{w}(\theta_{n+1})$ for some function $\tilde{w}$ on $[-1,1]$, then all zeroes of $\tilde{w}$ are in $(-1,1)$ and are simple.

Proof. (a) follows from Theorem 1.2 and (b) is the consequence of Lemma 2.2. In the following, we will prove (c). Let us first show that w does not vanish at the north pole. If it vanishes at the north pole, then the strong maximum principle implies that the north pole must be an accumulation point of zeroes of w on $\mathbb{S}^N$. Since w is G-invariant, this would imply that all derivatives of w at the north pole are zero. The unique continuation property then implies $w = 0$ on $\mathbb{S}^N$. The same argument gives $w \ne 0$ at the south pole. Therefore all zeroes of $\tilde{w}$ are in $(-1,1)$. Use (2.14) one can see that $\tilde{w}$ satisfies the ordinary differential equation

$$-(1-t^2)\tilde{w}'' + nt\tilde{w}' = \lambda\big((\tilde{w}+1)^p - \tilde{w} - 1\big) \quad \text{on } [-1,1].$$

Since $\tilde{w} \ne 0$ on $[-1,1]$, this implies that all zeroes of $\tilde{w}$ must be simple. $\square$

Proof of Theorem 1.3. We will use bifurcation theory to carry out the proof. We first formulate (2.14) as an operator equation. Since $\Delta_{\mathbb{S}^N}$ is $O(N+1)$-invariant, it follows from the theory of elliptic equations that $-\Delta_{\mathbb{S}^N} + I : C_G^{4,\alpha}(\mathbb{S}^N) \to C_G^{2,\alpha}(\mathbb{S}^N)$ is invertible. Let T denote its inverse, then $T : C_G^{2,\alpha}(\mathbb{S}^N) \to C_G^{2,\alpha}(\mathbb{S}^N)$ is a compact linear operator. Let

$$\mathcal{D} := \Big\{(w,\mu) \in C_G^{2,\alpha}(\mathbb{S}^N) \times \mathbb{R} : \mu > 1 \text{ and } w > -1 \text{ on } \mathbb{S}^N\Big\},$$

and set

$$\mu = (p-1)\lambda + 1 \quad \text{and} \quad g(w,\mu) := \frac{\mu-1}{p-1} T\Big((w+1)^p - pw - 1\Big).$$

Then, finding a G-invariant nonzero solution of (2.14) is equivalent to finding a non-zero solution of the operator equation

$$f(w,\mu) := w - \mu T w - g(w,\mu) = 0 \quad \text{in } \mathcal{D}. \tag{2.15}$$

It is clear that g is a non-linear compact map of $\mathcal{D}$ into $C_G^{2,\alpha}(\mathbb{S}^N)$, and $g(w,\mu) = o(\|w\|_{C^{2,\alpha}(\mathbb{S}^N)})$ uniformly on bounded μ interval.

Note that $1/\mu$ is an eigenvalue of T if and only if $\mu - 1$ is an eigenvalue of $-\Delta_{\mathbb{S}^n}$ restricted to $C_G^{4,\alpha}(\mathbb{S}^N)$. Therefore, from Lemma 2.3 we can see that all the eigenvalues of T are simple and given by $1/\mu_k$, where $\mu_k = 1 + k(N+k-1)$. It then follows from Krasnoselski's theorem (see [24, Theorem 3.3.1]) that each $(0,\mu_k)$ is a bifurcation point of $f(w,\mu) = 0$ in $\mathcal{D}$.

Let $\mathcal{S}$ denote the closure of nontrivial solutions (w,μ) of $f(w,\mu) = 0$ in $\mathcal{D}$. According to Lemma 2.4 (a), $\mathcal{S} \cap (C_G^{2,\alpha}(\mathbb{S}^N) \times (1,\mu_1)) = \emptyset$. Let $\mathcal{C}_k$ be the connected component of $\mathcal{S}$ containing $(0,\mu_k)$. Then, by the global bifurcation theory of Rabinowitz (see [26, Theorem 1.4] or [24, Theorem 3.4.1]), we know that either (i) $\mathcal{C}_k$ is not compact in $\mathcal{D}$ or (ii) $\mathcal{C}_k$ contains a point $(0,\mu_j)$ with $j \neq k$. We are going to rule out the case (ii).

For each $k \geq 1$ we define $\mathcal{S}_k$ as in (1.4); i.e.,

$$\mathcal{S}_k := \big\{ v \in C_G^{2,\alpha}(\mathbb{S}^N) : \tilde{v} \text{ has exactly } k \text{ zeroes, all of them in } (-1,1) \text{ and simple} \big\}.$$

It is clear that each $\mathcal{S}_k$ is an open set in $C_G^{2,\alpha}(\mathbb{S}^N)$. By the local bifurcation theorem of Crandall and Rabinowitz (see [7, Theorem 1.7]), near each bifurcation point $(0,\mu_k)$, $\mathcal{S}$ has the parameterization $(w_k(s),\mu_k(s))$, $|s| < a_k$ for some small $a_k > 0$, where $\mu_k(0) = \mu_k$, $w_k(s) = sp_k + s\psi_k(s)$ and $\psi_k(0) = 0$. According to Lemma 2.3, $p_k \in \mathcal{S}_k$, thus $w_k(s) \in \mathcal{S}_k$ for small $s \neq 0$. Therefore there exists a neighborhood $\mathcal{O}_k$ of $(0,\mu_k)$ in $\mathcal{D}$ such that if $(w,\mu) \in \mathcal{O}_k \cap \mathcal{S}$ and $w \neq 0$, then $w \in \mathcal{S}_k$. Let $\mathcal{B}_k := \{(w,\mu) \in \mathcal{C}_k : w \in \mathcal{S}_k\} \cup \{(0,\mu_k)\}$. If we can show that $\mathcal{C}_k = \mathcal{B}_k$ for each $k \geq 1$, then $\mathcal{C}_k$ can not contain a point $(0,\mu_j)$ with $j \neq k$, and we therefore rule out the case (ii).

In order to show $\mathcal{C}_k = \mathcal{B}_k$, it suffices to show that $\mathcal{B}_k$ is both open and closed in $\mathcal{C}_k$. It is clear that $\mathcal{B}_k$ is open in $\mathcal{C}_k$. Suppose now that $\{(w^{(l)},\mu^{(l)}\}$ is a sequence in $\mathcal{B}_k$ such that $(w^{(l)},\mu^{(l)}) \to (w,\mu)$ in $\mathcal{C}_k$. Note that (w,μ) is a solution of (2.15). If $w = 0$ on $\mathbb{S}^N$, then $\mu = \mu_j$ for some j. If $j = k$, $(0,\mu_k) \in \mathcal{B}_k$; if $j \neq k$, then $w^{(l)} \in \mathcal{S}_j \cap \mathcal{S}_k$ for large l which is impossible. Thus w is a non-zero solution of (2.14); (c) of Lemma 2.4 then implies $w \in \mathcal{S}_i$ for some i. If $i \neq k$, then the openness of $\mathcal{S}_i$ implies $w^{(l)} \in \mathcal{S}_i \cap \mathcal{S}_k$ for large

l which is again impossible. Hence, $w \in \mathcal{S}_k$ and $(w, \mu) \in \mathcal{B}_k$. Therefore, $\mathcal{B}_k$ is closed in $\mathcal{C}_k$.

The above argument has ruled out case (ii), therefore each $\mathcal{C}_k$ is non-compact in $\mathcal{D}$. Let

$$\Lambda = \sup \Big\{ \lambda > \mu_k : \mathcal{C}_k \cap \big(C_G^{2,\alpha}(\mathbb{S}^N) \times \{\mu\} \big) \neq \emptyset, \forall \mu_k \le \mu < \lambda \Big\}.$$

We will show that $\Lambda = \infty$. If not, say $\Lambda < \infty$. By connectedness of $\mathcal{C}_k$ and (a) of Lemma 2.4, $\mathcal{C}_k \subset \big(C_G^{2,\alpha}(\mathbb{S}^N) \times [\mu_1, \Lambda] \big)$. It follows, using (b) of Lemma 2.4 that $\mathcal{C}_k$ is compact, a contradiction. So $\Lambda = \infty$; i.e.,

$$\mathcal{C}_k \cap \big(C_G^{2,\alpha}(\mathbb{S}^N) \times \{\mu\} \big) \neq \emptyset,$$

for any $\mu > \mu_k$. The proof is thus complete. $\square$

3. Symmetric results on $\mathbb{S}^n$

3.1. **Some preliminary results.** Given a function v on $\mathbb{S}^n$, we will compare it with its Kelvin transform $v_{\mathbf{p},\lambda}$ defined by (1.7) for $\mathbf{p} \in \mathbb{S}^n$ and $0 < \lambda < \pi$. The first result indicates that the comparison is always possible if $\lambda > 0$ is small and v is regular at $\mathbf{p}$.

Lemma 3.1. *Assume that $n \ge 3$ and that Γ is a closed subset of $\mathbb{S}^n$. If $v \in C^1(\mathbb{S}^n \backslash \Gamma)$ and $v \ge c_0$ on $\mathbb{S}^n \backslash \Gamma$ for some constant $c_0 > 0$, then for each $\mathbf{p} \in \mathbb{S}^n \setminus \Gamma$ there exists $0 < \lambda_{\mathbf{p}} < \pi/2$ such that*

$$v_{\mathbf{p},\lambda} \le v \quad \text{on } \Sigma_{\mathbf{p},\lambda} \setminus \Gamma \text{ for each } 0 < \lambda < \lambda_{\mathbf{p}}.$$

Proof. Since $\mathbf{p} \notin \Gamma$ and Γ is closed in $\mathbb{S}^n$, there exists $0 < \lambda_0 < \frac{\pi}{2}$ such that $\Gamma \subset \Sigma_{\mathbf{p},\lambda_0}$. Then for $0 < r < \lambda_0$ we have

$$\begin{aligned}
&\frac{\partial}{\partial r} \Big\{ (1 + \cos^2 \lambda - 2 \cos \lambda \cos r)^{\frac{n-2}{4}} v(r, \omega) \Big\} \\
=&(1 + \cos^2 \lambda - 2 \cos \lambda \cos r)^{\frac{n-2}{4}} \Big[v_r + \frac{n-2}{2} \frac{\cos \lambda \sin r}{1 + \cos^2 \lambda - 2 \cos \lambda \cos r} v \Big].
\end{aligned}$$

Noting that $\sup_{B_{\lambda_0}(\mathbf{p})} |\nabla_{\mathbb{S}^n} v|$ is finite and $v \ge c_0$ on $\mathbb{S}^n \setminus \Gamma$, there exists $0 < \lambda_1 < \lambda_0$ such that for $0 < \lambda < r < \lambda_1$

$$\frac{\partial}{\partial r} \Big\{ (1 + \cos^2 \lambda - 2 \cos \lambda \cos r)^{\frac{n-2}{4}} v(r, \omega) \Big\} > 0.$$

This implies that for $0 < \lambda < r < \lambda_1$

$$\begin{aligned}
&(1 + \cos^2 \lambda - 2 \cos \lambda \cos h_\lambda(r))^{\frac{n-2}{4}} v(h_\lambda(r), \omega) \\
&\le (1 + \cos^2 \lambda - 2 \cos \lambda \cos r)^{\frac{n-2}{4}} v(r, \omega).
\end{aligned}$$

Since

$$1+\cos^2\lambda-2\cos\lambda\cos h_\lambda(r)=\frac{\sin^4\lambda}{1+\cos^2\lambda-2\cos\lambda\cos r},$$

we therefore conclude that

$$v_{\mathbf{p},\lambda}(r,\omega)\le v(r,\omega)\quad\text{if } 0<\lambda<r<\lambda_1 \text{ and } \omega\in\mathbb{S}^{n-1}. \tag{3.1}$$

Next, we can find a constant C_1 such that for $(r,\omega)\in\Sigma_{\mathbf{p},\lambda_1}\setminus\Gamma$

$$\begin{aligned}\frac{v_{\mathbf{p},\lambda}(r,\omega)}{v(r,\omega)}=&\Big(\frac{\sin^2\lambda}{1+\cos^2\lambda-2\cos\lambda\cos r}\Big)^{\frac{n-2}{2}}\frac{v(h_\lambda(r),\omega)}{v(r,\omega)}\\ \le& C_1\Big(\frac{\sin^2\lambda}{1+\cos^2\lambda-2\cos\lambda\cos r}\Big)^{\frac{n-2}{2}},\end{aligned}$$

where we used the facts that $v\ge c_0>0$ and $v\circ\varphi_{\mathbf{p},\lambda}$ is bounded on $\Sigma_{\mathbf{p},\lambda_1}$. So there exists $0<\lambda_2<\lambda_1$ such that

$$v_{\mathbf{p},\lambda}\le v\quad\text{on } \Sigma_{\mathbf{p},\lambda_1}\setminus\Gamma \text{ for each } 0<\lambda<\lambda_2.$$

Combining this with (3.1) we thus complete the proof. □

Lemma 3.2. *Let $\mathcal{O}$ be a domain in $\mathbb{S}^n$ and $\mathbf{q}\in\overline{\mathcal{O}}$. If $v\in C^2(\mathcal{O}\setminus\{\mathbf{q}\})\cap C^0(\overline{\mathcal{O}}\setminus\{\mathbf{q}\})$ is a non-negative function such that $-\mathcal{L}_{\mathbb{S}^n}v+Cv\ge 0$ in $\mathcal{O}\setminus\{\mathbf{q}\}$ for some non-negative constant C and $v\ge c_0$ on $\partial\mathcal{O}\setminus\{\mathbf{q}\}$ for some constant $c_0>0$, then $v>c_1$ on $\overline{\mathcal{O}}\setminus\{\mathbf{q}\}$ for some constant $c_1>0$.*

Proof. Using the stereographic projection with respect to $\mathbf{q}$, the conclusion is a consequence of Lemma 4.1. □

Fix a point $\mathbf{p}\in\mathbb{S}^n$ and let $\Gamma\subset\mathbb{S}^n\setminus B_{\pi/2}(\mathbf{p})$ be a set consisting of discrete points. Let $g:(\mathbb{S}^n\setminus\Gamma)\times(0,\infty)\to\mathbb{R}$ be a continuous function. We consider the equation

$$-\mathcal{L}_{\mathbb{S}^n}v=g(\theta,v)\quad\text{and}\quad v>0\quad\text{on } \mathbb{S}^n\setminus\Gamma. \tag{3.2}$$

If $v\in C^2(\mathbb{S}^n\setminus\Gamma)$ is a solution of (3.2), we define $0<\bar\lambda_{\mathbf{p}}\le\pi$ by

$$\bar\lambda_{\mathbf{p}}:=\sup\{\lambda\in(0,\pi]: v_{\mathbf{p},\mu}\le v \text{ in } \Sigma_{\mathbf{p},\mu}\setminus\Gamma \text{ for each } 0<\mu<\lambda\}.$$

Since Γ is discrete, if we assume $\inf_{\mathbb{S}^n\setminus\Gamma}v>0$, then, by using Lemma 3.1, $\bar\lambda_{\mathbf{p}}$ is well defined. The next result shows that $\lambda_{\mathbf{p}}\ge\pi/2$ if g satisfies (g3) and the following conditions:

(g7)$_{\mathbf{p}}$ For each $s>0$, $0<\lambda<\pi/2$ and $\theta\in\Sigma_{\mathbf{p},\lambda}$, $g(\theta,s)\ge g(\varphi_{\mathbf{p},\lambda}(\theta),s)$.

(g8)$_{\mathbf{p}}$ Either $g(\theta,s)>g(\varphi_{\mathbf{p},\lambda}(\theta),s)$ for any $s>0$, $0<\lambda<\pi/2$ and $\theta\in\Sigma_{\mathbf{p},\lambda}\setminus\Gamma$, or for $0<\lambda<\pi/2$ and $\theta\in\Sigma_{\mathbf{p},\lambda}$, the function $s\to s^{-(n+2)/(n-2)}g(\theta,s)$ is strictly decreasing.

Lemma 3.3. *For* $\mathbf{p} \in \mathbb{S}^n$*, let* $\Gamma \subset \mathbb{S}^n \setminus B_{\pi/2}(\mathbf{p})$ *be a discrete set. Assume* g *is continuous on* $(\mathbb{S}^n \setminus \Gamma) \times (0, \infty)$ *and* $g(\cdot, s)$ *is bounded below on* $\mathbb{S}^n \setminus \Gamma$ *for each* $s \in (0, \infty)$*. Assume also that* g *satisfies (g3),* $(g7)_{\mathbf{p}}$ *and* $(g8)_{\mathbf{p}}$*. If* $v \in C^2(\mathbb{S}^n \setminus \Gamma)$ *is a solution of* (3.2) *with* $\inf_{\mathbb{S}^n \setminus \Gamma} v > 0$*, then* $\bar{\lambda}_{\mathbf{p}} \geq \pi/2$*.*

Proof. In the following we will use the abbreviations

$$\varphi_\lambda := \varphi_{\mathbf{p},\lambda},\ v_\lambda := v_{\mathbf{p},\lambda},\ B_\lambda := B_\lambda(\mathbf{p}),\ \Sigma_\lambda := \Sigma_{\mathbf{p},\lambda},\ \bar{\lambda} := \bar{\lambda}_{\mathbf{p}}.$$

By Lemma 3.1, $\bar{\lambda}$ is well defined. We argue by contradiction and assume $\bar{\lambda} < \pi/2$. From the definition of $\bar{\lambda}$ we have

$$v_\lambda \leq v \quad \text{ in } \Sigma_\lambda \setminus \Gamma \text{ for each } 0 < \lambda \leq \bar{\lambda}. \tag{3.3}$$

Moreover, from (3.2), (1.7) and (1.8) it follows that on $\Sigma_{\bar{\lambda}} \setminus \Gamma$ there hold

$$-v^{-\frac{n+2}{n-2}} \mathcal{L}_{\mathbb{S}^n} v = v^{-\frac{n+2}{n-2}} g(\theta, v), \tag{3.4}$$

and

$$-v_{\bar{\lambda}}^{-\frac{n+2}{n-2}} \mathcal{L}_{\mathbb{S}^n} v_{\bar{\lambda}} = (v \circ \varphi_{\bar{\lambda}})^{-\frac{n+2}{n-2}} g(\varphi_{\bar{\lambda}}(\theta), v \circ \varphi_{\bar{\lambda}}). \tag{3.5}$$

We define

$$\mathcal{O} := \left\{\theta \in \Sigma_{\bar{\lambda}} \setminus \Gamma : v(\theta) < v \circ \varphi_{\bar{\lambda}}(\theta)\right\}.$$

Let $v_s = sv + (1-s)v_{\bar{\lambda}}$, for $0 \leq s \leq 1$. Using the technique developed in the proof of [22, Lemma 2.2], it follows from $(g7)_{\mathbf{p}}$, (g3), (3.4) and (3.5) that

$$\begin{aligned}
0 &\geq -(v \circ \varphi_{\bar{\lambda}})^{-\frac{n+2}{n-2}} g(\theta, (v \circ \varphi_{\bar{\lambda}})) + (v \circ \varphi_{\bar{\lambda}})^{-\frac{n+2}{n-2}} g(\varphi_{\bar{\lambda}}(\theta), (v \circ \varphi_{\bar{\lambda}}))\\
&\geq -v^{-\frac{n+2}{n-2}} g(\theta, v) + (v \circ \varphi_{\bar{\lambda}})^{-\frac{n+2}{n-2}} g(\varphi_{\bar{\lambda}}(\theta), v \circ \varphi_{\bar{\lambda}})\\
&= \int_0^1 \frac{d}{ds}\Big(v_s^{-\frac{n+2}{n-2}} \mathcal{L}_{\mathbb{S}^n} v_s\Big) ds \\
&= \Big(\int_0^1 v_s^{-\frac{n+2}{n-2}} ds\Big) \mathcal{L}_{\mathbb{S}^n}(v - v_{\bar{\lambda}}) - \frac{n+2}{n-2}\Big(\int_0^1 v_s^{-\frac{2n}{n-2}} \mathcal{L}_{\mathbb{S}^n} v_s ds\Big)(v - v_{\bar{\lambda}})
\end{aligned} \tag{3.6}$$

in $\mathcal{O}$. We first claim that

$$v - v_{\bar{\lambda}} > 0 \quad \text{in } \mathcal{O}. \tag{3.7}$$

In fact, by (3.3), (3.6) and the strong maximum principle, either (3.7) holds or $v - v_{\bar{\lambda}} \equiv 0$ in $\mathcal{O}$. If $\mathcal{O} \neq \Sigma_{\bar{\lambda}} \setminus \Gamma$, (3.7) is true since $\partial\mathcal{O} \cap (\Sigma_{\bar{\lambda}} \setminus \Gamma) \neq \emptyset$ and $v - v_{\bar{\lambda}} > 0$ on $\partial\mathcal{O} \cap (\Sigma_{\bar{\lambda}} \setminus \Gamma)$. So we may assume $\mathcal{O} = \Sigma_{\bar{\lambda}} \setminus \Gamma$. If $v = v_{\bar{\lambda}}$ in $\Sigma_{\bar{\lambda}} \setminus \Gamma$, then by (3.6) we must have

$$v^{-(n+2)/(n-2)} g(\theta, v) \equiv (v \circ \varphi_{\bar{\lambda}})^{-(n+2)/(n-2)} g(\varphi_{\bar{\lambda}}(\theta), v \circ \varphi_{\bar{\lambda}}) \quad \text{in } \Sigma_{\bar{\lambda}} \setminus \Gamma.$$

But $(g8)_{\mathbf{p}}$ implies that this can not happen. We therefore obtain (3.7).

We next claim that

$$\frac{\partial}{\partial r}(v - v_{\bar\lambda})\Big|_{\partial B_{\bar\lambda}} > 0. \tag{3.8}$$

From (3.3) we know the left-hand side of (3.8) is always non-negative. Suppose (3.8) is not true; then there exists $\mathbf{p}_0 \in \partial B_{\bar\lambda}$ such that $\frac{\partial}{\partial r}(v - v_{\bar\lambda})(\mathbf{p}_0) = 0$. However, direct calculation shows that

$$\frac{\partial}{\partial r}(v \circ \varphi_{\bar\lambda} - v_{\bar\lambda}) = (n-2)(\cot \bar\lambda) v(\mathbf{p}_0).$$

Since $0 < \bar\lambda < \pi/2$, we then have

$$\frac{\partial}{\partial r}(v - v \circ \varphi_{\bar\lambda})(\mathbf{p}_0) < 0.$$

Since $v = v \circ \varphi_{\bar\lambda}$ on $\partial B_{\bar\lambda}$, we conclude that $\mathcal{U}_{\mathbf{p}_0} \cap \Sigma_{\bar\lambda} \subset \mathcal{O}$ for some neighborhood $\mathcal{U}_{\mathbf{p}_0}$ of $\mathbf{p}_0$ in $\mathbb{S}^n$. Thus, from (3.6) and the Hopf lemma it follows that $\frac{\partial}{\partial r}(v - v_{\bar\lambda})(\mathbf{p}_0) > 0$ which is a contradiction. We therefore obtain (3.8).

Noting that v and $v_{\bar\lambda}$ are C^2 near $\partial \Sigma_{\bar\lambda}$, by using (3.8) it is easy to find $\bar\lambda < \lambda_0 < \pi/2$ such that

$$v_\lambda < v \quad \text{on } B_{\lambda_0} \setminus \overline{B}_\lambda \text{ for } \bar\lambda \le \lambda < \lambda_0. \tag{3.9}$$

We still need to consider the points in $\Sigma_{\lambda_0} \setminus \Gamma$. By using (1.7) and the definition of $\mathcal{O}$, it is easy to see that there is a positive constant α_0 such that

$$v - v_{\bar\lambda} \ge \alpha_0 \quad \text{on } \Sigma_{\lambda_0} \setminus (\mathcal{O} \cup \Gamma). \tag{3.10}$$

This together with (3.7) implies that

$$v - v_{\bar\lambda} \ge c_0 > 0 \quad \text{on } \partial(\mathcal{O} \cap \Sigma_{\lambda_0}) \setminus \Gamma, \tag{3.11}$$

for some constant $c_0 > 0$.

We observe that since v is C^2 on the compact set $\mathbb{S}^n \setminus \Sigma_{\lambda_0}$,

$$v_{\bar\lambda} \le v \le v \circ \varphi_{\bar\lambda} \le C \quad \text{in } \mathcal{O} \cap \Sigma_{\lambda_0}.$$

Since $g(\cdot, C)$ is bounded below in $\mathbb{S}^n \setminus \Gamma$, it follows from (g3) and (3.2) that

$$-\mathcal{L}_{\mathbb{S}^n} v = g(\theta, v) \ge C^{-\frac{n+2}{n-2}} g(\theta, C) v^{\frac{n+2}{n-2}} \ge -Cv \quad \text{in } \mathcal{O} \cap \Sigma_{\lambda_0}.$$

It is easy to see from (3.5) that

$$-\mathcal{L}_{\mathbb{S}^n} v_{\bar\lambda} \ge -C v_{\bar\lambda} \quad \text{in } \mathcal{O} \cap \Sigma_{\lambda_0}.$$

Noting that v and $v_{\bar\lambda}$ have positive lower and upper bounds in $\mathcal{O} \cap \Sigma_{\lambda_0}$, we can use the condition $\inf_{\mathbb{S}^n \setminus \Gamma} v > 0$ and (3.6) to obtain, for some positive constant C,

$$-\mathcal{L}_{\mathbb{S}^n}(v - v_{\bar\lambda}) + C(v - v_{\bar\lambda}) \ge 0, \quad \text{in } \mathcal{O} \cap \Sigma_{\lambda_0}.$$

Since Γ is discrete, by using (3.3), (3.11) and Lemma 3.2 we have

$$\inf_{\mathcal{O}\cap\Sigma_{\lambda_0}} \left(v - v_{\bar\lambda}\right) > 0.$$

This together with (3.10) implies that for some constant $c_1 > 0$,

$$v - v_{\bar\lambda} \geq c_1 \quad \text{on } \Sigma_{\lambda_0} \setminus \Gamma.$$

Using (1.7) then we can find $0 < \varepsilon < \lambda_0 - \bar\lambda$ such that

$$v_\lambda < v \quad \text{on } \Sigma_{\lambda_0} \setminus \Gamma \text{ for each } \bar\lambda \leq \lambda < \bar\lambda + \epsilon. \tag{3.12}$$

Combining (3.3), (3.9) and (3.12) we have

$$v_\lambda \leq v \text{ in } \Sigma_\lambda \setminus \Gamma \text{ for each } 0 < \lambda < \bar\lambda + \varepsilon.$$

This contradicts the definition of $\bar\lambda$. Hence, $\bar\lambda \geq \pi/2$. $\square$

3.2. **Proof of Theorem 1.4.** Now we are ready to give the proof of Theorem 1.4. Under conditions (g1) and (g2), one can see that $(g7)_{\mathbf{p}}$ and $(g8)_{\mathbf{p}}$ are satisfied with $\mathbf{p} = \mathbf{s}$. So we may apply Lemma 3.3 to conclude that

$$v \geq v_{\mathbf{s},\pi/2} \quad \text{on } \Sigma_{\mathbf{s},\pi/2} \setminus \{\mathbf{n}\}.$$

Noting that $\varphi_{\mathbf{s},\pi/2}$ is a mirror reflection and $|J_{\varphi_{\mathbf{s},\pi/2}}| = 1$, we have, from (3.2), (g1), (g2) and (g4) that

$$-\mathcal{L}_{\mathbb{S}^n}(v - v_{\mathbf{s},\pi/2}) = g(\theta, v) - g(\varphi_{\mathbf{s},\pi/2}(\theta), v_{\mathbf{s},\pi/2}) > 0 \quad \text{in } \Sigma_{\mathbf{s},\pi/2} \setminus \{\mathbf{n}\}. \tag{3.13}$$

Therefore, by the strong maximum principle and the Hopf lemma we have

$$v > v_{\mathbf{s},\pi/2} \quad \text{in } \Sigma_{\mathbf{s},\pi/2} \setminus \{\mathbf{n}\}, \tag{3.14}$$

and

$$\frac{\partial}{\partial r}(v - v_{\mathbf{s},\pi/2})\Big|_{\partial B_{\pi/2}(\mathbf{s})} > 0. \tag{3.15}$$

By Lemma 3.2,

$$\inf_{\theta\to\mathbf{n}} (v(\theta) - v_{\mathbf{s},\pi/2}(\theta)) > c, \tag{3.16}$$

for some constant $c > 0$.

For any $\mathbf{q} \in \partial B_{\pi/2}(\mathbf{s})$, $\mathbf{q} = (\pi/2, \omega_0)$ in the geodesic polar coordinate with respect to $\mathbf{s}$. Let $\mathbf{p}_t = (t\pi/2, \omega_0)$ for $0 \leq t \leq 2$.

Claim 1. There exists $\varepsilon > 0$, such that for any $0 \leq t < \varepsilon$, there holds

$$v \geq v_{\mathbf{p}_t,\pi/2} \quad \text{in } \Sigma_{\mathbf{p}_t,\pi/2} \setminus \{\mathbf{n}\}. \tag{3.17}$$

The claim follows easily from (3.14), (3.15) and (3.16). For the readers' convenience, we include a proof by contradiction argument. If it is not true, then there exists a sequence $0 < t_i \to 0$ and a sequence $\{\theta_i\}$ with $\theta_i \in \Sigma_{\mathbf{p}_{t_i},\pi/2} \setminus \{\mathbf{n}\}$ such that

$$v(\theta_i) < v_{\mathbf{p}_{t_i},\pi/2}(\theta_i). \tag{3.18}$$

By taking a subsequence if necessary, we may assume $\{\theta_i\}$ converges to some point $\theta_0 \in \overline{\Sigma_{\mathbf{s},\pi/2}}$. If $\theta_0 \in \Sigma_{\mathbf{s},\pi/2} \setminus \{\mathbf{n}\}$, then $v(\theta_0) \le v_{\mathbf{s},\pi/2}(\theta_0)$, which violates (3.14). If $\theta_0 = \mathbf{n}$, (3.16) implies that $(v - v_{\mathbf{p}_i,\pi/2})(\theta_i) > c/2$ for large i, a contradiction of (3.18). Finally, we assume $\theta_0 \in \partial B_{\pi/2}(\mathbf{s})$. Let $\bar{\theta}_i$ be the closest point on $\partial\Sigma_{\mathbf{p}_{t_i},\pi/2}$ to θ_i. Noting that $v(\bar{\theta}_i) = v_{\mathbf{p}_{t_i},\pi/2}(\bar{\theta}_i)$, by using (3.18) we have

$$\frac{\partial}{\partial r_i}(v - v_{\mathbf{p}_{t_i},\pi/2})(\tilde{\theta}_i) < 0,$$

for some $\tilde{\theta}_i$ between θ_i and $\bar{\theta}_i$ on the geodesic line connecting θ_i and $\bar{\theta}_i$, where r_i denotes the geodesic distance from $\mathbf{p}_{t_i}$. Note that $\tilde{\theta}_i \to \theta_0$. We obtain

$$\frac{\partial}{\partial r}(v - v_{\mathbf{s},\pi/2})(\theta_0) \le 0,$$

which violates (3.15). We have thus proved the claim.

We define

$$\bar{t} := \sup\left\{t \in (0,1) : v \ge v_{\mathbf{p}_\tau,\pi/2} \text{ in } \Sigma_{\mathbf{p}_\tau,\pi/2} \setminus \{\mathbf{n}\} \text{ for all } 0 \le \tau < t\right\}.$$

From Claim 1, $\bar{t}$ is well defined and $\bar{t} > 0$.

Claim 2. $\bar{t} = 1$.

Supposing $\bar{t} < 1$, by continuity of v, we have

$$v \ge v_{\mathbf{p}_{\bar{t}},\pi/2} \quad \text{in } \Sigma_{\mathbf{p}_{\bar{t}},\pi/2} \setminus \{\mathbf{n}\};$$

the conditions on g imply

$$-\mathcal{L}_{\mathbb{S}^n}(v - v_{\mathbf{p}_{\bar{t}},\pi/2}) > 0 \quad \text{in } \Sigma_{\mathbf{p}_{\bar{t}},\pi/2} \setminus \{\mathbf{n}\}, \tag{3.19}$$

and

$$v \not\equiv v_{\mathbf{p}_{\bar{t}},\pi/2} \quad \text{in } \Sigma_{\mathbf{p}_{\bar{t}},\pi/2} \setminus \{\mathbf{n}\}.$$

Similar to the proof of Claim 1, there exists $\varepsilon_{\bar{t}} > 0$ such that for all $\bar{t} \le \mu < \bar{t} + \varepsilon_{\bar{t}} < 1$,

$$v \ge v_{\mathbf{p}_\mu,\pi/2} \quad \text{in } \Sigma_{\mathbf{p}_\mu,\pi/2} \setminus \{\mathbf{n}\},$$

which contradicts the definition of $\bar{t}$. Claim 2 thus follows.

By continuity of v, we finally obtain $v \ge v_{\mathbf{q},\pi/2}$ in $\Sigma_{\mathbf{q},\pi/2}$ for $\mathbf{q} \in \partial B_{\pi/2}(\mathbf{s})$. Since $\mathbf{q}$ is arbitrarily chosen on $\partial B_{\pi/2}(\mathbf{s})$, the proof is complete.

3.3. **Proof of Theorem 1.5 and its corollaries.** We first use Lemma 3.3 to give the proof of Theorem 1.5.

Proof of Theorem 1.5. Using the conditions (g1), (g5) and (g6), it is easy to see that $(g7)_{\mathbf{p}}$ and $(g8)_{\mathbf{p}}$ are satisfied for every $\mathbf{p} \in \partial B_{\pi/2}(\mathbf{s})$. Therefore Lemma 3.3 with $\Gamma = \{\mathbf{n}, \mathbf{s}\}$ implies that $\bar{\lambda}_{\mathbf{p}} \geq \pi/2$ for each $\mathbf{p} \in \partial B_{\pi/2}(\mathbf{s})$. This in particular implies $v_{\mathbf{p},\pi/2} \leq v$ on $\Sigma_{\mathbf{p},\pi/2} \setminus \{\mathbf{n}, \mathbf{s}\}$ for each $\mathbf{p} \in \partial B_{\pi/2}(\mathbf{s})$. Consequently v is rotationally symmetric about the line through $\mathbf{n}$ and $\mathbf{s}$. □

In order to use Theorem 1.5 to prove Corollary 1.3, we will show $f(v) > 0$ on $\mathbb{S}^n$ for any solution $v \in C^2(\mathbb{S}^n)$ of (1.13), thus condition (1.10) is satisfied due to Lemma 3.2. This is given by the following simple observation.

Lemma 3.4. *Let f satisfy the conditions given in Corollary* 1.3. *If $v \in C^2(\mathbb{S}^n)$ is a solution of* (1.13), *then $f(v) > 0$ on $\mathbb{S}^n$.*

Proof. Supposing it is not true, then there is $\bar{\theta} \in \mathbb{S}^n$ such that $f(v(\bar{\theta})) \leq 0$. Let $\hat{\theta} \in \mathbb{S}^n$ be a point such that $v(\hat{\theta}) = \max_{\mathbb{S}^n} v$. Then, it follows from the condition on f that

$$v(\hat{\theta})^{-(n+2)/(n-2)} f(v(\hat{\theta})) \leq v(\bar{\theta})^{-(n+2)/(n-2)} f(v(\bar{\theta})) \leq 0.$$

On the other hand, by the maximality of v at $\hat{\theta}$ we have $\Delta v(\hat{\theta}) \leq 0$, it follows from (1.13) that

$$f(v(\hat{\theta})) \geq \frac{n(n-2)}{4} v(\hat{\theta}) > 0.$$

We thus derive a contradiction. □

Proof of Corollary 1.3. It is clear that $g(\theta, s) := f(s)$ satisfies (g1), (g3), (g5) and (g6). Since $v \in C^2(\mathbb{S}^n)$ and Lemma 3.4 implies $f(v) > 0$ on $\mathbb{S}^n$, we can use Theorem 1.5 to conclude v is rotationally symmetric about the line through $\mathbf{p}$ and $-\mathbf{p}$ for any $\mathbf{p} \in \mathbb{S}^n$. Therefore, v must be constant. □

Proof of Corollary 1.4. (i) When $\beta < n(n-2)/4$, the function $g(\theta, s)$ defined by (1.14) satisfies (g1), (g3), (g5) and (g6) and is positive for $s \in (0, \infty)$. Thus we can apply Theorem 1.5 to conclude that any smooth solution $v \in C^2(\mathbb{S}^n \setminus \{\mathbf{n}, \mathbf{s}\})$ of (1.15) is rotationally symmetric about the line through $\mathbf{n}$ and $\mathbf{s}$. Given a solution $v \in C^2(\mathbb{S}^n \setminus \{\mathbf{n}, \mathbf{s}\})$ of (1.13), the function $u(x) := \xi(x) v(\pi_{\mathbf{n}}^{-1}(x))$ is then radially symmetric in $\mathbb{R}^n \setminus \{0\}$. Writing $u(x) = u(r)$ with $r = |x|$, u then satisfies the ordinary differential equation

$$u'' + \frac{n-1}{r} u' + \frac{n(n-2) - 4\beta}{(1+r^2)^2} u + u^{\frac{n+2}{n-2}} = 0, \ u(r) > 0 \text{ for } 0 < r < \infty. \quad (3.20)$$

(ii) Now, we assume $\beta \le 0$ and (1.13) has a solution $v \in C^2(\mathbb{S}^n \backslash \{\mathbf{n}, \mathbf{s}\})$. This implies that (3.20) has a solution u. Let $\varphi(r) := (1+r^2)^{\frac{n-2}{2}} u(r)$; then

$$\Big(\frac{r^{n-1}}{(1+r^2)^{n-2}}\varphi'\Big)' = -\frac{r^{n-1}}{(1+r^2)^n}\Big(\varphi^{\frac{n+2}{n-2}} - 4\beta\varphi\Big) \quad \text{for } 0 < r < \infty. \tag{3.21}$$

Using $\beta \le 0$ we have for any $r > \varepsilon > 0$ that

$$\frac{r^{n-1}}{(1+r^2)^{n-2}}\varphi'(r) < \frac{\varepsilon^{n-1}}{(1+\varepsilon^2)^{n-2}}\varphi'(\varepsilon). \tag{3.22}$$

If

$$\lim_{r\to 0} \frac{r^{n-1}}{(1+r^2)^{n-2}}\varphi'(r) > 0,$$

then there exists a number $\alpha_0 > 0$ such that $\varphi'(r) \ge \alpha_0 r^{1-n}$ for small $r > 0$. Therefore, for any small $r > \varepsilon > 0$ there holds

$$\varphi(r) \ge \varphi(r) - \varphi(\varepsilon) \ge \frac{\alpha_0}{n-2}\Big(\varepsilon^{2-n} - r^{2-n}\Big).$$

Fixing r and letting $\varepsilon \to 0$ we then derive a contradiction. Therefore

$$\lim_{r\to 0} \frac{r^{n-1}}{(1+r^2)^{n-2}}\varphi'(r) \le 0.$$

It then follows from (3.22) that $\varphi'(r) < 0$ for all $r > 0$. Let $\tilde\varphi(r) := \varphi(\frac{1}{r})$, then $\tilde\varphi'(r) > 0$ for all $r > 0$. However, by direct calculation one can see that $\tilde\varphi$ is also a solution of (3.21). Therefore the above argument applies to $\tilde\varphi$ and shows that $\tilde\varphi'(r) < 0$ for all $r > 0$. We thus derive a contradiction.

(iii) Showing that (1.15) has infinitely many solutions is equivalent to showing that (3.20) has infinitely many solutions defined on $(0,\infty)$. To this end, for any function u defined on $(0,\infty)$ we define

$$w(t) = e^{-\frac{n-2}{2}t} u(e^{-t}), \quad t \in (-\infty,\infty).$$

By an easy calculation one can see that u satisfies (3.20) if and only if w satisfies

$$w'' - \Big(\frac{n-2}{2}\Big)^2 w + w^{\frac{n+2}{n-2}} + \frac{c_\beta e^{-2t}}{(1+e^{-2t})^2} w = 0, \quad w > 0 \text{ on } (-\infty,\infty), \tag{3.23}$$

where $c_\beta := n(n-2) - 4\beta$. Therefore, it suffices to show that (3.23) has infinitely many positive solutions.

Let us introduce a function $h(\cdot,\cdot) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ by

$$h(a,b) := b^2 - \Big(\frac{n-2}{2}\Big)^2 a^2 + \frac{n-2}{n} a^{\frac{2n}{n-2}}, \quad \forall (a,b) \in \mathbb{R}_+ \times \mathbb{R}.$$

Since $\beta > \frac{n-2}{2}$, there are infinitely many $(a,b) \in \mathbb{R}^+ \times \mathbb{R}$ such that

$$h(a,b) + \frac{1}{4} c_\beta a^2 < 0. \tag{3.24}$$

Let us fix one of them and consider the initial-value problem

$$\begin{cases} w'' - \left(\frac{n-2}{2}\right)^2 w + w^{\frac{n+2}{n-2}} + c_\beta \frac{e^{-2t}}{(1+e^{-2t})^2} w = 0 \\ w(0) = a, \quad w'(0) = b. \end{cases}$$

By the local existence theory for ordinary differential equations, it has a unique solution w defined on some interval containing $t = 0$. Let $(-B, A)$ be the largest interval on which w exists and $w(t) > 0$. Since $w(0) = a > 0$, A and B must be positive. It remains only to show that $A = \infty$ and $B = \infty$. In the following we will only prove $A = \infty$, since $B = \infty$ can be proven in the same way.

Suppose $A < \infty$ and consider the function

$$h(t) := h(w(t), w'(t)) \quad \text{on } [0, A).$$

From the equation satisfied by w it follows that

$$h'(t) = -c_\beta \frac{e^{-2t}}{(1+e^{-2t})^2} \left[w(t)^2 \right]'.$$

Therefore, for $t \in [0, A)$ one has, by integration by parts and noting that $\frac{e^{-2t}}{(1+e^{-2t})^2}$ is non-increasing on $[0, \infty)$,

$$\begin{aligned} h(t) - h(0) &= -c_\beta \Big\{ \frac{e^{-2t}}{(1+e^{-2t})^2} w(t)^2 - \tfrac{1}{4} w(0)^2 - \int_0^t w(s)^2 \Big[\frac{e^{-2s}}{(1+e^{-2s})^2} \Big]' ds \Big\} \\ &\le \frac{1}{4} c_\beta w(0)^2. \end{aligned}$$

Consequently, by using (3.24),

$$h(t) \le h(a,b) + \frac{1}{4} c_\beta a^2 < 0 \quad \text{on } [0, A). \tag{3.25}$$

This together with the equation for w implies that

$$w + |w'| + |w''| \le C \quad \text{on } [0, A),$$

for some positive constant C. Therefore $w(A)$ and $w'(A)$ are well defined and, hence, w is defined on a larger interval $[0, A+\varepsilon)$ for some $\varepsilon > 0$. But from (3.25) one can see that $w(A) > 0$. Thus, by continuity, $w > 0$ on $[A, A+\varepsilon)$ for some smaller $\varepsilon > 0$. This contradicts the definition of A. Therefore $A = \infty$. $\square$

3.4. **Proof of Theorem 1.8, Theorem 1.9 and Theorem 1.10.**

Proof of Theorem 1.8. Let $\mathbf{s}$ be the south pole of $\mathbb{S}^n$. As before we define

$$\bar{\lambda}_{\mathbf{s}} := \sup\left\{\lambda \in (0,\pi) : v_{\mathbf{s},\mu} \le v \text{ in } \Sigma_{\mathbf{s},\mu} \setminus \{\mathbf{n}\} \text{ for each } 0 < \mu < \lambda\right\}.$$

Let (r,ω) be the geodesic polar coordinates on $\mathbb{S}^n$ with respect to $\mathbf{s}$. Then the conditions on K are equivalent to saying that K is non-constant on $\mathbb{S}^n \setminus \{\mathbf{n}\}$ and for each fixed $\omega \in \mathbb{S}^{n-1}$ the function $r \to K(r,\omega)$ is non-decreasing for $0 < r < \pi$.

By condition (1.23), it follows from Lemma 3.1 that $\bar{\lambda} := \bar{\lambda}_{\mathbf{s}} > 0$ is well defined. We claim that $\bar{\lambda} = \pi$. If $\bar{\lambda} < \pi$, then from (1.22), (1.7) and (1.8) it follows that

$$-v^{-\frac{n+2}{n-2}} \mathcal{L}_{\mathbb{S}^n} v(r,\omega) + v_{\mathbf{s},\bar{\lambda}}^{-\frac{n+2}{n-2}} \mathcal{L}_{\mathbb{S}^n} v_{\mathbf{s},\bar{\lambda}}(r,\omega) = K(r,\omega) - K(h_{\bar{\lambda}}(r),\omega).$$

Following the proof of Lemma 3.3 and using the conditions on K we have

$$0 \ge \Big(\int_0^1 v_s^{-\frac{n+2}{n-2}} ds\Big) \mathcal{L}_{\mathbb{S}^n}(v - v_{\mathbf{s},\bar{\lambda}}) - \frac{n+2}{n-2}\Big(\int_0^1 v_s^{-\frac{2n}{n-2}} \mathcal{L}_{\mathbb{S}^n} v_s ds\Big)(v - v_{\mathbf{s},\bar{\lambda}}),$$

in $\Sigma_{\mathbf{s},\bar{\lambda}} \setminus \{\mathbf{n}\}$.

Note that $v - v_{\mathbf{s},\bar{\lambda}} \ge 0$ on $\Sigma_{\mathbf{s},\bar{\lambda}} \setminus \{\mathbf{n}\}$. Since K is non-constant on $\mathbb{S}^n \setminus \{\mathbf{n}\}$, we must have $v - v_{\mathbf{s},\bar{\lambda}} \not\equiv 0$ on $\Sigma_{\mathbf{s},\bar{\lambda}} \setminus \{\mathbf{n}\}$. Thus it follows from the strong maximum principle and the Hopf lemma that

$$v - v_{\mathbf{s},\bar{\lambda}} > 0 \quad \text{on } \Sigma_{\mathbf{s},\bar{\lambda}} \setminus \{\mathbf{n}\},$$

and

$$\frac{\partial}{\partial r}(v - v_{\mathbf{s},\bar{\lambda}}) > 0 \quad \text{on } \partial\Sigma_{\mathbf{s},\bar{\lambda}}.$$

Next, we show that

$$\inf_{\theta \to \mathbf{n}} \left(v - v_{\mathbf{s},\bar{\lambda}}\right) > 0.$$

Choose $0 < r_0$ small such that $\bar{\lambda} + r_0 < \pi$. Define

$$\mathcal{O} := \left\{\theta \in B_{r_0}(\mathbf{n}) \setminus \{\mathbf{n}\} : v(\theta) < 2 v_{\mathbf{s},\bar{\lambda}}(\theta)\right\}.$$

By the fact that v has a positive lower bound, we can obtain $v - v_{\mathbf{s},\bar{\lambda}} \ge c_1$ on $\overline{B_{r_0}(\mathbf{n})} \setminus \left(\mathcal{O} \cup \{\mathbf{n}\}\right)$ for some positive constant c_1; moreover, v and $v_{\mathbf{s},\bar{\lambda}}$ have lower and upper bounds in $\mathcal{O}$. Following the proof of Lemma 3.3, we can obtain that for some large positive constant C

$$-\mathcal{L}_{\mathbb{S}^n}(v - v_{\mathbf{s},\bar{\lambda}}) + C(v - v_{\mathbf{s},\bar{\lambda}}) \ge 0 \quad \text{in} \quad \mathcal{O}.$$

Lemma 3.2 implies that $v - v_{\mathbf{s},\bar{\lambda}} > c_2$ in $\mathcal{O}$ for some positive constant c_2. We thus obtain

$$v - v_{\mathbf{s},\bar{\lambda}} \geq \min(c_1, c_2) > 0 \quad \text{in } B_{r_0}(\mathbf{n}) \setminus \{\mathbf{n}\}.$$

Now, we can argue as in the proof of Lemma 3.3 to show that there exists $\varepsilon > 0$ such that $v \geq v_{\mathbf{s},\lambda}$ on $\Sigma_{\mathbf{s},\lambda} \setminus \{\mathbf{n}\}$ for each $0 < \lambda < \bar{\lambda} + \varepsilon$. This contradicts the definition of $\bar{\lambda}$. Therefore $\bar{\lambda}_{\mathbf{s}} = \pi$.

Next, we are going to show (1.24). In the above, we have shown that

$$v \geq v_{\mathbf{s},\lambda} \quad \text{on } \Sigma_{\mathbf{s},\lambda} \setminus \{\mathbf{n}\} \quad \text{for each } 0 < \lambda < \pi. \tag{3.26}$$

For $\theta \in B_{\pi/2}(\mathbf{n}) \setminus \{\mathbf{n}\}$, let (r, ω) be its geodesic polar coordinate with respect to the south pole $\mathbf{s}$. Then we take $\pi/2 < \lambda < \pi$ such that

$$2 \cos \lambda = (1 + \cos^2 \lambda) \cos r. \tag{3.27}$$

This implies that $\varphi_{\mathbf{s},\lambda}(\theta) \in \partial B_{\pi/2}(\mathbf{n})$. Moreover, $\lambda \to \pi$ as $\theta \to \mathbf{n}$. Letting $C_1 := \min_{\partial B_{\pi/2}(\mathbf{n})} v$, we have from (3.26) and (3.27) that

$$\liminf_{\theta \to \mathbf{n}} d(\theta, \mathbf{n})^{(n-2)/2} v(\theta) \geq C_1 \liminf_{r \to \pi} \Big(\frac{(\pi - r) \sin^2 \lambda}{1 + \cos^2 \lambda - 2 \cos \lambda \cos r} \Big)^{(n-2)/2} = C_1.$$

The proof is complete. $\square$

The proofs of Theorem 1.9 and Theorem 1.10 are based on the method used by Li and Li in [15].

Proof of Theorem 1.9. Since $0 < v \in C^2(\mathbb{S}^n)$, by Lemma 3.1 the moving sphere procedure can start from the south pole $\mathbf{s}$. So there exists $0 < \lambda_1 < \pi$, such that

$$v \geq v_{\mathbf{s},\lambda} \quad \text{in } \Sigma_{\mathbf{s},\lambda} \text{ for each } 0 < \lambda < \lambda_1.$$

Let

$$\bar{\lambda} = \sup\{\lambda \in (0, \pi) : v \geq v_{\mathbf{s},\mu} \text{ in } \Sigma_{\mathbf{s},\mu} \text{ for each } 0 < \mu < \lambda\}.$$

We will show that $\bar{\lambda} = \pi$. If not, say $0 < \bar{\lambda} < \pi$. Then by the conformal invariance and the condition on K we have

$$f\Big(\lambda(A_{v^{\frac{4}{n-2}} g_0})\Big) - f\Big(\lambda(A_{v_{\mathbf{s},\bar{\lambda}}^{\frac{4}{n-2}} g_0})\Big) = K(r, \omega) - K(h_{\bar{\lambda}}(r), \omega) \geq 0,$$

with strict inequality somewhere in $\Sigma_{\mathbf{s},\bar{\lambda}}$.

By the argument in [15, Lemma 2.1], there exists an elliptic operator L such that

$$L(v - v_{\mathbf{s},\bar{\lambda}}) \geq 0 \quad \text{in} \quad \Sigma_{\mathbf{s},\bar{\lambda}},$$

with strict inequality somewhere in $\Sigma_{\mathbf{s},\bar{\lambda}}$. Noticing that $v \geq v_{\mathbf{s},\bar{\lambda}}$ in $\Sigma_{\mathbf{s},\bar{\lambda}}$, we can obtain that

$$v - v_{\mathbf{s},\bar{\lambda}} > 0 \quad \text{in } \Sigma_{\mathbf{s},\bar{\lambda}}, \tag{3.28}$$

and

$$\frac{\partial(v - v_{\mathbf{s},\bar{\lambda}})}{\partial r} > 0 \quad \text{on } \partial\Sigma_{\mathbf{s},\bar{\lambda}}. \tag{3.29}$$

Once (3.28) and (3.29) are established, we can argue as in the proof of Lemma 3.3 to show that we can move spheres beyond $\bar{\lambda}$, which contradicts the definition of $\bar{\lambda}$. So $\bar{\lambda} = \pi$. Following the argument of Theorem 1.8, we can see that v blows up at $\mathbf{n}$. Therefore (1.31) has no positive $C^2(\mathbb{S}^n)$ solutions. □

Proof of Theorem 1.10. Let (r, ω) be the geodesic polar coordinates with respect to $\mathbf{s}$. Let $\Sigma^+_{\mathbf{s},\lambda} = \mathbb{S}^n_+ \cap \Sigma_{\mathbf{s},\lambda}$. We define

$$\bar{\lambda} = \sup\{\lambda \in (0,\pi) : v \geq v_{\mathbf{s},\mu} \text{ in } \Sigma^+_{\mathbf{s},\mu} \text{ for all } 0 < \mu < \lambda\}.$$

Since v has positive lower bound on $\mathbb{S}^n_+$, it follows from Lemma 3.1 that $\bar{\lambda}$ is well defined and $\bar{\lambda} > 0$. We will show that $\bar{\lambda} = \pi$. If not, say $0 < \bar{\lambda} < \pi$, then

$$v \geq v_{\bar{\lambda}} \quad \text{in} \quad \Sigma^+_{\mathbf{s},\bar{\lambda}}.$$

Moreover, by the conformal invariance we have

$$\begin{cases} f\Big((A_{v_{\mathbf{s},\bar{\lambda}}^{\frac{4}{n-2}} g_0})\Big) = K(h_{\bar{\lambda}}(r), \omega) & \text{in } \mathbb{S}^n_+, \\ \frac{\partial v_{\mathbf{s},\bar{\lambda}}}{\partial \nu} = H(h_{\bar{\lambda}}(r), \omega) v_{\mathbf{s},\bar{\lambda}}^{\frac{n}{n-2}} & \text{on } \partial\mathbb{S}^n_+. \end{cases}$$

By the argument in [15, Lemma 2.1] and the condition $\nabla_{\frac{\partial}{\partial\theta_{n+1}}} K \geq 0$ on $\mathbb{S}^n_+$, there exists a linear elliptic operator L such that

$$L\big(v - v_{\mathbf{s},\bar{\lambda}}\big) \geq 0 \quad \text{in } \Sigma^+_{\mathbf{s},\bar{\lambda}}.$$

Moreover,

$$\frac{\partial(v - v_{\mathbf{s},\bar{\lambda}})}{\partial \nu} = H(r,\omega) v^{\frac{n}{n-2}} - H(h_{\bar{\lambda}}(r),\omega) v_{\mathbf{s},\bar{\lambda}}^{\frac{n}{n-2}} \quad \text{on } \partial\mathbb{S}^n_+ \cap \Sigma^+_{\mathbf{s},\bar{\lambda}}.$$

Since K or H is non-constant and $\nabla_{\frac{\partial}{\partial\theta_{n+1}}} H \geq 0$ on $\partial\mathbb{S}^n_+$, it follows from the strong maximum principle and the Hopf lemma that

$$v > v_{\mathbf{s},\bar{\lambda}} \quad \text{in } \Sigma^+_{\mathbf{s},\bar{\lambda}}. \tag{3.30}$$

Using the Hopf lemma again and [22, Lemma 10.1] we can also obtain

$$\frac{\partial(v - v_{\mathbf{s},\bar{\lambda}})}{\partial r} > 0 \quad \text{on } \partial\Sigma_{\mathbf{s},\bar{\lambda}} \cap \mathbb{S}^n_+.$$

Thus, there exists $\bar{\lambda} < \lambda_0 < \pi$ such that

$$v \geq v_{\mathbf{s},\lambda} \quad \text{on} \quad \Sigma^+_{\mathbf{s},\lambda_0} \setminus \Sigma^+_{\mathbf{s},\lambda} \quad \text{for } \bar{\lambda} < \lambda < \lambda_0.$$

By using (3.30) and the definition of $v_{\mathbf{s},\lambda}$, we can find $0 < \epsilon < \lambda_0 - \bar{\lambda}$ such that

$$v \geq v_{\mathbf{s},\lambda} \quad \text{on } \Sigma^+_{\mathbf{s},\lambda_0} \text{ for } \bar{\lambda} < \lambda < \bar{\lambda} + \epsilon.$$

Therefore,

$$v \geq v_{\mathbf{s},\lambda} \quad \text{on } \Sigma^+_{\mathbf{s},\lambda} \text{ for } \bar{\lambda} < \lambda < \bar{\lambda} + \epsilon.$$

We thus derive a contradiction to the definition of $\bar{\lambda}$. Hence, $\bar{\lambda} = \pi$. Consequently (1.32) has no positive $C^2(\mathbb{S}^n_+)$ solutions. $\square$

3.5. **Proofs of Theorem 1.6 and Theorem 1.7.** Given a function v on $\mathbb{S}^2$, we define, for each fixed $\mathbf{p} \in \mathbb{S}^2$ and $0 < \lambda < \pi$, its Kelvin transform as

$$v_{\mathbf{p},\lambda} = v \circ \varphi_{\mathbf{p},\lambda} + \frac{1}{2} \log |J_{\varphi_{\mathbf{p},\lambda}}|.$$

By conformal invariance it is known that

$$-\Delta_{\mathbb{S}^2} v_{\mathbf{p},\lambda} + 1 = |J_{\varphi_{\mathbf{p},\lambda}}| \Big((-\Delta_{\mathbb{S}^n} v) \circ \varphi_{\mathbf{p},\lambda} + 1 \Big). \tag{3.31}$$

The proof of Theorem 1.6 follows essentially the same idea as in the proof of Theorem 1.5. We need to compare the functions v and $v_{\mathbf{p},\lambda}$. Similar to Lemma 3.1 we have the following.

Lemma 3.5. *Let Γ be a closed subset of $\mathbb{S}^2$. If $v \in C^1(\mathbb{S}^2 \backslash \Gamma)$ and $v \geq -C_0$ on $\mathbb{S}^2 \backslash \Gamma$ for some constant $C_0 > 0$, then for each $\mathbf{p} \in \mathbb{S}^2 \setminus \Gamma$ there exists $0 < \lambda_{\mathbf{p}} < \frac{\pi}{2}$ such that*

$$v_{\mathbf{p},\lambda} \leq v, \quad \text{on } \Sigma_{\mathbf{p},\lambda} \setminus \Gamma \text{ for each } 0 < \lambda < \lambda_{\mathbf{p}}.$$

The next two lemmas, similar to Lemma 3.2, are used to deal with singularities.

Lemma 3.6. *Let $\mathcal{O}$ be an open set in $\mathbb{S}^2$ and $\mathbf{q} \in \mathcal{O}$. If $v \in C^2(\mathcal{O} \setminus \{\mathbf{q}\})$ satisfies $-\Delta_{\mathbb{S}^2} v + 1 > 0$ in $\mathcal{O} \setminus \{\mathbf{q}\}$ and*

$$\limsup_{\theta \in \mathcal{O}, \theta \to \mathbf{q}} \frac{v(\theta)}{\log d(\theta, \mathbf{q})} \leq 0, \tag{3.32}$$

then $v > -C_0$ in $\mathcal{O} \setminus \{\mathbf{q}\}$ for some constant $C_0 > 0$.

Proof. Fix an open set $\mathcal{O}'$ such that $\mathbf{q} \in \mathcal{O}' \subset \overline{\mathcal{O}}' \subset \mathcal{O}$ and $|\pi_{\mathbf{q}}(\theta)| > 1$ for $\theta \in \mathcal{O}'$. For each $\varepsilon > 0$ consider the function

$$\alpha_\varepsilon(\theta) = v(\theta) + \log \frac{2}{1 + |\pi_{\mathbf{q}}(\theta)|^2} + C_0 + (2+\varepsilon)\log|\pi_{\mathbf{q}}(\theta)|, \quad \theta \in \overline{\mathcal{O}}'\backslash\{\mathbf{q}\},$$

where C_0 is a constant such that $v(\theta) + \log \frac{2}{1+|\pi_{\mathbf{q}}(\theta)|^2} + C_0 > 0$ on $\partial\mathcal{O}'$. One can check that

$$\Delta_{\mathbb{S}^2}\Big(\log \frac{2}{1 + |\pi_{\mathbf{q}}(\theta)|^2}\Big) = -1 \quad \text{and} \quad \Delta_{\mathbb{S}^2}(\log|\pi_{\mathbf{q}}(\theta)|) = 0 \quad \text{in } \mathcal{O}' \setminus \{\mathbf{q}\}.$$

Moreover, $\lim_{\theta\to\mathbf{q}} |\pi_{\mathbf{q}}(\theta)| = +\infty$. Noting that (3.32) implies

$$\liminf_{\theta\in\mathcal{O},\theta\to\mathbf{q}} \frac{v(\theta)}{\log|\pi_{\mathbf{q}}(\theta)|} \geq 0.$$

We, therefore, have

$$-\Delta_{\mathbb{S}^2}\alpha_\varepsilon > 0 \ \text{ in } \mathcal{O}'\backslash\{\mathbf{q}\}, \ \alpha_\varepsilon > 0 \text{ on } \partial\mathcal{O}' \ \text{ and } \lim_{\theta\in\mathcal{O},\theta\to\mathbf{q}} \alpha_\varepsilon = +\infty.$$

It then follows from the maximum principle that $\alpha_\varepsilon > 0$ on $\overline{\mathcal{O}}'\setminus\{\mathbf{q}\}$. Letting $\varepsilon \to 0$ we get

$$v(\theta) \geq -C_0 - \log \frac{2|\pi_{\mathbf{q}}(\theta)|^2}{1 + |\pi_{\mathbf{q}}(\theta)|^2} \geq -C_0 - \log 2 \quad \text{for } \theta \in \mathcal{O}' \setminus \{\mathbf{q}\},$$

which gives the desired assertion. □

Lemma 3.7. *Let $\mathcal{O}$ be an open set in $\mathbb{S}^2$ and $\mathbf{q} \in \overline{\mathcal{O}}$. If $v \in C^2(\mathcal{O} \setminus \{\mathbf{q}\}) \cap C^0(\overline{\mathcal{O}}\backslash\{\mathbf{q}\})$ is a non-negative function such that $-\Delta_{\mathbb{S}^2} v > 0$ in $\mathcal{O} \setminus \{\mathbf{q}\}$ and $v \geq c_0$ on $\partial\mathcal{O}\backslash\{\mathbf{q}\}$ for some constant $c_0 > 0$, then $v > c_1$ on $\overline{\mathcal{O}} \setminus \{\mathbf{q}\}$ for some constant $c_1 > 0$.*

Proof. First by the strong maximum principle we have $v > 0$ on $\overline{O}\backslash\{\mathbf{q}\}$. By shrinking $\mathcal{O}$ if necessary, we may assume $|\pi_{\mathbf{q}}(\theta)| > 1$ for $\theta \in \mathcal{O}$. For each $\varepsilon > 0$ consider the function

$$\beta_\varepsilon(\theta) := v(\theta) - c_0 + \varepsilon \log|\pi_{\mathbf{q}}(\theta)|, \quad \theta \in \overline{\mathcal{O}}\backslash\{\mathbf{q}\}.$$

Note that $\Delta_{\mathbb{S}^2}(\log|\pi_{\mathbf{q}}(\theta)|) = 0$ on $\mathcal{O}\backslash\{\mathbf{q}\}$, $\lim_{\theta\to\mathbf{q}} |\pi_{\mathbf{q}}(\theta)| = +\infty$ and $v \geq c_0 > 0$ on $\partial\mathcal{O}\backslash\{\mathbf{q}\}$, we have

$$-\Delta_{\mathbb{S}^2}\beta_\varepsilon > 0 \text{ in } \mathcal{O}\backslash\{\mathbf{q}\}, \ \beta_\varepsilon > 0 \text{ on } \partial\mathcal{O}\backslash\{\mathbf{q}\} \ \text{ and } \lim_{\theta\in\mathcal{O},\theta\to\mathbf{q}} \beta_\varepsilon(\theta) = +\infty.$$

Therefore, by the maximum principle we have $\beta_\varepsilon > 0$ on $\overline{\mathcal{O}}\backslash\{\mathbf{q}\}$. Letting $\varepsilon \to 0$ gives the desired conclusion. □

Now, we are ready to give the proofs of Theorem 1.6 and Theorem 1.7.

Proof of Theorem 1.6. Since $f \geq 0$ and $K \geq 0$, we have

$$-\Delta_{\mathbb{S}^2} v + 1 > 0 \quad \text{on } \mathbb{S}^2 \backslash \{\mathbf{n}, \mathbf{s}\}.$$

It then follows from Lemma 3.6 that

$$v \geq -C_0 \quad \text{on } \mathbb{S}^2 \backslash \{\mathbf{n}, \mathbf{s}\},$$

for some constant $C_0 > 0$. Lemma 3.5 then implies for each $\mathbf{p} \in \partial B_{\pi/2}(\mathbf{s})$ that there exists $0 < \lambda_{\mathbf{p}} < \pi/2$ such that

$$v_{\mathbf{p},\lambda} \leq v \quad \text{on } \Sigma_{\mathbf{p},\lambda} \backslash \{\mathbf{n}, \mathbf{s}\} \text{ for each } 0 < \lambda < \lambda_{\mathbf{p}}.$$

For each $\mathbf{p} \in \partial B_{\pi/2}(\mathbf{s})$ we can define $\bar{\lambda}_{\mathbf{p}}$ as before, then $\bar{\lambda}_{\mathbf{p}} > 0$. Using (1.16), (3.31), (K1), (f1), (Kf1), the symmetry properties of K and f, and Lemma 3.7, we may imitate the proof of Lemma 3.3 to conclude that

$$\bar{\lambda}_{\mathbf{p}} \geq \pi/2 \quad \text{for each } \mathbf{p} \in \partial B_{\pi/2}(\mathbf{s}).$$

The desired assertion thus follows. □

Proof of Theorem 1.7. Under the conditions in Theorem 1.7, we can argue as in the proof of Theorem 1.6 to show that

$$v - v_{\mathbf{s},\pi/2} \geq 0 \quad \text{on } \Sigma_{\mathbf{s},\pi/2} \setminus \{\mathbf{n}\}.$$

Together with condition (K2) and (f2), we obtain that

$$-\Delta_{\mathbb{S}^2}(v - v_{\mathbf{s},\pi/2}) \geq 0, \quad \text{on} \quad \Sigma_{\mathbf{s},\pi/2} \setminus \{\mathbf{n}\},$$

and, due to (Kf2), this inequality is strict somewhere in $\Sigma_{\mathbf{s},\pi/2} \setminus \{\mathbf{n}\}$. Hence, we can follow the proof of Theorem 1.4 to show that

$$v - v_{\mathbf{p},\pi/2} \geq 0, \quad \text{on} \quad \Sigma_{\mathbf{p},\pi/2} \setminus \{\mathbf{n}\},$$

for any $\mathbf{p} \in \partial B_{\pi/2}(\mathbf{s})$. The proof is complete. □

4. APPENDIX

In this section, we prove a lemma from which Lemma 3.2 follows. For $n \geq 3$, let $\mathcal{O}$ be an open set in $\mathbb{R}^n \setminus B_1(0)$. Consider

$$-\Delta v(y) + \sum_{i=1}^{n} \frac{b_i(y)}{|y|^3} v_i(y) + \frac{c(y)}{|y|^4} v(y) \geq 0, \quad v > 0, \quad \text{in} \quad \mathcal{O}, \tag{4.1}$$

where $b_i, c \in L^\infty(\mathcal{O})$.

Lemma 4.1. *For $n \geq 3$, let $\mathcal{O}$ be an open set in $\mathbb{R}^n \setminus B_1(0)$. Assume $v \in C^2(\mathcal{O}) \cap C^0(\bar{\mathcal{O}})$ is a solution of (4.1). If $b_i, c \in L^\infty(\mathcal{O})$ and there exists a constant $c_0 > 0$ such that*

$$v(y) \geq \frac{c_0}{|y|^{n-2}}, \quad \text{on} \quad \partial\mathcal{O},$$

and

$$\liminf_{|y|\to\infty} v(y) \geq 0,$$

then there exists a constant $c_1 > 0$ such that

$$v(y) \geq \frac{c_1}{|y|^{n-2}}, \quad \text{in} \quad \mathcal{O}.$$

Proof. The proof is based on the argument of [22, Lemma 2.1]. Let $\xi(y) = |y|^{2-n} + |y|^{1-n}$, then

$$-\Delta\xi(y) = -(n-1)|y|^{-n-1}.$$

By the condition on $b_i(x)$ and $c(x)$, there exists $R > 1$ large enough, such that

$$-\Delta\xi(y) + \sum_{i=1}^{n} \frac{b_i(y)}{|y|^3}\xi_i(y) + \frac{c(y)}{|y|^4}\xi(y) \leq 0, \quad \text{in} \quad \mathcal{O} \setminus B_R(0).$$

Since $v > 0$ in $\bar{\mathcal{O}}$, there exists some $\epsilon > 0$ such that

$$v(y) \geq \epsilon\xi(y), \quad \text{for} \quad |y| = R, \quad y \in \bar{\mathcal{O}},$$

and

$$v(y) \geq \frac{c_0}{|y|^{2-n}} \geq \epsilon\xi(y), \quad \text{on} \quad \partial\mathcal{O}.$$

We have

$$\begin{cases} -\Delta(v - \epsilon\xi) + \sum \frac{b_i}{|y|^3}(v_i - \epsilon\xi_i) + \frac{c}{|y|^4}(v - \epsilon\xi) \geq 0, \quad \mathcal{O} \setminus B_R(0) \\ v - \epsilon\xi \geq 0, \quad \text{on} \quad \partial(\mathcal{O} \setminus B_R(0)) \\ \liminf_{|y|\to\infty}(v - \epsilon\xi) \geq 0. \end{cases}$$

By the maximum principle, $v - \epsilon\xi \geq 0$ in $\mathcal{O} \setminus B_R(0)$. Thus for some constant $c_1 > 0$,

$$v \geq \frac{c_1}{|y|^{2-n}}, \quad \text{in} \quad \mathcal{O}. \qquad \square$$

We have the following equivalent lemma on a bounded open set of $\mathbb{R}^n$. For $n \geq 3$, let Ω be a bounded open set in $\mathbb{R}^n$, let $\mathbf{p} \in \bar{\Omega}$. Consider

$$-\Delta v(y) + |y-\mathbf{p}| \sum_{i=1}^{n} b_i(y) v_i(y) + c(y) v(y) \geq 0, \quad v > 0, \quad \text{in} \quad \Omega, \tag{4.2}$$

where $b_i, c \in L^\infty(\Omega)$.

Lemma 4.2. *For $n \geq 3$, let Ω be a bounded open set in $\mathbb{R}^n$, $\mathbf{p} \in \bar{\Omega}$. Assume that $v \in C^2(\Omega \setminus \{\mathbf{p}\}) \cap C^0(\bar{\Omega} \setminus \{\mathbf{p}\})$ is a solution of* (4.2)*. If $b_i, c \in L^\infty(\Omega)$ and $\inf_{\partial\Omega\setminus\{\mathbf{p}\}} v > 0$. Then $v > c_1$ in $\Omega \setminus \{\mathbf{p}\}$ for some constant $c_1 > 0$.*

Proof. Let

$$\tilde{\Omega} := \Big\{ z \in \mathbb{R}^n \,:\, \mathbf{p} + \frac{z-\mathbf{p}}{|z-\mathbf{p}|^2} \in \Omega \Big\},$$

and make a Kelvin transform with respect to $\mathbf{p}$,

$$\tilde{v}(z) = \frac{1}{|z-\mathbf{p}|^{n-2}} v\Big(\mathbf{p} + \frac{z-\mathbf{p}}{|z-\mathbf{p}|^2}\Big).$$

Then $\tilde{v} \in C^2(\tilde{\Omega}) \cap C^0(\bar{\tilde{\Omega}})$. It is easy to check that $\tilde{v}$ satisfies

$$-\Delta \tilde{v} + \sum_{i=1}^{n} \frac{\tilde{b}_i(z)}{|z-\mathbf{p}|^3} \tilde{v}_i(z) + \frac{\tilde{c}(z)}{|z-\mathbf{p}|^4} \tilde{v}(z) \geq 0, \quad \tilde{v} > 0, \quad \text{in} \quad \tilde{\Omega},$$

where $\tilde{b}_i, \tilde{c} \in L^\infty(\tilde{\Omega})$ and

$$\tilde{v}(z) \geq \frac{c_0}{|z-\mathbf{p}|^{n-2}}, \quad \text{on} \quad \partial\tilde{\Omega},$$

for some constant $c_0 > 0$. From Lemma 4.1, we get $\tilde{v}(z) \geq \frac{c_1}{|z-\mathbf{p}|^{n-2}}$ in $\tilde{\Omega}$ for some constant $c_1 > 0$. The conclusion of the lemma follows easily. □

Remark 4.1. It is easy to see from the proof that Δ can be replaced by $a_{ij}\partial_{ij}$ with $a_{ij} \in C^0(\bar{\Omega})$, $(a_{ij}) > 0$ in $\bar{\Omega}$. In fact, the same conclusion holds for more general operators.

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