Solutions

Contents

2 Some Elementary Logic

Problem 2.1

1.

p	q	$p \Rightarrow q$	$\neg p$	$\neg q$	$\neg q \Rightarrow \neg p$	$\neg p \vee q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$
T	T	T	F	F	T	T	F	T
T	F	F	F	T	F	F	T	F
F	T	T	T	F	T	T	F	T
F	F	T	T	T	T	T	F	T

2.

p	q	$p \vee q$	$\neg p$	$\neg q$	$\neg p \wedge \neg p$	$\neg(\neg p \wedge \neg p)$
T	T	T	F	F	F	T
T	F	T	F	T	F	T
F	T	T	T	F	F	T
F	F	F	T	T	T	F

3.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \vee q$	$\neg p$	$\neg q$	$(\neg p) \vee (\neg q)$
T	T	T	F	T	F	F	F
T	F	F	T	T	F	T	T
F	T	F	T	T	T	F	T
F	F	F	T	F	T	T	T

4.

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$	$p \Leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

5.

p	q	$p \Rightarrow q$	$\neg(p \Rightarrow q)$	$\neg q$	$p \wedge \neg q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

Problem 2.2 Suppose p is the greatest prime. Let q be the product of all primes $\leq p$, i.e. q is the product of *all* primes.

If $q+1$ is *not* prime then it must be divisible by some prime p^*, say. But q is divisible by p^* and so $q+1$ leaves a remainder 1 when divided by p^*. Hence $q+1$ *is* prime, but since $q+1 > p$ we have a contradiction to the assumption p is the greatest prime.

Thus there is no greatest prime.

Problem 2.3

1. (a) $\forall x\,(x \in \mathbb{Q} \Rightarrow x^2 \in \mathbb{Q})$ [1] *or*
   $\forall x \in \mathbb{Q}\,(x^2 \in \mathbb{Q})$.
   (b) $\exists x$ such that $(x \in \mathbb{Q} \wedge x^2 \notin \mathbb{Q})$ *or*
   $\exists x \in \mathbb{Q}$ such that $(x^2 \notin \mathbb{Q})$. [2]
   (c) There is a rational number whose square is irrational.
2. (a) $\neg\exists x$ such that $\big((x$ is an elephant$) \wedge (x$ can stand the sight of a mouse$)\big)$.
   (b) $\exists x$ such that $\big((x$ is an elephant$) \wedge (x$ can stand the sight of a mouse$)\big)$.
   (c) There is an elephant which can stand the sight of a mouse.

Comments

1. Quantifiers should generally *precede* the statements to which they refer, as otherwise the result will usually be ambiguous. For example, do not write a statement such as:

$$\exists y \text{ such that } (y > x) \quad \forall x \tag{1}$$

or

$$\exists y \text{ such that } (y > \text{all } x). \tag{2}$$

Does this mean

$$\exists y \text{ such that } \forall x\,(y > x)\ ? \tag{3}$$

or

$$\forall x \exists y \text{ such that } (y > x)\ ? \tag{4}$$

Note that (3) is false in $\mathbb{R}$ and that (4) is true in $\mathbb{R}$. You should always use either (3) or (4) (depending on the intended meaning), and not (1) or (2).

2. The statement

$$\exists x \text{ such that } (x \text{ is a rational}) \wedge \neg\ (x^2 \text{ is a rational})$$

is also ambiguous. More generally,

$$\exists x \text{ such that } P(x) \wedge Q(x),$$

is ambiguous. It could mean either

$$\big(\exists x \text{ such that } P(x)\big) \wedge Q(x)$$

or

$$\exists x \text{ such that } \big(P(x) \wedge Q(x)\big).$$

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[1] It is implicit from the context of this Question that the quantifiers $\forall$ and $\exists$ range over the set of real numbers, unless otherwise specified.

[2] We sometimes omit the words "such that" after the symbol $\exists$. In the present situation we could also write $\exists x\,(x \in \mathbb{Q} \wedge x^2 \notin \mathbb{Q})$ *or* $\exists x \in \mathbb{Q}\,(x^2 \notin \mathbb{Q})$.

These have different meanings. In particular, the first has *exactly* the same meaning as

$$\Bigl(\exists y \text{ such that } P(y)\Bigr) \wedge Q(x),$$

and its truth or falsity may change with the value of x in $Q(x)$. The second has *exactly* the same meaning as

$$\exists y \text{ such that } \Bigl(P(y) \wedge Q(y)\Bigr).$$

Problem 2.4

PROOF: Let $n = \frac{k(k+1)}{2}$.

1. Let $k = 6p$. Then

$$n = 3p(6p+1) = 3\Bigl(p(6p+1)\Bigr)$$

and so the remainder after division by 3 is 0.

2. Let $k = 6p - 1$. Then

$$n = (6p-1)3p = 3(6p-1)p$$

and so the remainder after division by 3 is 0.

3. Let $k = 6p - 2$. Then

$$n = (3p-1)(6p-1) = 18p^2 - 9p + 1 = 3(6p^2 - 3p) + 1$$

and so the remainder after division by 3 is 1.

4. Let $k = 6p - 3$. Then

$$n = (6p-3)(3p-1) = 3(2p-1)(3p-1)$$

and so the remainder after division by 3 is 0.

5. Let $k = 6p - 4$. Then

$$n = (3p-2)(6p-3) = 3(3p-2)(2p-1)$$

and so the remainder after division by 3 is 0.

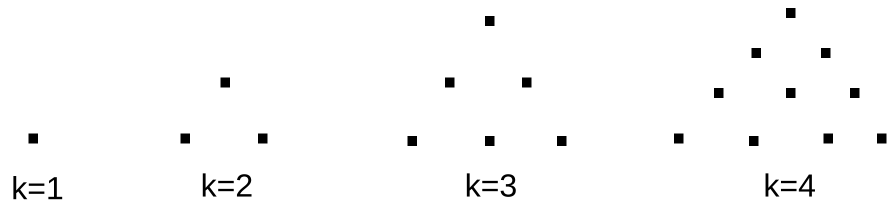
6. Let $k = 6p - 5$. Then

$$n = (6p-5)(3p-2) = 18p^2 - 27p + 10 = 3(6p^2 - 9p + 3) + 1$$

and so the remainder after division by 3 is 1.

This takes care of all possible cases. ∎

The following diagram indicates why numbers of the form $\frac{k(k+1)}{2}$ are called *triangular*.



Comment Note that $\frac{6}{4} = \frac{3}{2}$, but the remainder after division in each case is *not* the same.

Problem 2.5

1. **Definition** A function $f : A\,(\subset \mathbb{R}) \to \mathbb{R}$ is *uniformly continuous* if[3]

$$\forall \epsilon > 0\ \exists \delta > 0 \text{ such that } \Big(\forall x \in A\ \forall y \in A\ \big(|x-y| < \delta \Rightarrow |f(x)-f(y)| < \epsilon \big) \Big).$$

2. (a) A function $f : A\,(\subset \mathbb{R}) \to \mathbb{R}$ is *not* uniformly continuous iff: there is an $\epsilon > 0$ such that for each $\delta > 0$ there exist $x, y \in A$ for which $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$.

   (b) A function $f : A\,(\subset \mathbb{R}) \to \mathbb{R}$ is *not* uniformly continuous iff:

$$\exists \epsilon > 0 \text{ such that } \forall \delta > 0$$
$$\exists x \in A \text{ and } \exists y \in A \text{ for which } \big(|x - y| < \delta\ \wedge\ |f(x) - f(y)| \geq \epsilon \big),$$

   or more concisely

$$\exists \epsilon > 0\, \forall \delta > 0 \Big(\exists x \in A\, \exists y \in A\, \big(|x - y| < \delta\ \wedge\ |f(x) - f(y)| \geq \epsilon \big) \Big).$$

3. Let $f(x) = 1/x$ for $x \in (0, 1)$. Then f is not uniformly continuous on $(0, 1)$.

   PROOF: We will show that 2(a) and 2(b) are true by taking $\epsilon = 1$.

   For each $\delta > 0$ we can certainly choose $x, y \in (0, 1)$ such that $|x-y| < \delta$ and $|1/x-1/y| \geq 1$. For example, if $0 < \delta < 1$ let $x = \delta$ and $y = \delta/2$, and if $\delta \geq 1$ let $x = 1/2$ and $y = 1/4$. It follows from either 2(a) or 2(b) that f is not uniformly continuous. ∎

Comments

[3] Recall that in a definition it is conventional to write *if* when more precisely one should write *if and only if*.

1. The order of the quantifiers is critical. It does *not* change the meaning if two consecutive universal quantifiers are reversed (e.g. $\forall x \forall y$ is replaced by $\forall y \forall x$) or if two consecutive existential quantifiers are reversed (e.g. $\exists x \exists y$ is replaced by $\exists y \exists x$). But it is *incorect* to replace $\forall x \exists y$ by $\exists y \forall x$ or to replace $\exists y \forall x$ by $\forall x \exists y$.

2. Do not omit $\exists x \in A$ and $\exists y \in A$ in 2(b). If they are omitted, the convention is that *universal* quantifiers are intended.

3. You should not even omit $\forall x \in A$ and $\forall y \in A$ in 1. If you do, the convention is that universal quantifiers are intended. But it would still not be clear if the intended meaning is

   $$\forall \epsilon > 0\ \exists \delta > 0 \text{ such that } \Big(\forall x \in A\ \forall y \in A\ \big(|x-y| < \delta \Rightarrow |f(x)-f(y)| < \epsilon\big)\Big)$$

   or

   $$\forall x \in A\ \forall y \in A\ \forall \epsilon > 0\ \exists \delta > 0 \text{ such that } \big(|x-y| < \delta \Rightarrow |f(x)-f(y)| < \epsilon\big).$$

   The second does *not* give a correct definition of uniform continuity.(*Why?*)

   A common mistake is to omit the universal quantifiers and then ended up in 2 with the assertion that a function $f : A\,(\subset \mathbb{R}) \to \mathbb{R}$ is *not* uniformly continuous iff:

   $$\exists \epsilon > 0\ \forall \delta > 0 \big(|x-y| < \delta \ \wedge\ |f(x) - f(y)| \geq \epsilon\big).$$

   This is incorrect, as noted in 2.

Problem 2.6 1. **Definition** Suppose $f_1, f_2, \ldots, f_n, \ldots$ is a sequence of functions such that $f_n : [0,1] \to \mathbb{R}$ for all n. Suppose that $f : [0,1] \to \mathbb{R}$. Then *the sequence* $(f_n)_{n=1}^{\infty}$ *converges to* f *uniformly* if

$$\forall\, \epsilon > 0\ \ \exists\, N \text{ such that } \big(n \geq N \Rightarrow \forall\, x \in [0,1]\ (|f_n(x) - f(x)| < \epsilon)\ \big).$$

Note: one usually omits “such that”, and it is understood from context that n and N are integers.

2. Let

   $$f_n(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1/2 - 1/n\ , \\ 1 - n(1/2 - x) & \text{if } 1/2 - 1/n < x < 1/2, \\ 1 & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

   Draw a diagram!

   (a) If $0 \leq x < 1/2$ then $f_n(x) = f(x)$ provided n is sufficiently large, i.e. provided $x \leq 1/2 - 1/n$, i.e. provided $n > 2/(1-2x)$[4], and so certainly $f_n(x) \to f(x)$ for such x. Note that the closer x is to $1/2$, the larger we need to take n, so there is no “uniform” choice of n for all $x \in [0, 1/2)$.

[4]Since $x \leq 1/2 - 1/n$ iff $2nx \leq n-2$ iff $2 \leq n(1-2x)$ iff $n \geq 2/(1-2x)$ (as $0 \leq x < 1/2$ and so $1 - 2x > 0$).

(b) If $1/2 \le x \le 1$ then $f_n(x) = f(x) = 1$ for all n and so again $f_n(x) \to f(x)$ for such x.

Thus we see that the sequence $(f_n)_{n=1}^{\infty}$ converges to f *pointwise*, but not *uniformly*.

3. (a) **Definition** Suppose $f_1, f_2, \dots, f_n, \dots$ is a sequence of functions such that $f_n : [0,1] \to \mathbb{R}$ for all n. Suppose that $f : [0,1] \to \mathbb{R}$. Then *the sequence* $(f_n)_{n=1}^{\infty}$ *converges to* f *pointwise* if

   for all $x \in [0,1]$ and for every $\epsilon > 0$ there exists N such that $n \ge N$ implies $|f_n(x) - f(x)| < \epsilon$.

   (b) **Definition** Suppose $f_1, f_2, \dots, f_n, \dots$ is a sequence of functions such that $f_n : [0,1] \to \mathbb{R}$ for all n. Suppose that $f : [0,1] \to \mathbb{R}$. Then *the sequence* $(f_n)_{n=1}^{\infty}$ *converges to* f *pointwise* if

$$\forall x \in [0,1]\ \forall \epsilon > 0\ \exists N \text{ such that } \Big(n \ge N \Rightarrow |f_n(x) - f(x)| < \epsilon\Big).$$

   Note: one again usually omits "such that".

Remarks on Solutions

1. It is also correct in 2. to write

$$\forall \epsilon > 0\ \exists N \text{ such that } \forall x \in [0,1]\Big(n \ge N \Rightarrow |f_n(x) - f(x)| < \epsilon\Big).$$

2. *Important*: an even more complete version of 2. would be to insert the implicit quatifier for n and write

$$\forall \epsilon > 0\ \exists N \text{ such that } \forall x \in [0,1]\ \forall n\ \Big(n \ge N \Rightarrow |f_n(x) - f(x)| < \epsilon\Big),$$

   or equivalently

$$\forall \epsilon > 0\ \exists N \text{ such that } \forall x \in [0,1]\ \forall n \ge N\ \Big(|f_n(x) - f(x)| < \epsilon\Big).$$

   It is *essential* that all quantifiers be included in this way if one is to obtain the negation of this statement correctly; i.e.

$$\exists \epsilon > 0 \text{ s.t. } \forall N\ \exists x \in [0,1]\ \exists n \text{ s.t. } \neg\Big(n \ge N \Rightarrow |f_n(x) - f(x)| < \epsilon\Big).$$

   or equivalently

$$\exists \epsilon > 0 \text{ s.t. } \forall N\ \exists x \in [0,1]\ \exists n \text{ s.t. } \Big(n \ge N \wedge |f_n(x) - f(x)| \ge \epsilon\Big),$$

   or

$$\exists \epsilon > 0 \text{ s.t. } \forall N\ \exists x \in [0,1]\ \exists n \ge N \text{ s.t. } \Big(|f_n(x) - f(x)| \ge \epsilon\Big).$$

3 The Real Number System

Problem 3.1 Since $a + b \le \sup A + \sup B$ for all $a \in A$ and $b \in B$, i.e. $c \le \sup A + \sup B$ for all $c \in C$; it follows $\sup A + \sup B$ is an upper bound for C. Hence

$$\sup C \le \sup A + \sup B, \tag{5}$$

since $\sup C$ is the *least* upper bound.

Next suppose $\epsilon > 0$. Then there exists $a \in A$ such that $a \ge \sup A - \epsilon$ and there exists $b \in B$ such that $b \ge \sup B - \epsilon$. Hence

$$a + b \ge \sup A + \sup B - 2\epsilon.$$

But $a + b \in C$, and so

$$\sup C \ge a + b.$$

It follows that

$$\sup C \ge \sup A + \sup B - 2\epsilon.$$

Since $\epsilon > 0$ is otherwise arbitrary, it follows that

$$\sup C \ge \sup A + \sup B.$$

Hence, using (5),

$$\sup C = \sup A + \sup B.$$

Problem 3.2 Let

$$M = \sup_{x\in[a,b]} f(x), \quad K = \sup_{x\in[a,b]} g(x).$$

Then

$$f(x) \le M \text{ and } g(x) \le K \quad \forall x \in [a,b].$$

Hence

$$f(x) + g(x) \le M + K \quad \forall x \in [a,b];$$

i.e., $M + K$ is an upper bound for $S = \Big\{ f(x) + g(x) : x \in [a,b] \Big\}$. Since $\sup_{x\in[a,b]} \big(f(x) + g(x)\big)$ is the *least* upper bound for S, it follows

$$\sup_{x\in[a,b]} \Big(f(x) + g(x)\Big) \le M + K,$$

as required.

A simple counterexample to equality is given by $f(x) = x$ and $g(x) = 1 - x$ for $x \in [0,1]$. Then $f(x) + g(x) = 1$. The *sup* for all three functions is 1.

Comment It is *not* necessarily true that $\sup_{x\in[a,b]} f(x)$ equals $f(x)$ for some $x \in [a,b]$. For example, let

$$f(x) = \begin{cases} -|x| & x \in [-1,]\backslash\{0\} \\ -1 & x = 0 \end{cases}$$

Problem 3.3 First note that

$$\begin{aligned} a^{-1}a &= aa^{-1} \\ &= 1 \end{aligned}$$

by the commutative axiom for multiplication and the multiplicative inverse axiom. Thus

$$a^{-1}a = aa^{-1} = 1. \tag{6}$$

Similarly

$$1a = a1 = a \tag{7}$$

by the commutative axiom for multiplication and the multiplicative identity axiom.

(ai) One has

$$\begin{aligned} a(ba^{-1}) &= (ba^{-1})a && \text{commutative axiom for multiplication} \\ &= b(a^{-1}a) && \text{associative axiom for multiplication} \\ &= b1 && \text{from (6)} \\ &= b && \text{from (7).} \end{aligned}$$

Hence $ax = b$ *if* $x = ba^{-1}$.

ii. We need to show that ba^{-1} is the *only* value of x such that $ax = b$. In other words, we need to deduce from the assumption $ax = b$ that $x = ba^{-1}$. So assume $a \neq 0$ and $ax = b$. Then
$a^{-1}(ax) = a^{-1}b$ "=" means "is the same object as"
$\Rightarrow (a^{-1}a)x = a^{-1}b$ assoc. axiom for multiplication
$\Rightarrow 1x = a^{-1}b$ from (6)
$\Rightarrow x = a^{-1}b$ from (7)
$\Rightarrow x = ba^{-1}$ commutative axiom for multiplication

Thus we have shown that there exists one, and only one, number x such that $ax = b$. Moreover, $x = ba^{-1}$.

(b) a(0+0) = a 0 since 0+0 = 0 from the additive identity axiom
$\Rightarrow a0 + a0 = a0$ distributive axiom
$\Rightarrow (a0 + a0) + -(a0) = a0 + -(a0)$
$\Rightarrow a0 + (a0 + -(a0)) = a0 + -(a0)$ associative axiom for addition
$\Rightarrow a0 + 0 = 0$ additive inverse axiom applied twice
$\Rightarrow a0 = 0$ additive identity axiom

Problem 3.4 Let $\alpha = \sup A$ and $\beta = \inf B$. (We assume $A, B \neq \emptyset$. Note that we may have $\inf B = 0$, in which case we interpret $\sup A / \inf B$ as $+\infty$.)

Any $c \in C$ can be written as $c = a/b$ where $a \in A$ and $b \in B$. Since $a \leq \alpha$ and $b \geq \beta$, it follows that $c = a/b \leq \alpha/\beta$.[5] Hence α/β is an upper bound for C. Hence

$$\sup C \leq \alpha/\beta, \tag{8}$$

since $\sup C$ is the *least* upper bound.

Next suppose $\epsilon > 0$. Then[6] there exist $a \in A$ such that $a \geq \alpha - \epsilon$ and there exists $b \in B$ such that $b \leq \beta + \epsilon$. Hence

$$\frac{a}{b} \geq \frac{\alpha - \epsilon}{\beta + \epsilon}. \tag{9}$$

Given $\delta > 0$, it is possible to choose $\epsilon > 0$ so that

$$\frac{\alpha - \epsilon}{\beta + \epsilon} \geq \frac{\alpha}{\beta} - \delta. \tag{10}$$

(This is clear. More precisely, a calculation shows it is sufficient to choose

$$\epsilon \leq \frac{\beta\delta}{\alpha + \beta - \delta\beta},$$

provided $\alpha + \beta - \delta\beta > 0$. But this latter condition is true provided $\delta < \frac{\alpha+\beta}{\beta}$, and if (10) is true for some $\delta < \frac{\alpha+\beta}{\beta}$ it is certainly true for all larger δ.)

Hence given $\delta > 0$, it follows from (9) and (10) that there exists $c \in C$ for which

$$c \geq \frac{\alpha}{\beta} - \delta.$$

Hence

$$\sup C \geq \frac{\alpha}{\beta} - \delta.$$

Since $\delta > 0$ is otherwise arbitrary, it follows that

$$\sup C \geq \frac{\alpha}{\beta}.$$

Hence, using (8),

$$\sup C = \frac{\sup A}{\inf B}.$$

[5] You may assume the usual algebraic properties of "$<$", "$\leq$", etc. in this question.

[6] From the definition of "sup" we have (i) $a \leq \sup A$ for all $a \in A$, and (ii) for each $\epsilon > 0$ there exists $a \in A$ such that $a \geq \sup A - \epsilon$. Moreover, $\sup A$ is the *unique* real number with these two properties. This is a useful fact that you should remember, and which we use in this problem.

Problem 3.5 1. From the comments at the beginning of the question, $-(-a)$ is *uniquely* determined by the property

$$(-a) + (-(-a)) = 0.$$

But we also have

$$\begin{aligned} (-a) + a &= a + (-a) \quad \text{commutative axiom for addition} \\ &= 0 \quad \text{additive inverse axiom.} \end{aligned}$$

Hence $-(-a) = a$.

2. From the comments at the beginning of the question, $-x$ is *uniquely* determined by the property

$$x + (-x) = 0.$$

But we also have

$$\begin{aligned} x + (-1)x &= x.1 + (-1)x \quad \text{multiplicative identity axiom} \\ &= x.1 + x(-1) \quad \text{commutative axiom for multiplication} \\ &= x(1 + (-1)) \quad \text{distributive axiom} \\ &= x.0 \quad \text{additive inverse axiom} \\ &= 0 \quad \text{earlier problem.} \end{aligned}$$

Hence $(-1)x = -x$.

3. From the comments at the beginning of the question, $-(ab)$ is *uniquely* determined by the property

$$ab + (-(ab)) = 0.$$

But we also have

$$\begin{aligned} ab + a(-b) &= a(b + (-b)) \quad \text{distributive axiom} \\ &= a0 \quad \text{additive inverse axiom} \\ &= 0 \quad \text{earlier problem.} \end{aligned}$$

Hence $a(-b) = -(ab)$.

Also

$$\begin{aligned} ab + (-a)b &= ba + b(-a) \quad \text{commutative axiom twice} \\ &= b(a + (-a)) \quad \text{distributive axiom} \\ &= b0 \quad \text{additive inverse axiom} \\ &= 0 \quad \text{earlier problem.} \end{aligned}$$

Hence $-(ab) = (-a)b$.

Problem 3.6 1. We have that α is the least real number such that $\alpha \geq a$ for all $a \in A$. So if $\alpha \in A$ it must be the maximum element of A. Conversely if there is a maximum element $\beta \in A$, then certainly β is an upper bound for A and no lesser number can be, so $\beta = \sup A$.

2. Suppose that $\alpha \notin A$, and that $\varepsilon > 0$ is such that there are only finitely many elements of A greater than $\alpha - \varepsilon$, say $a_1, \ldots, a_k$. The largest of these, say, a_j, is clearly an upperbound for A, yet cannot equal α since $\alpha \notin A$. This contradiction shows no such ε exists.

Comment Alternatively, one can argue inductively that for each $n \in \mathbb{N}$, there is $a_n \in A$ with $a_n > \alpha - 1/n$, and $a_n > a_j$ for $1 \leq j < n$. Then for any $\varepsilon > 0$, $1/n < \varepsilon$ for $n > N$ and so the infinite set $(a_n)_{n\,N}$ lies in $A \cap (\alpha - \varepsilon, \alpha)$.

4 Set Theory

Problem 4.1 Since $S = \mathbb{Q} \times \mathbb{Q}$ and $\mathbb{Q}$ is countable, it follows that S is countable from Theorem 4.9.1.

Problem 4.2 The map $S \mapsto f$, where

$$f(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

defines a one-one map from $\mathcal{P}[a,b]$ into $F[a,b]$. Thus the cardinality of $F[a,b]$ is $\geq$ the cardinality of $\mathcal{P}[a,b]$, which as we saw in Theorem 4.10.4 is $> c$.

NOTE: One can show that $\mathcal{P}[a,b]$ and $F[a,b]$ have the same cardinality.

Problem 4.3 Let $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ be enumerations of A and B respectively.

1. If A and B are disjoint then

$$a_1, b_1, a_2, b_2, \ldots$$

is an enumeration of $A \cup B$.

2. If $A \cap B \neq \emptyset$ then let $c_1, c_2, \ldots$ be an enumeration of $C = B \setminus A$ (obtained by proceeding through the enumeration of B and only including terms in B which are not also in A). This enumeration may terminate (i.e. $B \setminus A$ is finite) or may not terminate (i.e. $B \setminus A$ is not finite).

Then $A \cup B = A \cup C$, but A and C are disjoint. We can thus enumerate $A \cup C$ as in (a), with an easy modification in case C is finite.

Problem 4.4 Let A_f be the family of all finite subsets of A. Let A_n be the family of all subsets of cardinality n (where n is any natural number), i.e. the family of all subsets of A with exactly n members.

Then

$$A_f = \{\emptyset\} \cup A_1 \cup A_2 \cup \cdots.$$

Thus to prove A_f is countable it is sufficient by Theorem 4.9.1(3) to prove that $A_1, A_2, \ldots$ are countable.

Let

$$a_1, a_2, \ldots$$

be an enumeration of A. Then

$$\{a_1\}, \{a_2\}, \ldots$$

is an enumeration of A_1.

To see that A_2 is countable, note that there is a one-one map from A_2 into $A \times A$ given by $\{a_i, a_j\}$ is mapped to (a_p, a_q) where p is the minimum of i and j, and q is the other index (e.g. $\{a_3, a_5\} = \{a_5, a_3\}$ maps to (a_3, a_5)). Since $A \times A$ is countable by Theorem 4.9.1(2), it follows A_2 is countable by Proposition 4.5.2 .

Similarly there is a one-one map from A_3 into $A \times A \times A$, obtained from arranging the indices of the members of $\{a_i, a_j, a_k\}$ in increasing order. But $A \times A \times A$ is countable by two applications of Theorem 4.9.1(2)[7]

Similarly A_n is countable for any integer n.

It now follows from Theorem 4.9.1(3) that A_f is countable, and hence is denumerable as it is certainly not finite.

Problem 4.5 Without loss of generality we may take the denumerable set to be $\mathbb{N}$.

There is a one-one correspondence between the set S_1 of all subsets of $\mathbb{N}$ and the set S_2 of all sequences of the form

$$a_1, a_2, a_3, \ldots \tag{11}$$

where every a_i is either 0 or 1. Namely, if $A \in S_1$ then the corresponding sequence (11) is given by $a_1 = 0, 1$ according as $1 \notin A, 1 \in A$; $a_2 = 0, 1$ according as $2 \notin A, 2 \in A$; $a_3 = 0, 1$ according as $3 \notin A, 3 \in A$; etc. Hence S_1 and S_2 have the same cardnality.

Claim: The set S_2 has cardinality c.

To see this first note that every real number in $[0, 1]$ corresponds to a member of S_2 by taking its binary expansion, i.e. expansion to base 2. The map is

$$\cdot a_1 a_2 a_3 \ldots \mapsto (a_1, a_2, a_3, \ldots).$$

If there is more than one expansion, which occurs only for numbers of the form

$$\cdot a_1 a_2 a_3 \ldots a_n 1000 \ldots = \cdot a_1 a_2 a_3 \ldots a_n 0111 \ldots,$$

we take the expansion ending in zeros. This gives a one-one map from $[0, 1]$ into S_2.

One simple way of getting a one-one map from S_2 into $[0, 1]$ is to use the usual *decimal* expansions to base 10 and take the map

$$(a_1, a_2, a_3, \ldots) \mapsto \cdot a_1 a_2 a_3.$$

This is one-one (but not of course onto).

[7] **Remark:** For any sets A, B and C there is a one-one correspondence between $A \times B \times C$ and $(A \times B) \times C$, namely $(a, b, c) \leftrightarrow ((a, b), c)$. If A, B and C are all countable then $A \times B$ is countable by Theorem 4.9.1(2) and then $(A \times B) \times C$ is countable by another application of Theorem 4.9.1(2). By the one-one correspondence it follows that $A \times B \times C$ is also countable.

Thus by Schröder-Bernstein the claim follows. Hence S_1 also has cardinality c.

Problem 4.6 1. We are given that A has cardinality c and $B \subset A$ is denumerable. Let B' be a denumerable subset of $A \setminus B$.

To construct B' first choose

$$x_1 \in A \setminus B,$$

then choose

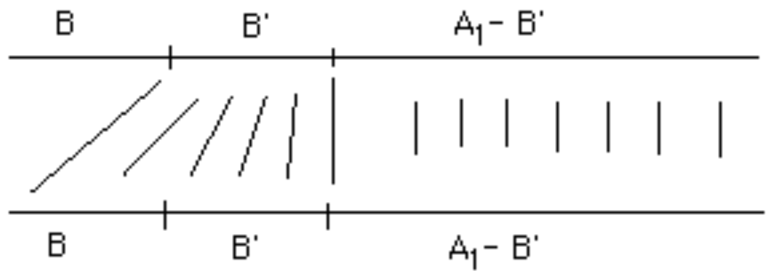
$$x_2 \in A \setminus (B \cup \{x_1\}),$$

then choose

$$x_3 \in A \setminus (B \cup \{x_1, x_2\}),$$

etc. This is always possible, as otherwise A is the union of two countable sets B and $\{x_1, \ldots, x_n\}$ (for some n) and so is countable. Now let $B' = \{x_1, x_2, \ldots\}$.

Since B and B' are denumerable, so is $B \cup B'$ by Problem 4.3, and so there is a one-one correspondence between B' and $B \cup B'$. Together with the identity one-one correspondence between $A \setminus B'$ and itself, this gives a one-one correspondence between A_1 and A.



2. Since the set of irrationals is $\mathbb{R} \setminus \mathbb{Q}$, the result follows from 1.

Problem 4.7 Let S be the set of all finite tuples of integers $(a_0, \ldots, a_n)$ for any natural number n. If $\alpha = (a_0, \ldots, a_n)$ let A_α be the set of real algebraic numbers which are solutions of $a_0 + a_1x^2 + \cdots + a_nx^n = 0$. There can be at most n solutions[8] and so the cardinality of A_α is at most n, and is certainly countable.

[8]Prove this by induction on n. If $n = 1$ it is clearly true. Assume the result for $n = k$. If λ is a solution of $Q(x) := a_0 + a_1x^2 + \cdots + a_{k+1}x^{k+1} = 0$ then the remainder after dividing $Q(x)$ by $x - \lambda$ must be zero and so $Q(x) = (x - \lambda)P(x)$ where $P(x)$ is a polynomial of degree k. Every solution of $Q(x) = 0$ other than $x = \lambda$ must thus be a solution of $P(x) = 0$. It follows from the inductive hypothesis that there can be at most $k + 1$ solutions of $Q(x) = 0$.

If A is the set of all real algebraic numbers, then $A = \bigcup_{\alpha \in S} A_\alpha$. Now S is countable by repeated applications of Theorem 4.9.1(1) of the Notes. Hence A is countable by Theorem 4.9.1(3). But A is certainly not finite (it contains the integers) and so must be denumerable.

Note The same result and proof shows that the set of all algebraic numbers (including the complex ones) is denumerable.

Problem 4.8

1. The set of all integer multiples of 5 is the set

$$A = \{\ldots, -15, -10, -5, 0, 5, 10, 15, \ldots\}.$$

This is in one-one corrrespondence with the set $\mathbb{Z}$ via the map

$$5n \leftrightarrow n.$$

Since we already know $\mathbb{Z}$ is denumerable, it follows that A is denumerable.

2. Since A is denumerable, we can write

$$A = \{a_1, a_2, a_3, \ldots\}.$$

We can also write

$$B = \{b_1, \ldots, b_n\}$$

for some natural number n (unless $B = \emptyset$, in which case the result is trivial).

Since A and B are disjoint,

$$A \cup B = \{b_1, \ldots, b_n, a_1, a_2, \ldots\}.$$

This immediately gives an enumeration of $A \cup B$, i.e. a one-one correspondence with $\mathbb{N}$, via the map

$$f(1) = b_1,\ f(2) = b_2, \ldots,\ f(n) = b_n,\ f(n+1) = a_1,\ f(n+2) = a_2, \ldots.$$

Thus $A \cup B$ is denumerable.

3. If A and B are not necessarily disjoint, then let $C = B \setminus A$.[9] It follows that $A \cup C = A \cup B$. But A and C are disjoint, and C is finite, and so $A \cup C$ is denumerable by part 2.

4. The set of all complex numbers of the form $a + bi$, where a and b are rational, is equivalent to the set $\mathbb{Q} \times \mathbb{Q}$ via the map

$$a + bi \leftrightarrow (a, b).$$

Since $\mathbb{Q} \times \mathbb{Q}$ is a product of denumerable sets, it is denumerable. This give the result.

[9] i.e. C consists of those elements of B *not* in A.

Comments

1. Remember that integers can be either positive or negative, whereas the natural numbers are $1, 2, 3, \ldots$.

2. The set of all complex numbers of the form $a + bi$, where a and b are rational, is not the *same* as the set $\mathbb{Q} \times \mathbb{Q}$; it is *equivalent* to the set $\mathbb{Q} \times \mathbb{Q}$.

3. Do not write A/B for $A \setminus B$. The first notation has a different meaning and, for example, in the theory of vector spaces is used to denote a certain "quotient space".

Problem 4.9 Since $f_1 : A \to B$ is one-one, $\overline{\overline{A}} \le \overline{\overline{B}}$ from the definition of $\le$. Since $f_2 : A \to B$ is onto, $\overline{\overline{B}} \le \overline{\overline{A}}$ by Theorem 4.8.4. It follows from the Schröder-Bernstein theorem that $\overline{\overline{A}} = \overline{\overline{B}}$.

Comment In the proof the Schröder-Bernstein theorem is needed. Since this is a deep and non-obvious result, you should explicitly note it in your proof.

Problem 4.10 1. First suppose that $x \in A \cup (B \cap C)$. Then $x \in A$ *or*[10] $x \in B \cap C$. In the first case it follows that $x \in A \cup B$ and $x \in A \cup C$, and so $x \in (A \cup B) \cap (A \cup C)$. In the second case $x \in B$ and $x \in C$, and so in particular $x \in A \cup B$ and $x \in A \cup C$, and hence $x \in (A \cup B) \cap (A \cup C)$.

2. (a) We first prove

$$f^{-1}[U \cup V] = f^{-1}[U] \cup f^{-1}[V].$$

Suppose that $x \in f^{-1}[U \cup V]$. This means $f(x) \in U \cup V$. Hence $f(x) \in U$ or $f(x) \in V$, i.e. $x \in f^{-1}[U]$ or $x \in f^{-1}[V]$, and so $x \in f^{-1}[U] \cup f^{-1}[V]$.

Conversely, suppose $x \in f^{-1}[U] \cup f^{-1}[V]$. Hence $x \in f^{-1}[U]$ or $x \in f^{-1}[V]$, i.e. $f(x) \in U$ or $f(x) \in V$. Hence $f(x) \in U \cup V$, i.e. $x \in f^{-1}[U \cup V]$.

(b) We next prove

$$f^{-1}\Big[\bigcup_{\lambda \in J} U_\lambda\Big] = \bigcup_{\lambda \in J} f^{-1}\left[U_\lambda\right].$$

(The proof is essentially the same as for the previous case, and you should carefully note the similarities.)

[10]As always in mathematics, *or* includes the possibility that both alternatives are true.

Suppose $x \in f^{-1}\Big[\bigcup_{\lambda\in J} U_\lambda\Big]$. This means $f(x) \in \bigcup_{\lambda\in J} U_\lambda$. Hence $f(x) \in U_\lambda$ for some (i.e. at least one) $\lambda \in J$, i.e. $x \in f^{-1}[U_\lambda]$ for the same $\lambda \in J$, and so $x \in \bigcup_{\lambda\in J} f^{-1}[U_\lambda]$.

Conversely, suppose $x \in \bigcup_{\lambda\in J} f^{-1}[U_\lambda]$. Hence $x \in f^{-1}[U_\lambda]$ for some $\lambda \in J$, i.e. $f(x) \in U_\lambda$ for some $\lambda \in J$. Hence $f(x) \in \bigcup_{\lambda\in J} U_\lambda$, i.e. $x \in f^{-1}\Big[\bigcup_{\lambda\in J} U_\lambda\Big]$.

3. (a) We first prove

$$f[C \cap D] \subset f[C] \cap f[D].$$

To do this, suppose $y \in f[C \cap D]$. This means $y = f(x)$ for some $x \in C \cap D$. In particular, $x \in C$ and so $y\big(= f(x)\big) \in f[C]$. Similarly, $x \in D$ and so $y\big(= f(x)\big) \in f[D]$. It follows that $y \in f[C] \cap f[D]$. This proves the result.

(b) We next prove

$$f\Big[\bigcap_{\lambda\in J} C_\lambda\Big] \subset \bigcap_{\lambda\in J} f[C_\lambda].$$

(The proof is essentially the same as for the previous case, and you should carefully note the similarities.)

To do this, suppose $y \in f\Big[\bigcap_{\lambda\in J} C_\lambda\Big]$. This means $y = f(x)$ for some $x \in \bigcap_{\lambda\in J} C_\lambda$. Hence $x \in C_\lambda$ for every $\lambda \in J$ and so $y\big(= f(x)\big) \in f[C_\lambda]$ for every $\lambda \in J$. It follows that $y \in \bigcap_{\lambda\in J} f[C_\lambda]$. This proves the result.

4. Let $f : A \to B$, where $A = \{a, b, c\}$ and $B = \{x, y\}$, be given by $f(a) = x$, $f(b) = y$ and $f(c) = x$. Let $C = \{a, b\}$ and $D = \{b, c\}$. Then $f[C \cap D] = \{y\}$ and $f[C] \cap f[D] = B$.

Comments

1. It does not make any sense to "use induction on J" in part 2. First of all, J need not be countable. And even if J were denumerable, induction is still of no use. Using induction could only help us to prove the result for J being of arbitrary *finite* cardinality.

2. The inverse function f^{-1} may not exist; and your proof should not assume that it does exist.

3. It is logically incorrect in 2. to say:

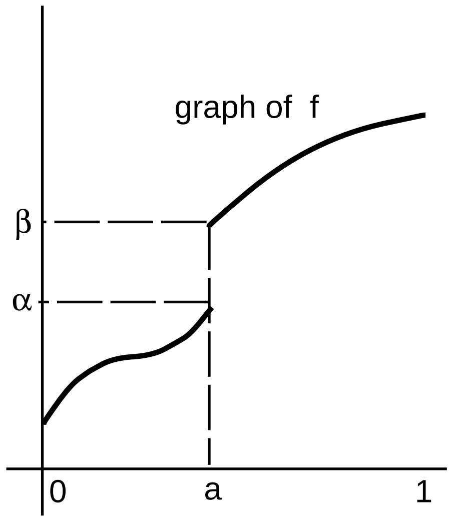
   Suppose $f(x) \in f[C \cap D]$. Then $x \in C \cap D$.

   It *is* true that "any member of $f[C \cap D]$ can be written in the form $f(x)$ for some $x \in C \cap D$". But this is *not* logically equivalent to "if $f(x) \in f[C \cap D]$ then $x \in C \cap D$". (In fact the second statement may be false. If we modify the example in 3 so that $f(a) = y$, then $f(a) \in f[C \cap D]$ but $a \notin C \cap D$.)

Problem 4.11 1. *Note:* If $a = 1$ then we only define $\lim_{x\to a^-} f(x)$, while if $a = 0$ then we only define $\lim_{x\to a^-} f(x)$.

(a) First suppose $a \in (0, 1]$ and define

$$\alpha = \sup\{f(x) : x \in [0, a)\}.$$



Note that the *sup* does exist, since $\{f(x) : x < a\}$ is bounded above by $f(a)$ (as f is increasing). We *claim* that $\lim_{x\to a^-} f(x)$ exists and equals α.

Suppose $\epsilon > 0$. From the definition of *sup* there exists $x \in [0, a)$ such that

$$\alpha - \epsilon < f(x) \leq \alpha. \tag{12}$$

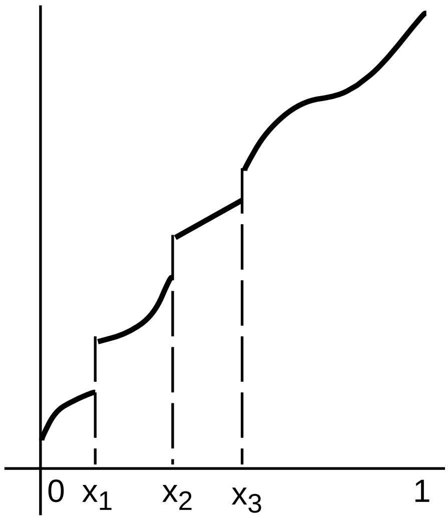
Let x_0 be one such x. Since f is increasing, it follows that (12) is true for all $x \in [x_0, a)$. It follows from the definition of $\lim_{x\to a^-} f(x)$ that $\lim_{x\to a^-} f(x) = \alpha$.

Similarly, if $a \in [0, 1)$, by considering $\beta = \inf\{f(x) : x \in (a, 1]\}$ it follows that $\lim_{x\to a^+} f(x) = \beta$.

(b) Suppose $0 < x_1 < x_2 < \ldots < x_n < 1$. Then

$$\begin{aligned} f(0) \leq \lim_{x\to x_1{}^-} f(x) \leq \lim_{x\to x_1{}^+} f(x) \leq \lim_{x\to x_2{}^-} f(x) \leq \lim_{x\to x_2{}^+} f(x) \\ \leq \ldots \leq \lim_{x\to x_n{}^-} f(x) \leq \lim_{x\to x_n{}^+} f(x) \leq f(1). \end{aligned} \tag{13}$$

(This is easy to see. For example, choose $a \in (x_1, x_2)$. Then since f is increasing, it follows that $\lim_{x\to x_1{}^+} f(x) \leq f(a) \leq \lim_{x\to x_2{}^-} f(x)$.)



If $\lim_{x\to x_i^+} f(x) - \lim_{x\to x_i^-} f(x) > \epsilon$ for $i = 1,\ldots,n$, then it follows from (13) that

$$f(1) - f(0) > n\epsilon,$$

i.e.

$$n < \frac{f(1) - f(0)}{\epsilon}.$$

Hence there are at most $\big(f(1)-f(0)\big)/\epsilon$ numbers a such that $\lim_{x\to a^+} f(x) - \lim_{x\to a^-} f(x) > \epsilon$

(c) Let

$$E_j = \{a : \lim_{x\to a^+} f(x) - \lim_{x\to a^-} f(x) > 1/j\},$$

where $j = 1, 2, \ldots$. Then f is discontinuous at a iff $a \in E_j$ for some j *(why?)*, i.e. iff $a \in \bigcup_{j\ge 1} E_j$. But each E_j is finite by the previous result, and so $\bigcup_{j\ge 1} E_j$ is countable, being a union of a countable family of countable (in fact finite) sets.

2. Let

$$f(x) = \begin{cases} 0 & x \in [0,1] \cap \mathbb{Q} \\ 1 & x \in [0,1] \setminus \mathbb{Q} \end{cases}$$

Then f is discontinuous at each $a \in [0,1]$ since there are points arbitrarily close to a at which f takes the value 0, and there are points arbitrarily close to a at which f takes the value 1.

Comment The set of discontinuities need not be finite. For example, let

$$f(x) = 1 - \frac{1}{n} \quad \text{if} \quad x \in \left[1 - \frac{1}{n}, 1 - \frac{1}{n+1}\right)$$

where $n = 1, 2, \ldots$, and let $f(1) = 1$. Then f is increasing, and f is discontinuous if $x = \frac{1}{n}$ where $n = 1, 2, \ldots$.

Note, incidentally, that f *is* continuous at 1 *(why?)*.

Sketch the graph of f.

Problem 4.12 1. Let $f : X \to Y$ where $A \subset Y$.

Suppose $y \in f\,[f^{-1}[A]]$ (we want to show $y \in A$). Then $y = f(x)$ for some $x \in f^{-1}[A]$. But $x \in f^{-1}[A]$ means $f(x) \in A$, i.e. $y \in A$. Hence $f\,[f^{-1}[A]] \subset A$.

2. Let $X = \{x\}$ and $Y = A = \{p, q\}$. Let $f(x) = p$. Then $f^{-1}[A] = \{x\}$ and $f\,[f^{-1}[A]] = \{p\} \neq A$.

3. (i) It is easiest to use polar coordinates $x = r\cos\theta,\ y = r\sin\theta$. Then

$$f[A] = \{(r, r\cos\theta + r\sin\theta) : 0 \le r \le a,\ 0 \le \theta \le 2\pi\}.$$

But

$$\begin{aligned} r\cos\theta + r\sin\theta &= r(\cos\theta + \sin\theta) \\ &= \sqrt{2}r(\cos\frac{\pi}{4}\cos\theta + \sin\frac{\pi}{4}\sin\theta) \\ &= \sqrt{2}r\cos(\theta - \frac{\pi}{4}). \end{aligned}$$

Since $-\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$, we see $\cos(\theta - \frac{\pi}{4})$ takes all values in $[-1, 1]$.

Hence

$$f[A] = \{(r, s) : 0 \le r \le a,\ -\sqrt{2}r \le s \le \sqrt{2}r\}.$$

See the following diagram.

(ii) We have

$$f^{-1}[A] = \left\{(x, y) : \left((x^2 + y^2) + (x + y)^2\right)^{1/2} \le a\right\}.$$

But

$$\begin{aligned} \left((x^2 + y^2) + (x + y)^2\right)^{1/2} &\le a \\ \text{iff} \quad x^2 + y^2 + (x + y)^2 &\le a^2 \\ \text{iff} \quad x^2 + y^2 + xy &\le a^2/2. \end{aligned}$$

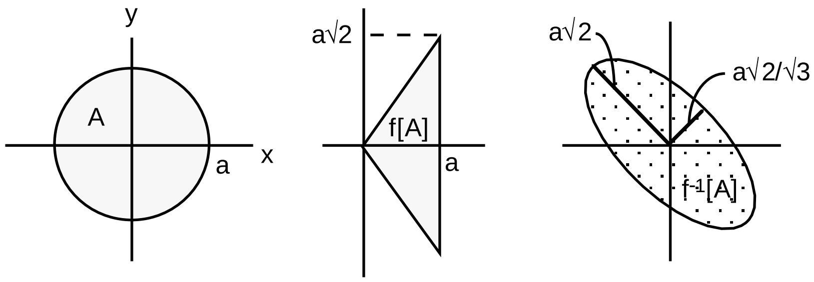
Thus

$$f^{-1}[A] = \left\{(x, y) : x^2 + y^2 + xy \le a^2/2\right\}.$$

This can be written in the form

$$f^{-1}[A] = \left\{(x, y) : \frac{3}{4}(x + y)^2 + \frac{1}{4}(x - y)^2 \le \frac{a^2}{2}\right\},$$

which shows that $f^{-1}[A]$ is bounded by an ellipse with major and minor axes of length $a\sqrt{2}/\sqrt{3}$ and $a\sqrt{2}$ respectively, as shown in the following diagram.



Remarks

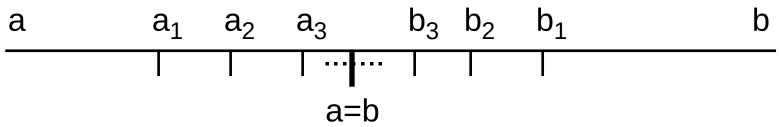
1. $f^{-1}(x)$ does not make sense unless the function f is is one-one and onto, and hence has an inverse. But $f^{-1}[A]$ makes sense for any f, provided A is a subset of the codomain.

Problem 4.13 1. Suppose $i < j$. Then

$$[a_i, b_i] \subset [a_j, b_j]$$

and so

$$a_i \le a_j \le b_j \le b_i.$$



It follows

$$a_1 \le a_2 \le a_3 \le \cdots \le b_3 \le b_2 \le b_1.$$

In particular, the set $\{a_1, a_2, a_3, \ldots\}$ is bounded above by any b_n and so has a l.u.b. a, say. Similarly, $\{b_1, b_2, b_3, \ldots\}$ is bounded below by any a_n and so has a g.l.b. b, say. Moreover $a \le b$.

Proof of $a \le b$. We know a is the *least* upper bound of $\{a_1, a_2, a_3, \ldots\}$. But any b_n is also an upper bound and so $a \le b_n$ for all n. Hence a is a *lower* bound for $\{b_1, b_2, b_3, \ldots\}$. Hence $a \le b$ as b is the *greatest* lower bound of $\{b_1, b_2, b_3, \ldots\}$. ∎

(*Note* that so far we have not used the fact that the intervals I_n are closed.)

Since

$$a_n \le a \le b \le b_n$$

for all n we see that

$$[a, b] \subset [a_n, b_n] = I_n$$

for all n. As $a \leq b$ it follows there exists $x \in I_n$ for all n, just take any $x \in [a, b]$. (*Note* that the last few lines use the fact that the I_n are closed. What goes wrong if the I_n are open?)

To see that there is a *unique* $x \in I_n$ for all n, assume $x_1, x_2 \in I_n$ for all n and $x_1 < x_2$. Then $[x_1, x_2] \subset I_n$ for all n (as each I_n is an interval). But this implies

$$\text{length}\, I_n \geq x_2 - x_1$$

for all n, which contradicts the fact length $I_n \to 0$.

2. We can in fact show that the result in 1. is false if $\mathbb{R}$ is replaced by $\mathbf{Q}$ and the intervals I_n have *rational* endpoints.

   For example, take an increasing sequence of rational numbers $a_n \to \sqrt{2}$ and a decreasing sequence of rational numbers $b_n \to \sqrt{2}$.[11] Then there is no rational number x belonging to all the $I_n = [a_n, b_n]$, since the unique number in all the I_n is $\sqrt{2}$ and this is irrational.

3. (Although not explicitly stated, it is intended that the intervals (a_n, b_n) in the counterexample should be non-empty, as otherwise the result is trivial.)

   Let $I_n = (0, 1/n)$. Then the intersection of all the I_n is empty.

We finally prove $[a, b]$ is uncountable.

Suppose (in order to obtain a contradiction) that $[a, b]$ is countable. Let $x_1, x_2, x_3, \ldots, x_n, \ldots$ be a sequence which enumerates $[a, b]$. Divide $[a, b]$ into 3 intervals $[a, a+(b-a)/3]$, $[a+(b-a)/3, a+2(b-a)/3]$ and $[a+2(b-a)/3, b]$. Then for at least one of these intervals, which we call I_1, we have $x_1 \notin I_1$ (why do we need to divide $[a, b]$ into 3, and not 2, parts for this to be true?).

Now divide I_1 into 3 intervals. For at least one of these intervals, which we call I_2, we have $x_2 \notin I_1$.

Continuing in this way we obtain a decreasing sequence of closed intervals $I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$ and such that length$I_n \to 0$ as $n \to \infty$. By the previous part of the question, there exist an x such that $x \in I_n$ for every n. It follows that for each n, $x \neq x_n$, since $x_n \notin I_n$. Hence x is not a term in the sequence $x_1, x_2, x_3, \ldots, x_n, \ldots$. Thus for *any* sequence of numbers from $[a, b]$ there is a member of $[a, b]$ not in the sequence. Thus $[a, b]$ is not countable.

Remarks

[11] This is possible. For example, let

$$a_n = \text{the } n\text{th decimal approximation to } \sqrt{2}$$

and let

$$b_n = 2 - \Big(\text{the } n\text{th decimal approximation to } 2 - \sqrt{2}\Big).$$

Why does this work?

1. It is incorrect to use induction in this Problem and argue along the following lines:

   Let P_n be the property "$\exists x$ such that $x \in I_1 \cap \cdots \cap I_n$". Since P_1 is true and since $P_n \Rightarrow P_{n+1}$, then "$\exists x$ such that $x \in$ every I_n".

   This is totally erroneous. It is indeed the case that P_n is true for every n, but this does *not* imply "$\exists x$ such that $x \in I_1 \cap I_2 \cap \cdots \cap I_n \cap \cdots$".

   For example, let $I_n = (0, 1/n)$. Then $I_1 \cap I_2 \cap \cdots \cap I_n \cap \cdots = \emptyset$. But P_n is true for every n.

2. Do not used undefined notation such as $\lim_{n\to\infty} I_n$. It is not at all clear what this means.

   (a) Write length $I_n \to 0$ if this is what you mean, and not $I_n \to 0$.

   (b) Does

   $$\lim_{n\to\infty} [a_n, b_n] \quad \text{mean} \quad \left[\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n\right]?$$

   If so, say it. And then you must justify the existence of $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$.

   (c) And does $\lim_{n\to\infty}(-1/n, 1/n) = \{0\}$ or $= \emptyset$?

   All this indicates the need to be very precise.

Problem 4.14 Let $\{A_i\}_{i\geq 1}$ be a countable family of countable sets. Write

$$\begin{array}{rcl} A_1 & = & \{a_{11}\ a_{12}\ a_{13}\ a_{14}\ \ldots\} \\ A_2 & = & \{a_{21}\ a_{22}\ a_{23}\ a_{24}\ \ldots\} \\ A_3 & = & \{a_{31}\ a_{32}\ a_{33}\ a_{34}\ \ldots\} \\ \vdots & = & \qquad \vdots \end{array}$$

Modifications: If any A_i is empty, omit it from the sequence. If any A_i is finite, say $A_i = \{a_{i1}, \ldots, a_{in}\}$, take $a_{i\,n+1}, a_{i\,n+2}, \ldots = a_{in}$. If there are only a finite number of A_i's, say $A_1, \ldots, A_k$, set $A_{k+1}, A_{k+2}, \ldots = A_k$. The only case not included is if all the A_i are empty, but the result is trivial in this case.

Define

$$g : \mathbb{N} \times \mathbb{N} \to \bigcup_{i\geq 1} A_i$$

by

$$g(i, j) = a_{ij}.$$

Since g is clearly onto, we have from Proposition 4.8.4 and the fact $\mathbb{N} \times \mathbb{N}$ is countable that

$$\overline{\overline{\bigcup_{i\geq 1} A_i}} \leq \overline{\overline{\mathbb{N} \times \mathbb{N}}} = d.$$

Thus $\bigcup_{i\geq 1} A_i$ is countable.

Remarks Directly writing down some enumeration of $\bigcup_{i\geq 1} A_i$ is not answering the Question *as posed*. The Question was to *use* Proposition 4.8.4 and the fact $\mathbb{N}\times\mathbb{N}$ is countable *in order to* prove the countability of $\bigcup_{i\geq 1} A_i$, i.e. *in order to* prove that there is indeed an enumeration of $\bigcup_{i\geq 1} A_i$.

Problem 4.15

1. Let B^* be the set of all elements in B which are *not* in A. Then
$$A \cup B = A \cup B^*$$
and B^* is *disjoint* from A.

   Choose a denumerable set $B' \subset A$ (as in the proof of Problem 4.6).

   Then
$$A = (A \setminus B') \cup B',$$
where $A \setminus B'$ and B' are disjoint. Hence
$$A \cup B = A \cup B^* = (A \setminus B') \cup B' \cup B^*$$
where $A \setminus B'$ and $B' \cup B^*$ are disjoint.

   There is a one-one correspondence between $A \setminus B'$ and itself (just the identity map); and a one-one correspondence between B' and $B' \cup B^*$, since both are denumerable.

   This gives a one-one correspondence between A and $A \cup B$. Thus $\overline{\overline{A \cup B}} = \overline{\overline{A}}$.

2. Let A be the set of irrationals and $B = \mathbb{Q}$. Then from part 1, since $\mathbb{Q}$ has cardinality d,
$$\overline{\overline{A}} = \overline{\overline{A \cup B}} = \overline{\overline{\mathbb{R}}} = c.$$

Remarks Do not assume that A and B were disjoint from each other, nor that $B \subset A$. Neither need be the case!

Problem 4.16

1. Let
$$S = S_1 \cup S_2,$$
where S_1 is the set of those sequences which do *not* end in an infinite sequence of 1's, and S_2 is the set of those sequences which *do* end in an infinite sequence of 1's. Then every real number in $[0, 1]$ has a *unique* binary expansion corresponding to a member of S_1.[12] Hence $\overline{\overline{S_1}} = c$. On the other hand, the members of S_2 are in one-one correspondence with certain *rational* numbers in $[0, 1]$, and so S_2 is countable.

   It follows, that S has cardinality c from Question 2.

[12] For example, the number with expansion .10111 . . . also has the expansion .11000 . . . .

2. Let
$$S_0 = \bigcup_{n\geq 1} E_n,$$
where E_n is the set of sequences of length n. Then E_n is finite (with cardinality 2^n). Hence S_0 is the union of a denumerable number of finite sets, and so is countable by Theorem 4.9.1(3). It is clearly not finite (*why?*), and hence it is denumerable.

3. We define a one-one correspondence between $\mathcal{P}(\mathbb{N})$ (the set of all subsets of $\mathbb{N}$) and the set S as follows. If $A \subset \mathbb{N}$ then the corresponding element of S is $(a_1, a_2, a_3, \ldots, a_i, \ldots)$ where for each n, $a_n = 1$ if $n \in A$ and $a_n = 0$ if $n \notin A$.[13] Thus $\mathcal{P}(\mathbb{N})$ has cardinality c since S has cardinality c from part 1.

   There is also a map from S_0 *onto* the set of all *finite* subsets of $\mathbb{N}$, essentially defined as above. For example, the sequence $(1, 1, 0, 0, 1, 1, 0, 0)$ is mapped to the set $\{1, 2, 5, 6\}$.[14] It follows that the cardinality of the set of all *finite* subsets of $\mathbb{N}$ is $\leq$ the cardinality of S_0, which is d. Since the cardinality of the set of all *finite* subsets of $\mathbb{N}$ is clearly not finite (*why?*), it must equal d.

Problem 4.17 1. Let $x_1, x_2, \ldots, x_n, \ldots$ be an enumeration of $\mathbb{Q} \cap (0, 1)$ (this is possible as $\mathbb{Q}$ is denumerable). Suppose $\epsilon > 0$.

(a) Let $b_1 = x_1$ and let $I_1 \subseteq (0, 1)$ be an open interval containing b_1 with length $\leq \epsilon/2$ and irrational end-points.

(b) Let b_2 be the first x_i not in I_1 and let $I_2 \subseteq (0, 1)$ be an open interval containing b_2 with length $\leq \epsilon/4$ and irrational end-points which is disjoint from I_1.[15]

(c) Let b_3 be the first x_i not in $I_1 \cup I_2$ and let $I_3 \subseteq (0, 1)$ be an open interval containing b_3 with length $\leq \epsilon/8$ and irrational end-points which is disjoint from $I_1 \cup I_2$.

(d) Let b_4 be the first x_i not in $I_1 \cup I_2 \cup I_3$ and let $I_4 \subseteq (0, 1)$ be an open interval centred at b_4 with length $\leq \epsilon/16$ and irrational end-points which is disjoint from $I_1 \cup I_2 \cup I_3$.

(e) etc.

In this way we obtain a sequence of open intervals $\{I_n\}$ containing all the rationals in $(0, 1)$ and for which the sum of the lengths is $\leq \epsilon$.

[13] For example, the set $\{1, 2, 5, 6, 8, \ldots\}$ corresponds to the sequence $(1, 1, 0, 0, 1, 1, 0, 1, \ldots)$.

[14] This map is not one-one. For example, the sequence $(1, 1, 0, 0, 1, 1)$ is also mapped to the set $\{1, 2, 5, 6\}$.

[15] Since the end-points of I_1 are irrational, b_2 is *not* an end-point. We need this fact, since if b_2 *were* an end-point we could not select I_2 containing b_2 and disjoint from I_1. A similar point is rlevant in the rest of the discussion.

2. Since any open interval contains a rational number, this is clear.

3. (*Sketch*) Let

$$C_n = [0,1] \setminus \bigcup_{i=1}^{n} (a_i, b_i).$$

Then C_n consists of $n+1$ disjoint closed intervals. Moreover,

$$C_1 \supset C_2 \supset \cdots \supset C_n \supset \cdots.$$

Note that C_{n+1} is obtained from C_n by replacing one of the disjoint closed intervals corresponding to C_n by two disjoint closed subintervals.

Note also that

$$A^c = [0,1] \setminus \bigcup_{i=1}^{\infty} (a_i, b_i) = \bigcap_{n=1}^{\infty} C_n,$$

why?

Suppose $x = (x_1, x_2, \ldots, x_n, \ldots) \in S$, where S is as in Question 4.16. We first define a decreasing sequence of *closed intervals* $\{K_j\}_{j=1}^{\infty}$ as follows:

(a) According as $x_1 = 0$ or $x_1 = 1$, let K_1 be the left or right interval in $[0,1] \setminus (a_1, b_1)$.

(b) In the process of constructing $\{C_n\}_{n=1}^{\infty}$, the interval K_1 is at some stage replaced by two disjoint subintervals. Let K_2 be the left or right interval according as $x_2 = 0$ or $x_2 = 1$.

(c) In the process of constructing $\{C_n\}_{n=1}^{\infty}$, the interval K_2 is at some stage replaced by two disjoint subintervals. Let K_3 be the left or right interval according as $x_3 = 0$ or $x_3 = 1$.

(d) etc.

Then the intersection of the sets in the sequence $\{K_j\}_{j=1}^{\infty}$ is a singleton. To see this, use Problem 4.13 (the fact the length of the K_j's is converging to zero uses the fact any given rational is not in K_j for all j sufficiently large). Let $f(x)$ be the member of this singleton. Then $f(x) \in A^c$.

It is clear that f is one-one, *why?* Hence A^c has cardinality $\geq c$. But as A^c is a subset of $[0,1]$ it follows it has cardinality equal to c.

5 Vector Space Properties of $\mathbb{R}^n$

Problem 5.1 Let
$$\mathbf{x} = \epsilon_1 \mathbf{v}_1 + \cdots + \epsilon_n \mathbf{v}_n \,,$$
and
$$\mathbf{x} = \delta_1 \mathbf{v}_1 + \cdots + \delta_n \mathbf{v}_n \,.$$
where $\epsilon_1, \ldots, \epsilon_n, \delta_1 \ldots, \delta_n = 0$ or 1, be two vertices of the n-cube.

Then

$$\mathbf{x} - \mathbf{y} = \gamma_1 \mathbf{v}_1 + \cdots + \gamma_n \mathbf{v}_n$$

where each γ_i can take the values $0, 1$ or -1. It follows that

$$|\mathbf{x} - \mathbf{y}| = \sqrt{\gamma_1^2 + \cdots + \gamma_n^2}$$

can take any of the valaues $1, \sqrt{2}, \ldots \sqrt{n}$.

Problem 5.2 (a) Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be a basis for $\mathbb{R}^n$ such that $\mathbf{x}_1, \ldots, \mathbf{x}_k$ is a basis for V.

Apply the Gram-Schmidt process to $\mathbf{x}_1 \ldots, \mathbf{x}_n$ to obtain an orthonormal basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ for $\mathbb{R}^n$. Note that the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are precisely those obtained from the Gram-Schmidt process applied to $\mathbf{x}_1, \ldots, \mathbf{x}_k$ and so $\mathbf{v}_1, \ldots, \mathbf{v}_k$ give an orthonormal basis for V. If $i > k$ then $\mathbf{v}_i$ is orthogonal to $\mathbf{v}_j$ for each $j \leq k$. Since $\mathbf{v}_i$ is thus orthogonal to every member of a basis for V, it easily follows (*Exercise*) that $\mathbf{v}_i$ is orthogonal to every member of V, that is, $\mathbf{v}_i \in V^\perp$. Thus we have $n - k$ linearly independent (and orthonormal) vectors $\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n$ in $V^\perp$.

We *claim* that in fact the vectors $\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n$ *span* $V^\perp$ and thus form a basis. To see this suppose that $\mathbf{x} \in V^\perp$, say

$$\mathbf{x} = \alpha_1 \mathbf{v}_1 \cdots + \alpha_k \mathbf{v}_k + \alpha_{k+1} \mathbf{v}_{k+1} + \cdots + \alpha_n \mathbf{v}_n \,.$$

Since $\mathbf{x} \cdot \mathbf{v}_i = 0$ for $i \leq k$ it follows from the orthonormality of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ that $\alpha_i = 0$ for $i \leq k$. Thus

$$\mathbf{x} = \alpha_{k+1} \mathbf{v}_{k+1} + \cdots + \alpha_n \mathbf{v}_n \,.$$

and so $\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n$ span $V^\perp$ as claimed.

(b) If $\mathbf{x} \in \mathbb{R}^n$ then we can write

$$\mathbf{x} = \beta_1 \mathbf{v}_1 + \cdots + \beta_k \mathbf{v}_k + \beta_{k+1} \mathbf{v}_{k+1} + \cdots + \beta_n \mathbf{v}_n \,.$$

and so
$$\mathbf{x} = \mathbf{y} + \mathbf{z}$$

where

$$\mathbf{y} = \beta_1 \mathbf{v}_1 + \cdots + \beta_k \mathbf{v}_k \in V$$

and

$$\mathbf{z} = \beta_{k+1} \mathbf{v}_{k+1} + \cdots + \beta_n \mathbf{v}_n \in V^\perp .$$

We *claim* that the representation above is unique. For suppose

$$\mathbf{z} = \mathbf{y}_1 + \mathbf{z}_1 = \mathbf{y}_2 + \mathbf{z}_2$$

where $\mathbf{y}_1, \mathbf{y}_2 \in V$ and $\mathbf{z}_1, \mathbf{z}_2 \in V^\perp$. Then

$$\mathbf{y}_1 - \mathbf{y}_2 = \mathbf{x}_1 - \mathbf{z}_2 .$$

Since the left side lies in V and the right side lies in $V^\perp$, and the sides equal each other, each lies in $V \cap V^\perp$. But this latter is $\{\mathbf{0}\}$, since if $\mathbf{v} \in V \cap V^\perp$, $\mathbf{v} \cdot \mathbf{v} = 0$, so that $\mathbf{v} = \mathbf{0}$. Hence $\mathbf{y}_1 = \mathbf{y}_2$ and $\mathbf{z}_1 = \mathbf{z}_2$.

Problem 5.3 1. These all come from substituting $||z||^2 = (z, z)$ in the right hand sides for suitable choices of z. (2) is the *parallelogram law.*

2. It is clear that $(x, x) = ||x||^2$. Now the parallelogram law gives

$$\begin{aligned} ||u+v+w||^2 = ||u+v-w||^2 &= 2||u+v||^2 + 2||w||^2 \\ ||u-v+w||^2 = ||u-v-w||^2 &= 2||u-v||^2 + 2||w||^2 \end{aligned}$$

Subtracting the two gives

$$\begin{aligned} ||u+v+w||^2 - ||u-v+w||^2 &+ ||u+v-w||^2 - ||u-v-w||^2 \\ &= 2||u+v||^2 - 2||u-v||^2 \end{aligned}$$

Thus

$$(u+w, v) + (u-w, v) = 2(u, v) .$$

In particular, for $w = u$ we see that $(2u, v) = 2(u, v)$. Now take $u + v = x, u - w = y, v = z$ to obtain

$$(x, z) + (y, z) = 2(\frac{x+y}{2}, z) = (x + y, z) .$$

A simple induction now shows that $(mx, y) = m(x, y)$ and $n(x/n, y) = (nx/n, y) = (x, y)$ so that

$$\frac{m}{n}(x, y) = m(\frac{x}{n}, y) = \frac{m}{n}(x, y) ,$$

and $(\cdot, \cdot)$ is positive rational linear. But $(\cdot, \cdot)$ is continuous and so $\lambda(x, y) = (\lambda x, y)$ for $\lambda \geq 0$. For $\lambda < 0$,

$$\begin{aligned} \lambda(x, y) - (\lambda x, y) &= \lambda(x, y) - (|\lambda|(-x), y) = \lambda(x, y) - |\lambda|(-x, y) \\ &= \lambda(x, y) + \lambda(-x, y) = \lambda(0, y) = 0 . \end{aligned}$$

Thus $(\cdot, \cdot)$ is in fact real linear.

6 Metric Spaces

Problem 6.1 1. $A = \{\mathbf{x} : 0 < |\mathbf{x} - \mathbf{x}_0| \le \delta\}, \delta > 0$. Then

$$\begin{aligned} \text{int}A &= \{\mathbf{x} : 0 < |\mathbf{x} - \mathbf{x}_0| < \delta\}, \\ \partial A &= \{\mathbf{x}_0\} \cup \{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| = \delta\}, \\ \overline{A} &= \{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| \le \delta\}. \end{aligned}$$

The arguments are similar to those for Propositon 6.3.7 of the Notes. In particular, $\mathbf{x}_0 \in \partial A$ since every $B_r(\mathbf{x}_0)$ clearly contains a member of A^c, namely $\mathbf{x}_0$, together with members of A.

2. $A = \{(r\cos\theta, r\sin\theta) : 0 < r < 1, 0 < \theta < 2\pi\}$. Then

$$\begin{aligned} \text{int}A &= A \\ \partial A &= \{(r, 0) : 0 \le r \le 1\} \cup \{(\cos\theta, \sin\theta) : 0 \le \theta \le 2\pi\}, \\ \overline{A} &= \{(r\cos\theta, r\sin\theta) : 0 \le r \le 1, 0 \le \theta \le 2\pi\}. \end{aligned}$$

Thus ∂A is the positive x-axis from 0 to 1 inclusive together with the unit circle centred at the origin, while $\overline{A}$ is the "closed unit disc (ball)". All cases easily follow from the definitions.

3. For this part recall that any real number x can be approximated arbitrarily closely by rational numbers.

Moreover, x can be approximated arbitrarily closely by irrational numbers. (Add a small rational if x is irrational, add a small irrational of x is rational.)

Now let $A = \{(x, y) :$ at least one of x or y is irrational $\}$. Then

$$\text{int}A = \emptyset$$

since for $(x, y) \in A$ every $B_r((x, y))$ contains points both of whose coordinates are rational.

$$\partial A = \mathbb{R}^2$$

since for $(x, y) \in A$ every $B_r((x, y))$ contains both points of A and points of A^c.

$$\overline{A} = \mathbb{R}^2$$

by the previous comment.

Problem 6.2 1. A is not open since some points of A are not interior points, that is, $A \ne \text{int}A$. A is not closed since some limit points of A are not in A, that is $\overline{A} \not\subset A$ (Theorem 6.4.6).

2. A is open (every point is an interior point), but is not closed since some limit points of A are not in A.

3. A is neither open or closed.

Problem 6.3 Let $H = \{\mathbf{x} : \mathbf{z} \cdot \mathbf{x} < c\}$. We want to show that if $\mathbf{y} \in H$ then $B_r(\mathbf{y}) \subset H$ for some $r > 0$.

Suppose $\mathbf{y} \in H$, so that $\mathbf{z} \cdot \mathbf{y} = c' < c$. Let $\mathbf{x} \in B_r(\mathbf{y})$ for some $r > 0$. Using the triangle inequality and the Hint,

$$\begin{aligned} |\mathbf{z} \cdot \mathbf{x} - bz \cdot \mathbf{y}| &= |\mathbf{z} \cdot (\mathbf{x} - \mathbf{y})| \\ &\leq |\mathbf{z}|r. \end{aligned}$$

That is,

$$|\mathbf{z} \cdot \mathbf{x} - c'| \leq |\mathbf{z}|r\,.$$

and so

$$\mathbf{z} \cdot \mathbf{x} \leq c' + |\mathbf{z}|r < c$$

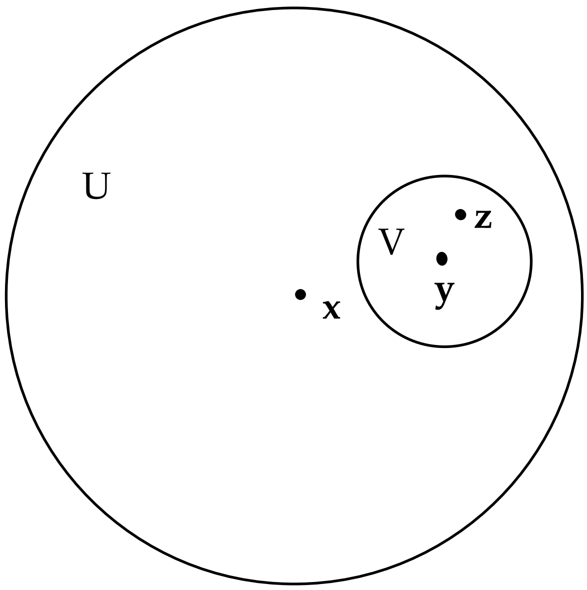
provided r is chosen sufficiently small. Thus $B_r(\mathbf{y}) \subset H$ for some $r > 0$ as required, and hence H is open.

Problem 6.4 1. We have that $x \in \partial A$ iff every $B_r(x)$ contains points of both A and A^c, that is, of A^c and $(A^c)^c$, that is, iff $x \in \partial A^c$.

2. For any set B, $B \subset \overline{B}$ from the definition of $\overline{B}$. So certainly

$$\overline{A} \subset \overline{(\overline{A})}\,.$$

Now let $a \in \overline{(\overline{A})}$, and consider $U = B_r(x)$. Then U contains a point $y \in \overline{A}$, and for $s > 0$ sufficiently small $V = B_s(y) \subset U$ (by the triangle inequality).



But V must contain a point $z \in A$, so that $z \in A \cap U$. This being the case for any $r > 0$, it follows that $a \in \overline{A}$. Hence

$$\overline{(\overline{A})} \subset \overline{A}\,.$$

It follows that $\overline{A} = \overline{(\overline{A})}$.

Problem 6.5 Just take $A = (0,1) \cup (1,2) \subset \mathbb{R}$. Then $\text{int}A = A$, but $\text{int}\overline{A} = (0,2)$.

Problem 6.6 We have A is open and B is closed. Thus $A \backslash B = A \cap B^c$ is the intersection of two open sets and so is open.

Problem 6.7

1. $\text{int}(A \cap B) \subset \text{int}A \cap \text{int}B$. If $x \in \text{int}(A \cap B)$, then $B_r(x) \subset A \cap B$ for some $r > 0$. Thus $B_r(x) \subset A$ and $B_r(x) \subset B$, so that x is an interior point of A and of B as required.
2. If $x \in \text{int}A \cap \text{int}B$, then there exist $B_{r_1}(x) \subset A$ and $B_{r_2}(x) \subset B$. But then $B_r(x) \subset A \cap B$ for $r = \min\{r_1, r_2\}$.

Problem 6.8 Suppose that $x \in \text{int}A \cup \text{int}B$, so that $x \in \text{int}A$ or $x \in \text{int}B$. It clearly suffices to consider $x \in \text{int}A$. So there is $r > 0$ such that $B_r(x) \subset A \subset A \cup B$. Thus $x \in \text{int}(A \cup B)$.

To see that equality need not hold, take $A = [0,1] \subset \mathbb{R}$, $B = [1,2] \subset \mathbb{R}$. Then $\text{int}A = (0,1)$, $\text{int}B = (1,2)$, yet $\text{int}A \cup B = (0,2) \neq (0,1) \cup (1,2)$.

Problem 6.9 It is clear that $\overline{d}$ satisfies positivity and symmetry.

The triangle inequality for $\overline{d}$ asserts

$$\overline{d}(x,y) \leq \overline{d}(x,z) + \overline{d}(x,z),$$

i.e.

$$\frac{d(x,y)}{1+d(x,y)} \leq \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)},$$

i.e.

$$\frac{d(x,y)}{1+d(x,y)} \leq \frac{d(x,z)+d(z,y)+2d(x,z)d(z,y)}{1+d(x,z)+d(z,y)+d(x,z)d(z,y)},$$

i.e.

$$\frac{d(x,y)}{1+d(x,y)} \leq \frac{\big(d(x,z)+d(z,y)+d(x,z)d(z,y)\big)+d(x,z)d(z,y)}{1+d(x,z)+d(z,y)+d(x,z)d(z,y)}. \qquad (14)$$

From the triangle inequality for d we have

$$d(x,y) \leq d(x,z) + d(z,y)$$

and so

$$d(x,y) \le d(x,z) + d(z,y) + d(x,z)d(z,y).$$

By letting

$$a = d(x,y)$$

and

$$b = d(x,z) + d(z,y) + d(x,z)d(z,y)$$

we see from (14) it is sufficient to prove that if $0 \le a \le b$ then

$$\frac{a}{1+a} \le \frac{b}{1+b}.$$

But this is equivalent to

$$a + ab \le b + ab,$$

which is certainly true.

This completes the proof of the triangle inequality for $\overline{d}$.

It remains to prove that d and $\overline{d}$ give the same collection of open sets.

As noted in the *Exercise* following Theorem 6.4.2 and concerning the Euclidean and the sup metric, it is sufficient to show *every d-ball centred at x contains a $\overline{d}$-ball centred at x, and conversely.*

Since

$$\overline{d}(x,y) = \frac{d(x,y)}{1+d(x,y)},$$

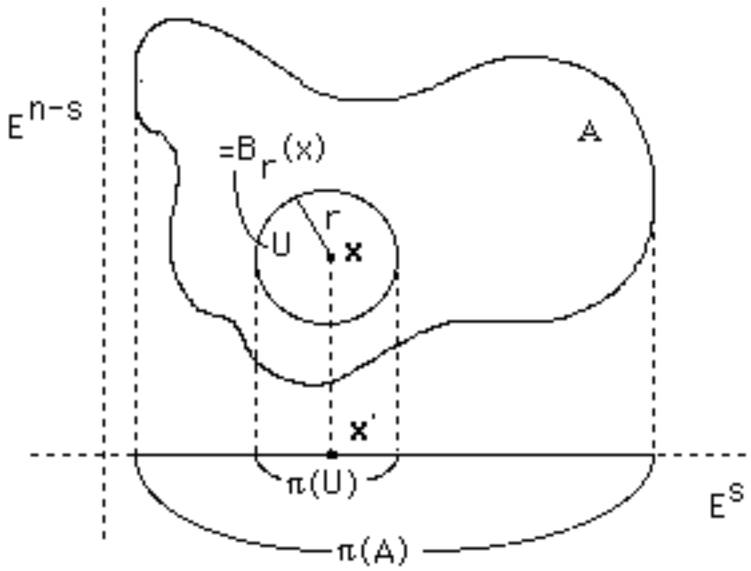
it follows

$$\{y : d(x,y) < r\} = \{y : \overline{d}(x,y) < r/(1+r)\}.$$

On the other hand, $d = \frac{\overline{d}}{1-\overline{d}}$ and so any $\overline{d}$-ball around x of radius $r < 1$ is also a d-ball around x of radius $r/(1-r)$. The $\overline{d}$-balls of radius $r \ge 1$ are the whole space and in particular contain the d-balls of radius 1.

Thus we have established the claim in italics, and so the open sets corresponding to both metrics are the same.

Problem 6.10 Write $\mathbb{R}^n = \mathbb{R}^s \times \mathbb{R}^{n-s}$. For any $\mathbf{x} \in \mathbb{R}^n$ write $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$ where $\mathbf{x}' = (x^1, \ldots, x^s)$ and $\mathbf{x}'' = (x^{s+1}, \ldots, x^{n-s})$. Let $\pi(\mathbf{x}) = \mathbf{x}'$.



Assume A is open. We claim $\pi[A]$ is open.

Take any point in $\pi[A]$, which without loss of generality we denote by $\mathbf{x}'$. Then for some $\mathbf{x}'' \in \mathbb{R}^{n-s}$ the point $\mathbf{x} := (\mathbf{x}', \mathbf{x}'') \in A$. Choose $r > 0$ such that $B_r(\mathbf{x}) \subset A$ (this is possible as A is open).

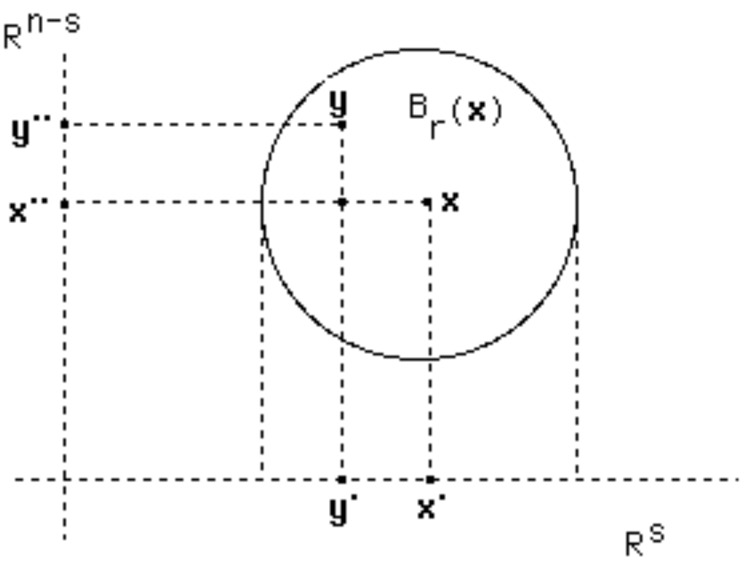
Then $\pi[B_r(\mathbf{x})] \subset \pi[A]$.[16] In the following lemma we show that

$$\pi[B_r(\mathbf{x})] = B'_r(\mathbf{x}')$$

where $B'_r(\mathbf{x}')$ is the ball in $\mathbb{R}^s$ about $\mathbf{x}'$ of radius r. It follows that $\pi[A]$ is open, as $\mathbf{x}'$ was an arbitrary point in $\pi[A]$.

Lemma *With the previous notation,*

$$\pi[B_r(\mathbf{x})] = B'_r(\mathbf{x}').$$



[16]If $A \subset B$ then $f[A] \subset f[B]$ as is easily checked.

Proof: ($\subset$): Let $\mathbf{y}'$ be any point in $\pi[B_r(\mathbf{x})]$. Thus there exists $\mathbf{y} \in [B_r(\mathbf{x})]$ such that $\mathbf{y}' = \pi(\mathbf{y})$, and so $\mathbf{y} = (\mathbf{y}', \mathbf{y}'')$ for some $\mathbf{y}'' \in \mathbb{R}^{n-s}$. Since

$$|\mathbf{y} - \mathbf{x}|^2 = |\mathbf{y}' - \mathbf{x}'|^2 + |\mathbf{y}'' - \mathbf{x}''|^2$$

it follows

$$|\mathbf{y}' - \mathbf{x}'| \le |\mathbf{y} - \mathbf{x}| < r.$$

Thus $\mathbf{y}' \in B'_r(\mathbf{x}')$.

($\supset$): Let $\mathbf{y}'$ be any point in $B'_r(\mathbf{x}')]$. Then

$$\pi(\mathbf{y}', \mathbf{x}'') = \mathbf{y}'.$$

But $(\mathbf{y}', \mathbf{x}'') \in B_r(\mathbf{x})$ since

$$|(\mathbf{y}', \mathbf{x}'') - \mathbf{x}| = |\mathbf{y}' - \mathbf{x}'| < r.$$

It follows $\mathbf{y}' \in \pi[B_r(\mathbf{x})]$. ∎

Problem 6.11 1. $S = [a, c) \cup (c, b]$, $A = [a, c)$. Then $A = (a-1, c) \cap S$ and so is open in S as it is the intersection of S with an open set.

A is *also* closed in S since $A = [a, c] \cap S$.

2. $S = (0, 1]$ and $A = \{1, 1/2, 1/3, \dots\}$. Then $A = E \cap S$ where $E = \{0\} \cup \{1, 1/2, 1/3, \dots\}$. Since E is closed, it follows A is closed in S.

A is not open in S. For assume (by way of obtaining a contradiction) that

$$A = S \cap E \tag{15}$$

where E is open. Then $1/2 \in E$ and so $I := (1/2 - \epsilon, 1/2 + \epsilon) \subset E$ for some $\epsilon > 0$ which we can choose to be $< 1/2 - 1/3 = 1/6$. But also $I \subset S$ and so $I \subset A$ as $A = S \cap E$. This is false and so (15) is not possible for an open set E. Thus A is not open in S.

3. $S = [0, 1]$ and $A = \{1, 1/2, 1/3, \dots\}$. The same argument as in (b) shows that A is not open in S.

Moreover A is not closed in S. For assume (by way of obtaining a contradiction) that

$$A = S \cap E \tag{16}$$

where E is closed. Then A is also closed since it is the intersection of two closed sets. But on the other hand A is not closed as 0 is a limit point of A and $0 \notin A$. This contradiction implies (16) is not possible for a closed set E. Thus A is not closed in S.

[*Note:* We will see in the next Problem that since S is closed in $\mathbb{R}$, a subset of S is closed in S iff it is closed in $\mathbb{R}$.]

Problem 6.12 We are given that S is an open subset of X, where (X, d) is a metric space.

If $A \subset S$ is open in S then $A = S \cap E$ for some open $E \subset X$. Since both S and E are open (in X) it follows that A is open in X.

Conversely, if $A \subset S$ is open in X then A is certainly open in S, as $A = S \cap A$ and so is the intersection of two sets which are open in X.

The argument in the closed case is similar.

Problem 6.13 1. Positivity and Symmetry are immediate. For the triangle inequality, note that

$$d(x, y) \leq d(x, z) + d(x, y)$$

since

(i) $x = y \Longrightarrow d(x, y) = 0$ and so result must be true as right side ≥ 0.

(ii) $x \neq y \Longrightarrow d(x, y) = 1$, and <u>at</u> <u>least</u> one of $d(x, z)$ and $d(z, y)$ equal 1 (since we cannot have both $z = x$ and $z = y$. Hence result is true.

2. $B_r(x) = \{y : d(y, x) < r\}$. Hence $B_r(x) = \{x\}$ if $r \leq 1$. $B_r(x) = X$ if $r > 1$.

NOTE: $B_1(x) = \{x : d(y, x) < 1\} = \{x\}$.

3. Since $B_{1/2}(x) = \{x\}$, we see $B_{1/2}(x) = \{x\}$. Thus x is an interior point of $\{x\}$. Hence $\text{int}\{x\} \supset \{x\}$ and so $\text{int}\{x\} = \{x\}$.

If $y \notin \{x\}$, i.e. $y \neq x$, then

$$B_{1/2}(y) = \{y\} \subset X \sim \{x\}$$

Hence $\text{ext}\{x\} \supset X \sim \{x\}$ and so $\text{ext}\{x\} = X \sim \{x\}$.

$$\begin{aligned} \partial\{x\} &= X \sim (\text{int}\{x\} \cup \text{ext}\{x\}) = \phi \\ \overline{\{x\}} &= \text{int}\{x\} \cup \partial\{x\} = \{x\} \end{aligned}$$

Problem 6.14 1. Positivity and symmetry are immediate. To prove the triangle inequality we have to show

$$(*) \qquad \min\{1, d(x, y)\} \leq \min\{1, d(x, z)\} + \min\{1, d(z, y)\}$$

We do this by considering various cases. One way is as follows:

(a) Suppose $d(x, z) \geq 1$ or $d(z, y) \geq 1$. Then the right side of (*) is ≥ 1. But the left side of (*) is ≤ 1. Hence result (*) is true.

(b) Next suppose $d(x, z) < 1$ and $d(z, y) < 1$. Then the right side of (*) is $d(x, z) + d(z, y)$. But the left side is $\leq d(x, y)$. Hence, by the triangle inequality for d, we see (*) is true.

2. We will use the notation

$$\begin{aligned}\overline{B}_r(x) &= \{y : \overline{d}(x,y) < r\} \\ B_r(x) &= \{y : d(x,y) < r\}\end{aligned}$$

Suppose $r \leq 1$, then

$$d(x,y) < r \qquad \text{iff} \quad d(x,y) < r$$

Suppose $r > 1$, then

$$\overline{d}(x,y) < r \qquad \text{for all} \quad y \in \mathbb{R}^2$$

Hence

$$\overline{B}_r(0) = \begin{cases} \text{usual } B_r(0) & \text{if} \quad r > 1 \\ \mathbb{R}^2 & \text{if} \quad r > 1 \end{cases}$$

3. Suppose $A \subset \mathbb{R}^2$ is open in the d metric. Then for each $x \in A$, $B_r(x) \subset A$ for some $r > 0$.

By taking a smaller r if necessary, we may assume $0 < r < 1$. But then $B_r(x) = B_r(x)$ and so $\overline{B}_r(x) \subset A$. Hence x is an interior point in the $\overline{d}$ metric.

Conversely, if A is open in the $\overline{d}$-metric, a similar argument shows A is open in the d-metric.

Problem 6.15 Suppose

$$\begin{aligned}d_1(x,y) &\leq \alpha d_2(x,y) \\ d_2(x,y) &\leq \beta d_1(x,y)\end{aligned}$$

1. If $y \in B^2_r(x)$ then $d_2(x,y) < r$. Hence $d_1(x,y) < \alpha r$. Hence $y \in B^1_{\alpha r}(x)$.

   i.e. $\quad B^2_r(x) \subset B^1_{\alpha r}(x)$.

   Similarly, $B^1_r(x) \subset B^2_{\beta r}(c)$.

2.

$$\begin{aligned}d_\infty(x,y) &= \max\{|x^1 - y^1|, \cdots, |x^n - y^n|\} \\ d_2(x,y) &= \sqrt{\sum_{i=1}^{n}(x^i - y^i)^2}\end{aligned}$$

Thus

$$d_\infty(x,y) \le d_2(x,y) \le \sqrt{\sum_{i=1}^{n}(d_\infty(x,y))^2} \le \sqrt{n}d_\infty(x,y)$$

This proves the result.

3. There is no α such that

$(*)$ $$d_2(x,y) \le \alpha\overline{d}(x,y)$$

for all x, $y \in \mathbb{R}^2$.

For suppose there were such an α. The right side of (*) is at most α. But by selecting suitable x, $y \in \mathbb{R}^2$, we can ensure the left side of (*) is greater than α.

This contradicts (*).

4.

$$d_\infty(f,g) = \max_{a\le x\le b}|f(x)-g(x)|(\quad = \quad \sup_{a\le x\le b}|f(x)-f(x)| \text{ by continuity}$$

$$d_1(f,g) \quad = \quad \int_a^b |f-g|$$

(i) Thus

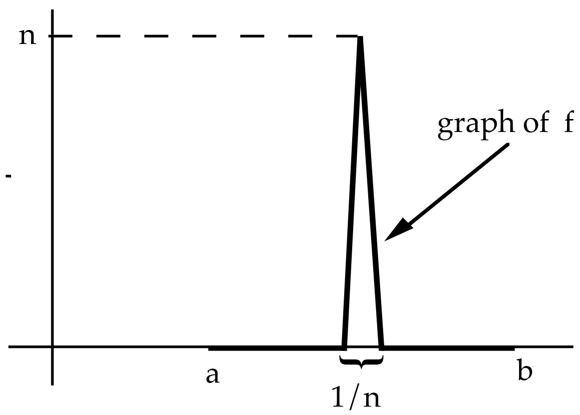
$$d_1(f,g) \le \int_a^b d_\infty(f,g) = (b-a)d_\infty(f,g)$$

(ii) However, there is no α such that

$(*)$ $$d_\infty(f,g) \le \alpha d_1(f,g)$$

for all f, $g \in C[a,b]$

To see this, suppose there were such an α that (*) is true.



By choosing $g = 0$ in $[a, b]$; and if with the graph as shown (we could easily write down an expression for f), we see

$$\begin{aligned} d_\infty(f, g) &= n \\ d_1(f, g) &= 1/2 \end{aligned}$$

By choosing n sufficiently large, we get a contradiction to (*). Hence there is no α such that (*) is true for all f and $g \in C[a, b]$.

5. Let $A = B_1^\infty(\mathbb{O})$ be the "unit ball" in the sup metric about zero function $\mathbb{O}$, i.e.

$$A = \{f \in C[a, b] : \sup_{a \le x \le b} |f(x)| < 1\}$$

A is open in the sup metric (since the open balls in any metric are indeed open sets with respect to that metric). But A is not open in the L^1 metric. To see this, first note that $\mathbb{O} \in A$ (where $\mathbb{O}$ is the zero function.

For any $\varepsilon > 0$, we can find a function $f \in C[a, b]$, $f \notin A$, with

$$d_1(f, \mathbb{O}) \le \varepsilon \,.$$

Hence A is not open in the L^1 metric; as $\mathbb{O} \in A$ and there are functions arbitrarily close to $\mathbb{O}$ in the sup metric which are NOT in A.

Problem 6.16 1.

1. Let $A = \{1, 1/2, 1/3, \ldots\} \in \mathbb{R}$.

2. Consider a ball $B_\varepsilon(x)$.

   Choose $x_1 \in A \cap (B_\varepsilon(x) \sim \{x\})$.

   Choose $x_2 \ne x_1$; $x_2 \in A \cap (B_\varepsilon(x) \sim \{x\})$. (This is possible by choosing $x_2 \in A \cap (B_{r_1}(x) \sim \{x\})$ where $r_1 < \min\{\varepsilon, d(x_1, x)\}$)

   Choose $x_3 \ne x_1, x_2$; $x_3 \in A \cap (B_\varepsilon(x) \sim \{x\})$. (This is possible by choos-
   ing $x_3 \in A \cap (B_{r_2}(x) \sim \{x\})$ where $r_2 < \min\{\varepsilon, d(x_1, x), d(x_2, x)\}$).

   Choose $x_4 \ne x_1, x_2, x_3$; $x_4 \in A \cap (B_\varepsilon(x) \sim \{x\})$ etc.

3. Trivial

4. Suppose $x \in \overline{A}$. If x is not a limit point of A then $x \in A$ (by definition of $\overline{A}$). Since x is not a limit point of A, it follows from Definition 6.9 that x is isolated. Thus every $x \in \overline{A}$ is either a limit point or an isolated point. It follows from Definition 6.9 that x cannot be both.

5. If $x \in \overline{A}$, then x is either a limit point or an isolated point. In either case, it follows from Definition 6.9 that every $B_r(x)$ contains a point from A. Conversely, suppose every $B_r(x)$ contains a point from A. If $x \in A$ then certainly $x \in \overline{A}$. If $x \notin A$, then x is a limit point from A (as
follows from Definition 6.9).

2. Let $A_\lambda (\lambda \in S)$ be a collection of closed sets. Then A_λ^c are all open, and so $\bigcup_{\lambda \in S} A_\lambda^c$ is open by Theorem 6.16.

But $\left[\bigcap_{\lambda \in S} A_\lambda\right]^c = \left[\bigcup_{\lambda \in S} A_\lambda^c\right]$, and so $\left[\bigcap_{\lambda \in S} A_\lambda\right]^c$ is open. Hence $\bigcap_{\lambda \in S} A_\lambda$ is closed.

Problem 6.17 1. (a) Suppose $B \subset A$, B open.

Take $x \in B$ and choose $r > 0$ so $B_r(x) \subset B$. Then $V B_r(x) \subset A$, and so x is an interior point of A. i.e. $B \subset \text{int } A$.

(b) Let $\mathcal{F}$ be the family of all open subsets of A. From (a), if $O \in \mathcal{F}$ then $O \subset \text{int } A$. Hence

$$\bigcup_{O \in \mathcal{F}} O \subset \text{int } A \quad .$$

But $\bigcup_{O \in \mathcal{F}} O \supset \int A$ is trivial, since int A is itself a member of $\mathcal{F}$.

This shows

$$\bigcup_{O \in \mathcal{F}} O = \text{int } A$$

2. We have

$$\begin{aligned} \overline{A^c} &= \text{int}(A^c) \cup \partial(A^c) \quad \text{[from (6.6)]} \\ &= \text{ext}A \cup \partial A \quad \text{[by (6.3), and the fact } \partial A = \partial A^c] \\ &= (\text{int}A)^c \quad \text{[from (6.2) and last line of Prop.6.8]} \end{aligned}$$

i.e. $\overline{A^c} = (\text{int}A)^c$ and so

$$\overline{A^c}^c = \text{int}A$$

This proves half the question.

Now replace A by A^c in 6.17. Then

$$\overline{A}^c = \text{int}A^c$$

and so

$$\overline{A} = (\text{int}A^c)^c$$

3. $\overline{A}$ is the smallest closed set containing A, in the sense that

(i) If $B \supset A$ and B is closed then $B \supset \overline{A}$.

(ii) $\overline{A} = \bigcup_{C \in \mathcal{G}} C$, where $\mathcal{G}$ is the family of all closed sets containing A.

PROOF: (i) Suppose $B \supset A$ and B is closed. Then $B^c \subset A^c$ and B^c is open. Therefore $B^c \subset \text{int}A^c$ by part 1(a), which equals $\overline{A}^c$ by part 2. Thus, taking complements, $B \supset \overline{A}$.

(ii) By part 2,

$$\overline{A} = (\text{int}A^c)^c = \left(\bigcup_{O \in \mathcal{F}} O\right)^c$$

by part 1b], where $\mathcal{F}$ is the family of all open subsets of A^c. Thus by de Morgan's laws,

$$\overline{A} = \bigcup_{O \in \mathcal{F}} O^c = \bigcup_{C \in \mathcal{G}} C$$

since O is an open subset of A^c iff O^c is a closed set containing A.

∎

Problem 6.18 1. If $f(a) \le b$, then from the first diagram

$$\underbrace{ab}_{\text{area of rectangle}} \quad \le \quad \underbrace{\int_0^a f + \int_0^b g}_{\text{area of rectangle + a little bit more}}$$

Similarly, if $f(a) > b$, use the second diagram.

2. Let $f(x) = x^{p-1}$. Note that f satisfies the conditions of (1). Moreover, the inverse g is given by

$$g(y) = x \quad \text{iff} \quad x^{p-1} = y \quad \text{iff} \quad x = y^{\frac{1}{p-1}}$$

Hence from (1)

$$\begin{aligned} ab &\le \textstyle\int_0^a x^{p-1}dx + \int_0^b y^{\frac{1}{p-1}}dy \\ &= p^{-1}x^p|_0^a + \tfrac{p-1}{p} y^{\frac{p}{p-1}}|_0^b \\ &= \tfrac{a^p}{p} + \tfrac{b^{p'}}{p'} \end{aligned}$$

3. <u>First</u> assume

$$\sum_i |a_i|^p = \sum |b_i|^{p'} = 1$$

Then from (2)

$$\begin{aligned} \textstyle\sum_i |a_i|\ |b_i| &= \textstyle\sum_i \left(\frac{1}{p}|a_i|^p + \frac{1}{p'}|b_i|^{p'}\right) \\ &= \tfrac{1}{p}\textstyle\sum_i |a_i|^p + \frac{1}{p'}\sum_i |b_i|^{p'} \\ &= \tfrac{1}{p} + \tfrac{1}{p'} = 1 \end{aligned}$$

In the general case, let

$$\alpha = \left(\sum_i |a_i|^p\right)^{1/p} \quad \text{and} \quad \beta = \left(\sum_i |b_i|^{p'}\right)^{\frac{1}{p'}}$$

Then

$$\sum_i \left|\frac{a_i}{\alpha}\right|^p = \sum_i \left|\frac{b_i}{\beta}\right|^{p'} = 1$$

and so by the previous special case

$$\sum_i \left|\frac{a_i b_i}{\alpha\beta}\right| \leq 1$$

that is,

$$\sum_i |a_i b_i| \leq \alpha\beta = \left(\sum_i |a_i|^p\right)^{1/p} \left(\sum_i |b_i|^{p'}\right)^{\frac{1}{p'}}$$

4. (This is same argument as for (3).)

   First assume

$$\int_a^b |f|^p = \int_a^b |g|^{p'} = 1$$

   Then from (2)

$$\begin{aligned} \textstyle\int_a^b |f\,g| &\leq \textstyle\int_a^b \frac{1}{p}|f|^p + \frac{1}{p'}|g|^{p'} \\ &= \textstyle\frac{1}{p} + \frac{1}{p'} \\ &= 1 \end{aligned}$$

   In the general case, let

$$\alpha = \left(\int_a^b |f|^p\right)^{1/p} \quad , \quad \beta = \left(\int_a^b |g|^{p'}\right)^{\frac{1}{p'}}$$

   Then

$$\int_a^b \left|\frac{f}{\alpha}\right|^p = \int_a^b \left|\frac{g}{\beta}\right|^{p'} = 1$$

   Then

$$\int_a^b \left|\frac{f\,g}{\alpha\,\beta}\right| \leq 1$$

   Therefore

$$\int_a^b |f\,g| \leq \alpha\,\beta = \left(\int_a^b |f|^p\right)^{1/p} \left(\int_a^b |g|^{p'}\right)^{1/p'}$$

5. Note that

   (a) $\|x\|_p \geq 0$ and $\|x\|_p = 0$ iff $x = \mathbb{O}$.

   (b) $\|\alpha\, x\|_p = |\alpha|\ \|x\|_p$ if $\alpha \in \mathbb{R}$.

Moreover,

$$
\begin{aligned}
\|x+y\|_p^p &= \textstyle\sum_i |x_i+y_i|^p \\
&= \textstyle\sum_i |x_i+y_i|\ |x_i+y_i|^{p-1} \\
&\le \textstyle\sum_i \left[|x_i|\ |x_i+y_i|^{p-1} + |y_i|\ |x_i+y_i|^{p-1}\right] \\
&\qquad \text{(by triangle inequality in } \mathbb{R}) \\
&\le (\textstyle\sum_i\ |x_i|^p)^{1/p}\left(\textstyle\sum_i\ |x_i+y_i|^{p'(p-1)}\right)^{1/p'} \\
&\qquad + (\textstyle\sum_i |y_i|^p)\left(\textstyle\sum |x_i+y_i|^{p'(p-1)}\right)^{1/p'} \\
&\qquad \text{(by Holder's inequality)} \\
&= (\|x\|_p + \|y\|_p)(\|x+y\|_p)^{1/p'}
\end{aligned}
$$

6. This is exactly the same as (5).

Problem 6.19 1. Symmetry and positivity are clear. The triangle inequality is immediate from the triangle inequality for real numbers, i.e.

$$
\begin{aligned}
d(p_1,p_2) &= 1\theta_1-\theta_2 1 \\
&\le 1\theta_1-\theta_3 1 + 1\theta_3-\theta_2 1 \\
&= d(p_1,p_3)+d(p_3,p_2)
\end{aligned}
$$

where $p_3 = (\cos\theta_3, \sin\theta_3)$.

2. From Theorem 6.3.6, $\overline{A} = \text{int}A \cup \partial A$. Since $\text{int}A$ and ∂A are mutually disjoint from Proposition 6.3.2, it follows that

$$\partial A = \overline{A} \setminus \text{int}A$$

3.

$$B_2^X(0) = X\ ;\ B_{1/2}^X(0) = [0, 1/2)$$

4.

$$B_2^X(0) = \{-1, 0, 1\}\ ;\ B_{1/2}^X(0) = \{0\}$$

5. Let

$$S = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \cup \{1\frac{1}{2}, 1\frac{1}{3}, 1\frac{1}{4}, \ldots\} \cup \{1\frac{1}{2}, 2\frac{1}{3}, 2\frac{1}{4}, \ldots\}$$

Limit points are 0, 1, 2.

6. (a) $(-1, 1, -1, 1, -1, 1, \ldots)$

   (b) Let (x_n) be an enumeration of Q. Then there exists a subsequence converging to <u>any</u> real number a (e.g. - take a subsequence of the n^{th} approximations in the decimal expansion of a). (The latter subsequence is needed to ensure we end up with a subsequence of the original (x_n).)

7 Sequences and Convergence

Problem 7.1 Since $2^{-m} \to 0$, $m^2/m! \to 0$ and $3^m/m! \to 0$ as $m \to \infty$ by standard properties of limits, it follows $(x_m, y_m) \to (1, 0)$ as $m \to \infty$.

Problem 7.2 Let $A \subset \mathbb{R}^s$ and $B \subset \mathbb{R}^{n-s}$ be closed.

In order to show $A \times B$ is closed let $(\mathbf{x}_k)_{k=1}^{\infty} \subset A \times B$ with $\mathbf{x}_k \to \mathbf{x}$ (we want to show $\mathbf{x} \in A \times B$). Write $\mathbf{x}_k = (\mathbf{x}'_k, \mathbf{x}''_k)$ for $k = 1, 2, \ldots$, and $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$, where $\mathbf{x}'_k, \mathbf{x}' \in \mathbb{R}^s$ and $\mathbf{x}''_k, \mathbf{x}'' \in \mathbb{R}^{n-s}$.

Since $|\mathbf{x}'_k - \mathbf{x}'| \le |\mathbf{x}_k - \mathbf{x}|$ and $|\mathbf{x}''_k - \mathbf{x}''| \le |\mathbf{x}_k - \mathbf{x}|$ it follows that $\mathbf{x}'_k \to \mathbf{x}'$ and $\mathbf{x}''_k \to \mathbf{x}''$. Since A and B are closed it follows $\mathbf{x}' \in A$ and $\mathbf{x}'' \in B$ and so $\mathbf{x} \in A \times B$. Thus $A \times B$ is closed.

Problem 7.3 Let $x_m \to x_0$ and $y_m \to y_0$ as $m \to \infty$. Assume $y_0 \neq 0$ and $y_m \neq 0$ for all $m \ge 1$. We want to show $x_m/y_m \to x_0/y_0$ (note that the sequences are in $\mathbb{R}$). As noted in the Question it is sufficient to show $y_m^{-1} \to y_0^{-1}$ since then by the multiplication property of limits the required result follows.

Suppose $\epsilon > 0$. Then

$$\begin{aligned} |y_m^{-1} - y_0^{-1}| &= \left| \frac{y_0 - y_m}{y_0 y_m} \right| \\ &= \frac{|y_0 - y_m|}{|y_0 y_m|}. \end{aligned}$$

Choose N so $m \ge N$ implies $|y_0 - y_m| < \epsilon$ and $|y_m| \ge |y_0|/2$.[17] Then for $m \ge N$ it follows

$$|y_m^{-1} - y_0^{-1}| < \frac{2\epsilon}{|y_0|^2}.$$

Since ϵ is arbitrary, this gives the result.[18]

Problem 7.4 From the Example in Section 7.4 we have

$$\begin{aligned} x_m &= \left(1 + \frac{1}{m}\right)^m \\ &= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{m}\right) + \frac{1}{3!}\left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right) \\ &\quad + \cdots + \frac{1}{m!}\left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)\cdots\left(1 - \frac{m-1}{m}\right), \qquad (17) \\ y_m &= 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!}. \end{aligned}$$

[17] The latter is possible as $y_0 \neq 0$ and $y_m \to y_0$.

[18] We could replace ϵ throughout the proof by $\epsilon\frac{|y_0|^2}{2}$ and thereby end up with $|y_m^{-1} - y_0^{-1}| < \epsilon$, but we would not normally bother doing this.

Moreover we also have from there that (x_m) and (y_m) are increasing sequences, $x_m \to x_0$ (say), $y_m \to y_0$ (say), and

$$x_m \le y_m \le y_0 \le 3.$$

Since $x_m \le y_m$ for all m it follows from the Comparison Test that

$$x_0 \le y_0. \tag{18}$$

On the other hand[19] if $n < m$ then by taking the first $n+1$ terms in (1) we have

$$\begin{aligned} x_m \quad \ge \quad & 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{m}\right) + \frac{1}{3!}\left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right) \\ & + \cdots + \frac{1}{n!}\left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)\cdots\left(1 - \frac{n-1}{m}\right). \end{aligned}$$

If we *fix* n and let $m \to \infty$ then it follows from the Comparison Test that

$$x_0 \ge 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}.$$

This is true for *all* n and so

$$x_0 \ge 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots = y_0. \tag{20}$$

From (18) and (20) it follows that $x_0 = y_0$, as required.

Problem 7.5 A singleton $A = \{\mathbf{x}\}$ from $\mathbb{R}^n$ is closed since any sequence from A must trivially be constant and so have its limit in A. From Section 7.6 of the Notes it follows that A is closed.

(Alternatively, if $\mathbf{y} \ne \mathbf{x}$ and $r = |\mathbf{y} - \mathbf{x}|$ then $B_r(\mathbf{y}) \subset A^c$, and so A^c is open, i.e. A is closed.)

As noted in the Question, any finite set is a finite union of singletons, and so is closed.

---

[19]It is *not* sufficient to just say that the $(n+1)$th term

$$\frac{1}{n!}\left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)\cdots\left(1 - \frac{n-1}{m}\right)$$

in (17) approaches $1/n!$ as $m \to \infty$ and so the right side of (17) approaches

$$1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!}.$$

The problem is that this is equivalent to saying that

$$\lim_{m\to\infty} \sum_{n=1}^{m} a_{nm} = \sum_{n=1}^{\infty} \lim_{m\to\infty} a_{nm}. \tag{19}$$

But if $a_{nm} = 0$ if $n \ne m$ and $a_{nm} = 1$ if $n = m$ then the left side of (19) is 1 and the right side is 0.

The basic rule is that it is *not* justifiable to interchange limits or infinite sums without further argument.

Problem 7.6 Suppose $A \subset \mathbb{R}^2$ is open.

Let S be the family of all balls $B_r(\mathbf{x})$ such that r is rational and the components of $\mathbf{x}$ are both rational. Then there is a one-one correspondence between S and $(\mathbb{Q} \cap \{r : r > 0\}) \times \mathbb{Q} \times \mathbb{Q}$, namely $B_r((x, y)) \leftrightarrow (r, x, y)$. But $(\mathbb{Q} \cap \{r : r > 0\}) \times \mathbb{Q} \times \mathbb{Q}$ is countable by Theorem 4.9.1(1) applied twice. Hence S is countable.

Let $\mathcal{S}_\mathcal{A}$ be the family of balls in S which are subsets of A. Note that S_A is countable, being a subset of a countable set. We *claim* that

$$A = \bigcup \mathcal{S}_\mathcal{A},$$

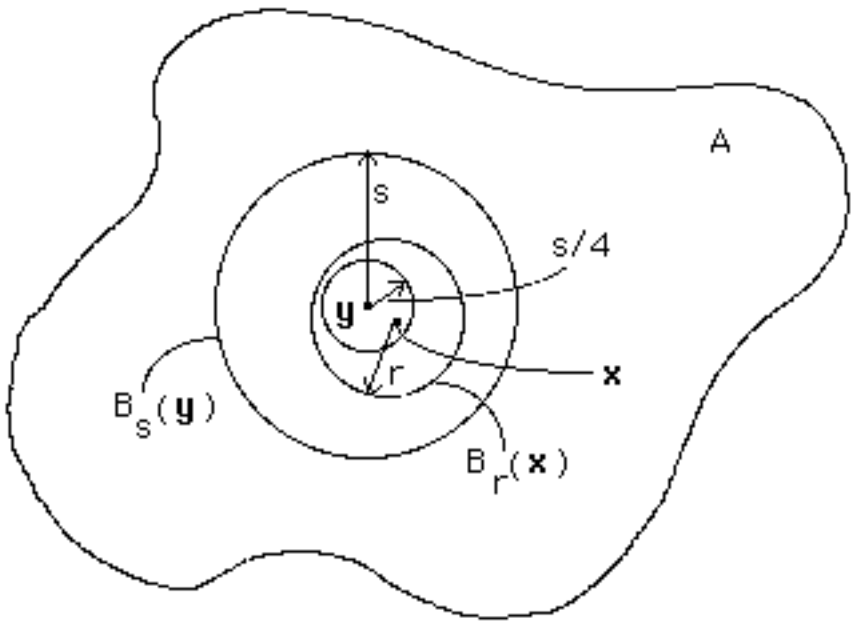
where $\bigcup \mathcal{S}_\mathcal{A}$ is the union of all balls in $\mathcal{S}_\mathcal{A}$.

Since $B_r(\mathbf{x}) \subset A$ for any $B_r(\mathbf{x}) \in \mathcal{S}_\mathcal{A}$, it follows $\bigcup \mathcal{S}_\mathcal{A} \subset A$.

On the other hand if $\mathbf{y} \in A$ then since A is open it follows $B_s(\mathbf{y}) \subset A$ for some $s > 0$ (see the following diagram). Choose a point $\mathbf{x} \in B_{s/4}(\mathbf{y}) \subset A$ both of whose coordinates are rational.[20] Choose $r > 0$ rational so $s/4 \leq r < s/2$. Then

$$\mathbf{y} \in B_r(\mathbf{x}) \subset B_s(\mathbf{y}) \subset A,$$

as can be easily checked from the triangle inequality. In particular $B_r(\mathbf{x}) \in \mathcal{S}_\mathcal{A}$ and so $\mathbf{y} \in \bigcup \mathcal{S}_\mathcal{A}$. Since $\mathbf{y}$ was an arbitrary element of A it follows $A \subset \bigcup \mathcal{S}_\mathcal{A}$.



This completes the proof. ∎

Problem 7.7 1. $|\|x_n\| - \|x\|| \leq \|x_n - x\|$ (by the comment after Definition 5.3), and we are done.

[20]This is possible. Let $\mathbf{y} = (y^1, y^2)$. Choose x^1 rational where $|x^1 - y^1| < s/8$ and choose x^2 rational where $|x^2 - y^2| < s/8$. Let $\mathbf{x} = (x^1, x^2)$. Then $\mathbf{x} \in B_{s/4}(\mathbf{y})$.

2. (i) Suppose A is open.

   We need to show that <u>if</u> $x \in A$ and $x_n \to x$ <u>then</u> $x_n \in A$ for all sufficiently large n] So assume $x \in A$ and $x_n \to x$. Now A is open, so that $B_r(x) \subset A$ for some $r > 0$. Since $x_n \to x$, $x_n \in B_r(x)$ for all sufficiently large n. <u>Hence</u>, $x_n \in A$ for all sufficiently large n.

   (ii) Suppose A is <u>not</u> open. Then for some $x \in A$, <u>no</u> $B_r(x) \subset A$. In particular, $B_{1/n}(x) \not\subset A$ for $(n = 1, 2, 3, \ldots)$. Choose $x_n \in B_{1/n}(x)$, $x_n \notin A$. Then $x \in A$, $x_n \to x$; but it is <u>not</u> the case that $x_n \in A$ for all sufficiently large n.

Problem 7.8 1. Let $A = B_1 \cup B_2$

(a) Since $B_1 \subset A$, it follows that $\overline{B_1} \subset \overline{A}$. Similarly $\overline{B_2} \subset \overline{A}$, so that

$$\overline{B_1} \cup \overline{B_2} \subset \overline{A}$$

(b) Suppose $x \in \overline{A}$, so there is a sequence $(x_n) \subset A$ such that $x_n \to x$. Then either B_1 or B_2 (possibly both) must contain an infinite subsequence (x_{i_n}); say it is B_i. Then $x \in \overline{B_i}$. It follows that

$$\overline{B_1} \cup \overline{B_2} \supset \overline{A}$$

2. The proof is almost identical to that of 1, the pidgeonhole principle giving an infinite subsequence in at least one of the B_i.

3. Again, the same proof as before, but it only works one way this time.

4. Taking $B_i = [1/i, 1]$, $A = (0, 1]$ so that $\overline{A} = [0, 1]$. But $\bigcup_{i=1}^{\infty} B_i = (0, 1] \neq \overline{A}$.

Problem 7.9 1. Suppose that $\alpha_n \to \alpha$ in $\mathbb{R}$, and $x_n \to x$ in X. Then

$$\begin{aligned} |\alpha x - \alpha_n x_n| &= |(\alpha(x - x_n) + (\alpha - \alpha_n)x| \\ &\leq |\alpha||x - x_n|| + |\alpha - \alpha_n|||x_n|| \\ &\to 0\,. \end{aligned}$$

2. (a)

$$\log(n+1) - \log(n)| = \log\frac{n+1}{n} = \log(1 + 1/n)$$

But $1 + 1/n \to 0$ as $n \to \infty$, and the logarithm function os continuous (at 1), so that

$$\log(1 + 1/n) \to \log(1) = 0\,.$$

(b) Certainly not. $|\log m - \log n| = |\log \frac{m}{n}|$ and the latter has no limit as $m, n \to \infty$. For example, choosing $m = kn$ for some fixed $k \in \mathbb{N}$, $|\log \frac{m}{n}| = \log k$ no matter how large n may be. And we can choose any such k.

8 Cauchy Sequences

Problem 8.1 Let $\mathcal{V}$ be the set of infinite sequences

$$\mathbf{x} = (x^1, x^2, \ldots)$$

for which $\sum_{n=1}^{\infty}(x^n)^2$ is finite. Define

$$||\mathbf{x}|| = \left[\sum_{n=1}^{\infty}(x^n)^2\right]^{1/2}.$$

The set of *all* infinite sequences is easily checked to be a vector space with zero vector $\mathbf{0} = (0, 0, 0, \ldots)$.[21] In order to show that $\mathcal{V}$ is a subspace (and hence a vector space) we have to show that $\mathcal{V}$ contains the zero sequence (which is trivial) and is closed under addition and scalar multiplication. In other words, if $||\mathbf{x}||$ and $||\mathbf{y}||$ are finite then so are $||\alpha\mathbf{x}||$ (for any $\alpha \in \mathbb{R}$) and $||\mathbf{x} + \mathbf{y}||$. But $||\alpha\mathbf{x}|| = |\alpha|\, ||\mathbf{x}||$ and

$$||\mathbf{x} + \mathbf{y}||^2 = \sum_{n=1}^{\infty}(x^n + y^n)^2 \leq^{22} 2\sum_{n=1}^{\infty}(x^n)^2 + 2\sum_{n=1}^{\infty}(y^n)^2 = 2||\mathbf{x}||^2 + 2||\mathbf{y}||^2.$$

Thus $||\mathbf{x} + \mathbf{y}||$ is finite if $||\mathbf{x}||$ and $||\mathbf{y}||$ are finite.

1. To check that $||\cdot||$ is a norm , note that positivity and homogeneity are easy. For the triangle inequality let

   $$\mathbf{y} = (y^1, y^2, \ldots).$$

   Note that

   $$||\mathbf{x} + \mathbf{y}|| = \left[\sum_{n=1}^{\infty}(x^n + y^n)^2\right]^{1/2} = \lim_{n\to\infty}\left[\sum_{k=1}^{n}(x^k + y^k)^2\right]^{1/2}.$$

   But

   $$\left[\sum_{k=1}^{n}(x^k + y^k)^2\right]^{1/2} \leq \left[\sum_{k=1}^{n}(x^k)^2\right]^{1/2} + \left[\sum_{k=1}^{n}(y^k)^2\right]^{1/2}$$

   by the triangle inequality in $\mathbb{R}^k$. It follows from the Comparison Test Theorem that

   $$||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||.$$

---

[21] This is also a particular case of the fact that the set of *all* real-valued functions defined on any set S is a vector space with the zero vector being the zero function. Here take $S = \mathbb{N}$.

[22] Since $(a + b)^2 \leq 2(a^2 + b^2)$ as is easily checked.

2. Let $(\mathbf{x}_k)_{k=1}^{\infty}$ be a Cauchy sequence in $\mathcal{V}$. Each $\mathbf{x}_k$ is itself a sequence and so we can write

$$\begin{aligned}
\mathbf{x}_1 &= x_1^1, x_1^2, x_1^3, \ldots \\
\mathbf{x}_2 &= x_2^1, x_2^2, x_2^3, \ldots \\
\mathbf{x}_3 &= x_3^1, x_3^2, x_3^3, \ldots \\
&\vdots \\
\mathbf{x}_k &= x_k^1, x_k^2, x_k^3, \ldots \\
&\vdots
\end{aligned}$$

We want to show that $\mathbf{x}_k \to \mathbf{x}$ as $k \to \infty$ for some $\mathbf{x} = (x^1, x^2, x^3, \ldots) \in \mathcal{V}$, where

$$\mathbf{x}_k \to \mathbf{x} \text{ means } ||\mathbf{x}_k - \mathbf{x}|| \to 0.^{23} \tag{21}$$

Since $(\mathbf{x}_k)$ is a Cauchy sequence, this means that $||\mathbf{x}_j - \mathbf{x}_k|| \to 0$ as $j, k \to \infty$. For each n it is easy to see that

$$|x_j^n - x_k^n| \le ||\mathbf{x}_j - \mathbf{x}_k||$$

and so $|x_j^n - x_k^n| \to 0$ as $j, k \to \infty$. That is, for each n the sequence $(x_k^n)_{k=1}^{\infty}$ is a Cauchy sequence of real numbers and so converges to x^n, say.

Let $\mathbf{x} = (x^1, x^2, x^3, \ldots)$ (thus $\mathbf{x}_k \to \mathbf{x}$ as $k \to \infty$, in the componentwise sense). We *claim* that $\mathbf{x}_k \to \mathbf{x}$ as $k \to \infty$, in the sense of (21) *(this is the main point)*.

So suppose $\epsilon > 0$. Choose K so

$$j, k \ge K \text{ implies } ||\mathbf{x}_j - \mathbf{x}_k|| < \epsilon,$$

i.e.

$$\sum_{n=1}^{\infty} \left(x_j^n - x_k^n\right)^2 \le \epsilon^2. \tag{22}$$

[23]This is sometimes called *norm convergence* to distinguish it from *componentwise convergence*. Componentwise convergence means that for each n we have $x_k^n \to x^n$ as $k \to \infty$.

It is not true that componentwise convergence implies norm convergence. For example let

$$\begin{aligned}
\mathbf{x}_1 &= 1, 0, 0, \ldots \\
\mathbf{x}_2 &= 0, 1, 0, \ldots \\
\mathbf{x}_3 &= 0, 0, 1, \ldots \\
&\vdots
\end{aligned}$$

Then for each fixed n we see $x_k^n \to 0$ as $k \to \infty$ and so $\mathbf{x}_k \to \mathbf{x} = (0, 0, 0, \ldots)$ in the componentwise sense as $k \to \infty$. But $||\mathbf{x}_k - \mathbf{x}|| = 1$ for all k and so it is *not* true that $\mathbf{x}_k \to \mathbf{x}$ in the norm sense.

On the other hand it is easy to see that norm convergence implies componentwise convergence.

Hence for each N

$$\sum_{n=1}^{N}\left(x_j^n - x_k^n\right)^2 \le \epsilon^2.$$

Fixing j and letting $k \to \infty$, it follows from the Comparison Test that

$$\sum_{n=1}^{N}\left(x_j^n - x^n\right)^2 \le \epsilon^2.$$

Since this is true for each N it follows by another application of the Comparison Test that

$$\sum_{n=1}^{\infty}\left(x_j^n - x^n\right)^2 \le \epsilon^2. \tag{23}$$

Thus for $j \ge K = K(\epsilon)$ we have

$$||\mathbf{x}_j - \mathbf{x}|| \le \epsilon.$$[24]

Since $\epsilon > 0$ was arbitrary it follows that $\mathbf{x}_j \to \mathbf{x}$ (in the norm sense) as $j \to \infty$. This proves the *claim* and so we are done.

3. Since $||\mathbf{x}|| = 1$ for all $\mathbf{x} \in A$ it follows A is bounded.

   To show A is closed let $(\mathbf{x}_k)_{k=1}^{\infty}$ be a convergent sequence of elements from A. Since $||\mathbf{e}_p - \mathbf{e}_q|| = \sqrt{2}$ if $p \ne q$ *(check)* and since any convergent sequence is Cauchy, it follows that for all $k \ge K$, say, we must have $\mathbf{x}_k = \mathbf{x}_K$. In other words, a convergent sequence from A is in fact constant beyond some term in the sequence. This constant value must be the limit of the sequence, and in particular the limit is in A.

   From Corollary 7.6.2 it follows that A is closed.

Problem 8.2 Let $\mathbf{x}_1 + \mathbf{x}_2 + \ldots$ be an infinite series in $\mathbb{R}^k$. Assume that the series of real numbers $|\mathbf{x}_1| + |\mathbf{x}_2| + \ldots$ converges.

---

[24]We cannot without further justification just let $k \to \infty$ in (22) and so deduce (23). The problem is that it is not necessarily true that

$$\lim_{k\to\infty}\sum_{n=1}^{\infty} y_k^n = \sum_{n=1}^{\infty}\lim_{k\to\infty} y_k^n.$$

For example let

$$\begin{array}{rcccl}\mathbf{y}_1 &=& y_1^1, y_1^2, y_1^3, \ldots &=& 1, 0, 0, \ldots\\ \mathbf{y}_2 &=& y_2^1, y_2^2, y_2^3, \ldots &=& 0, 1, 0, \ldots\\ \mathbf{y}_3 &=& y_3^1, y_3^2, y_3^3, \ldots &=& 0, 0, 1, \ldots\\ &\vdots&&&\end{array}$$

Then $\sum_{n=1}^{\infty} y_k^n = 1$ and so $\lim_{k\to\infty}\sum_{n=1}^{\infty} y_k^n = 1$; but $\lim_{k\to\infty} y_k^n = 0$ for each n and so $\sum_{n=1}^{\infty}\lim_{k\to\infty} y_k^n = 0$.

Let $\mathbf{s}_n = \mathbf{x}_1 + \cdots + \mathbf{x}_n$ be the corresponding sequence of partial sums.[25] Then for $m > n$,

$$\begin{aligned} |\mathbf{s}_m - \mathbf{s}_m| &= |\mathbf{x}_{n+1} + \cdots + \mathbf{x}_m| \\ &\leq |\mathbf{x}_{n+1}| + \cdots + |\mathbf{x}_m|. \end{aligned} \tag{24}$$

Now the series of real numbers $|\mathbf{x}_1|+|\mathbf{x}_2|+\ldots$ converges, i.e. the corresponding sequence of partial sums converges, and so this sequence of partial sums must be Cauchy. But this means that for each $\epsilon > 0$ there exists N such that $m > n \geq N$ implies

$$|\mathbf{x}_{n+1}| + \cdots + |\mathbf{x}_m| \leq \epsilon.$$

From (24) it follows

$$|\mathbf{s}_m - \mathbf{s}_n| \leq \epsilon$$

if $m > n \geq N$. Thus $(\mathbf{s}_n)$ is Cauchy and so converges to a point in $\mathbb{R}^k$ (since $\mathbb{R}^k$ is complete), i.e. the original series converges. ∎

The converse is: "if $\mathbf{x}_1+\mathbf{x}_2+\ldots$ converges then $|\mathbf{x}_1|+|\mathbf{x}_2|+\ldots$ converges". This is **FALSE**.

A counterexample in $\mathbb{R}$ is given by the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n+1}\frac{1}{n} + \cdots.$$

This converges but the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

diverges.

Problem 8.3 Suppose $a < b$ and $a, b \in I$ where I is an interval from $\mathbb{R}$. Suppose f is differentiable and $|f'(x)| \leq \lambda$ for all $x \in I$.

(i) If $x, y \in I$, $x < y$, it follows from the Mean Value Theorem of Calculus that for some $c \in (x, y)$

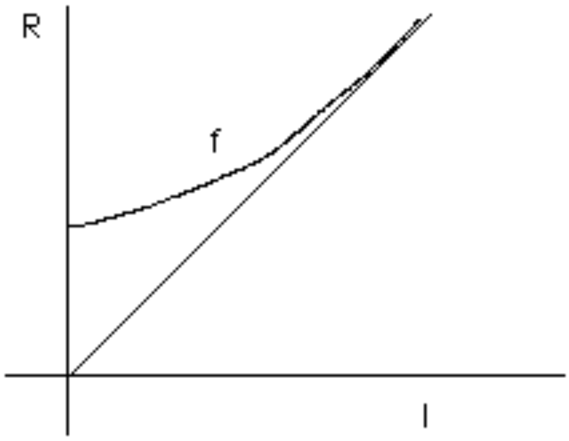
$$|f(x) - f(y)| = |f'(c)|\,|x - y| \leq \lambda|x - y|.$$

Hence f is a contraction map if $\lambda < 1$.

(ii) It follows immediately from the Contraction Mapping Theorem that $f(x) = x$ has a unique solution if $\lambda < 1$.

[25] Remember that convergence of any infinites series, by definition, means convergence of the corresponding sequence of partial sums.

Problem 8.4 Define $f: I \to I$ where $I = [0, \infty)$ so that f has a graph as shown in the next diagram.



For example let

$$f(x) = x + (x+1)^{-1}.$$

Then

$$f'(x) = 1 - (x+1)^{-2}.$$

Since $|f'(c)| < 1$ for all $c \in I$, it follows from the Mean Value Theorem (see the previous Question) that $|f(x) - f(y)| < |x - y|$ for all $x, y \in I$ and $x \neq y$.

On the other hand, $f(x) > x$ for all $x \in I$ and so $f(x) = x$ has no solutions.

This does not contradict the Contraction Mapping Principle since there is no single $\lambda < 1$ such that $|f(x) - f(y)| \leq \lambda |x - y|$ for *all* $x, y \in I$.

Problem 8.5 Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(x, y) = (\frac{1}{3}\sin x - \frac{1}{3}\cos y + 2, \frac{1}{6}\cos x - \frac{1}{2}\sin y - 1).$$

Let $\mathbf{x} = (x, y)$ and $\mathbf{u} = (u, v)$. Then

$$f(\mathbf{x}) - f(\mathbf{u}) = (\frac{1}{3}\sin x - \frac{1}{3}\sin u - \frac{1}{3}\cos y + \frac{1}{3}\cos v, \frac{1}{6}\cos x - \frac{1}{6}\cos u - \frac{1}{2}\sin y + \frac{1}{2}\sin v).$$

From the Mean Value Theorem, since $\sin' x = \cos x$ and $\cos' x = -\sin x$, it follows that $|\sin x - \sin u| \leq |x - u|$, $|\cos x - \cos u| \leq |x - u|$, and similarly for y and v. Thus[26]

$$\begin{aligned}
|f(\mathbf{x}) - f(\mathbf{u})|^2 &= |\tfrac{1}{3}\sin x - \tfrac{1}{3}\sin u - \tfrac{1}{3}\cos y + \tfrac{1}{3}\cos v|^2 \\
&\quad +|\tfrac{1}{6}\cos x - \tfrac{1}{6}\cos u - \tfrac{1}{2}\sin y + \tfrac{1}{2}\sin v|^2 \\
&\le \tfrac{2}{9}(\sin x - \sin u)^2 + \tfrac{2}{9}(\cos y - \cos v)^2 \\
&\quad +\tfrac{2}{36}(\cos x - \cos u)^2 + \tfrac{2}{4}(\sin y - \sin v)^2 \\
&\le \tfrac{2}{9}|x-u|^2 + \tfrac{2}{9}|y-v|^2 + \tfrac{2}{36}|x-u|^2 + \tfrac{2}{4}|y-v|^2 \\
&= \tfrac{5}{18}|x-u|^2 + \tfrac{13}{18}|y-v|^2 \\
&\le \tfrac{13}{18}|\mathbf{x}-\mathbf{u}|^2.
\end{aligned}$$

Thus f is a contraction mapping with contraction constant $\sqrt{13/18}$. It follows that f has a fixed point.

Problem 8.6 1. $A_n = [n, \infty)$.

2. Choose $a_n \in A_n$ for each $n \in \mathbb{N}$. Then given m, n,

$$x_m, x_n \in A_{\min\{m,n\}}$$

so that

$$d(x_m, x_n) \le \mathrm{diam} A_{\min\{m,n\}} \to 0$$

as $m, n \to \infty$. Thus (x_n) is a Cauchy sequence, and so converges to some $x \in \mathbb{R}$ by completeness. We *claim* that $x \in \bigcap_{n=1}^{\infty} A_n$. But for any $p \in \mathbb{N}$, we have $x_n \in A_p$ for all $n \ge p$, so that $x = \lim_n x_n \in A_p$. Thus $\bigcap_{n=1}^{\infty} A_n \ne \emptyset$.

Problem 8.7 1. (a) Take $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{x}' = (x'_1, \ldots, x'_n)$. Then

$$\begin{aligned}
|F(\mathbf{x}) - F(\mathbf{x}')| &= \max_i |\sum_{j=1}^{n} a_{ij}x_j - a_i j x'_j)| \\
&\le \max_i \sum_{j=1}^{n} |a_{ij}| x_j - x'_j| \\
&= \max_i \sum_{j=1}^{n} |a_{ij}| \; ||\mathbf{x} - \mathbf{x}'||_\infty \\
&\le \Big(\max_i \sum_{j=1}^{n} |a_{ij}|\Big) ||\mathbf{x} - \mathbf{x}'||_\infty \\
&\le \lambda ||\mathbf{x} - \mathbf{x}'||_\infty
\end{aligned}$$

provided $\sum_{j=1}^{n} |a_{ij}| \le \lambda$ for each i. The result is now clear.

[26] Using $(a+b)^2 \le 2a^2 + 2b^2$ in the first inequality. This is easily checked and worth remembering.

(b) Using the standard metric instead

$$\begin{aligned} |F(\mathbf{x}) - F(\mathbf{x}')|^2 &= \sum_i \sum_j (a_{ij}x_j - a_ijx'_j)^2 \\ &\le \sum_i [\sum_j a_{ij}^2 \sum_j |x_j - x'_j|^2 \\ &= (\sum_i \sum_j a_{ij}^2)(\sum_j |x_j - x'_j|^2) \\ &\le \lambda^2 ||\mathbf{x} - \mathbf{x}'||_2^2 \end{aligned}$$

where we have used the Cauchy-Schwarz inequality. The result follows.

(c) Immediate from the Contraction Mapping Theorem.

2. Suppose that $G = F^n$ is a contraction map. Then G has a unique fixed point, say x_0. Then

$$G(F(x_0)) = F^{n+1}(x_0) = F(G(x_0)) = F(x_0)\,,$$

so that $F(x_0)$ is also a fixed point of G. By uniqueness we thus have $F(x_0) = x_0$, so that F has a x_0 as a fixed point. Further, any fixed point of F is certainly also one of G, so by uniqueness of x_0 as a fixed point of G, F has unique fixed point x_0.

Problem 8.8 1. From the previous problem, assuming the a_{ij} are constants, the condition that $\alpha_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22} = \lambda < 1$ suffices.

2. Solving $F(\mathbf{x}) = \mathbf{x}$ we have, with

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

$$\mathbf{x} = (I - A)^{-1} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

3. The condition from 1. is just $\lambda_1^2 + \lambda_2^2 < 1$.

4. We have

$$\begin{aligned} |F(\mathbf{x} - F\mathbf{x}'|^2 &= |\lambda_1(x_1 - x_2)^2 + \lambda(x_2 - x'_2)^2| \\ &\le \max\{\lambda_1^2, \lambda_2^2\}((x_1 - x_2)^2 + (x_2 - x'_2)^2| \\ &= \max\{|\lambda_1|, |\lambda_2|\}^2 ||\mathbf{x} - \mathbf{x}'||_2^2 \end{aligned}$$

So F is a contraction if $\max\{|\lambda_1|, |\lambda_2|\} < 1$. On the other hand, it is easily seen, by looking at the standard basis vectors, that this condition is also necessary.

9 Sequences and Compactness

Problem 9.1 Let (X, d) be a metric space.

We will first show that *any* compact subset of X is closed.

Assume $A \subset X$ is compact. In order to show that A is closed, let $x_n \to x$ as $n \to \infty$, where $(x_n) \subset A$. We want to show $x \in A$.

Since A is compact, some *subsequence* of (x_n) converges to a, say, where $a \in A$. But this subsequence must also converge to x, by Theorem 9.1.1. Hence $x = a$, by Theorem 7.3.1. Thus $x \in A$, and so A is closed.

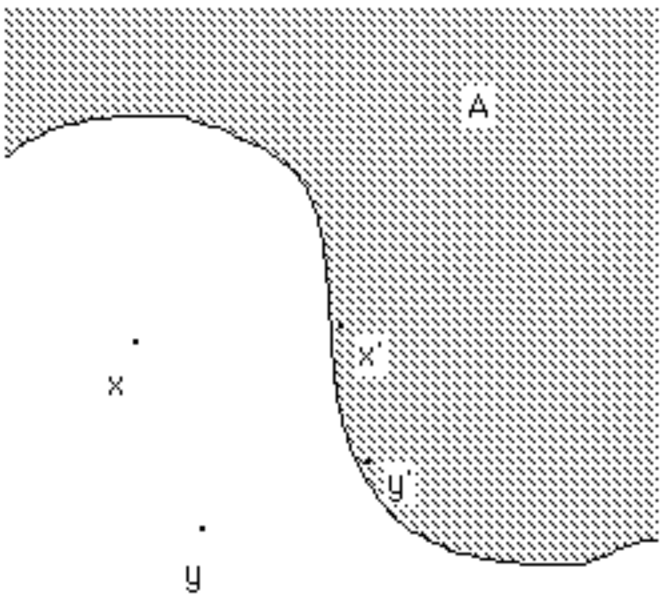
Assume next that $A \subset C$, where C is compact and A is closed. In order to show A is compact, let $(x_n)_{n=1}^{\infty} \subset A$. Since C is compact, some subsequence of (x_n) converges to c, say, where $c \in C$. Since A is closed, it follows that $c \in A$. Hence A is compact.

Problem 9.2 Let $x, y \in X$. Suppose $\epsilon > 0$ and choose $x', y' \in A$ such that[27]

$$d(x, x') \le d(x, A) + \epsilon$$

and

$$d(y, y') \le d(y, A) + \epsilon. \tag{25}$$



Then

$$f(x) - f(y) \quad = \quad d(x, A) - d(y, A)$$

[27]If A were closed, we could do this with $\epsilon = 0$; and obtain $d(x, x') = d(x, A)$, $d(y, y') = d(y, A)$. You should first think of this particular case. That is how I came up with the solution.

$$\begin{aligned} &\leq d(x,y') - d(y,A) \quad \ldots \text{ as } d(x,A) \leq d(x,y') \\ &\leq d(x,y') - d(y,y') + \epsilon \quad \ldots \text{ from (25)} \\ &\leq d(x,y) + \epsilon \quad \ldots \text{ from the triangle inequality.} \end{aligned}$$

Since $\epsilon > 0$ is otherwise arbitrary, it follows that

$$f(x) - f(y) \leq d(x,y).$$

Similarly,

$$f(y) - f(x) \leq d(x,y).$$

Hence

$$|f(x) - f(y)| \leq d(x,y).$$

Thus f is Lipschitz with Lipschitz constant 1.

Problem 9.3 1[NOTE: (i) A need not be bounded

(ii) We are intending the standard metric in $\mathbb{R}^n$; otherwise the result is false. For example, let $A = [0,1] \times [0,1] \subset \mathbb{R}^2$ Let $x = (2,0)$. Then $d_\infty(x,A) = 1$; and $d_\infty(x,y) = 1$ for any $y \in L$!!]

We know x has at best one nearest point in A.

Suppose

$$\begin{aligned} d(x,A) &= \lambda \\ d(x,y') &= (d(x,y'') = \lambda \ ; \end{aligned}$$

where $y' \in A$, $y'' \in A$, $y' \notin y''$.

Let $y = \frac{1}{2}y' + \frac{1}{2}y'' \in A$ as A is convex.

Suppose $a \neq b$ are real numbers.

Then

$$(a+b)^2 < 2a^2 + 2b^2 \ \text{ (and " = " if } a = b)$$

(since $2a^2 + 2b^2 - (a+b)^2 = a^2 + b^2 - 2ab = (a-b)^2 > 0$)

Replacing a by $a/2$ and b by $b/2$, we get

$$\left(\frac{a}{2} + \frac{b}{2}\right)^2 < \frac{a}{2}^2 + \frac{b}{2}^2$$

if $a \neq b$, and "=" is $a = b$. Now

$$\begin{aligned} d(x,y) &= \textstyle\sum_{i=1}^n (x_i - y_i)^2 \\ &= \textstyle\sum_{i=1}^n \left(\frac{x_i - y_i'}{2} + \frac{x_i - y_i''}{2}\right)^2 \\ &< \textstyle\sum_{i=1}^n \left[\frac{(x_i - y_i')^2}{2} + \frac{(x_i - y_i'')^2}{2}\right] \end{aligned}$$

since for at leastone i we have$y_i' \neq y_i''$, and hence $x_i - y_i' \neq x_i - y_i''$. Hence

$$\begin{aligned} d(x,y) &< \tfrac{1}{2}\textstyle\sum_{i=1}^n (x_i - y_i')^2 + \tfrac{1}{2}\sum_{i=1}^n (x_i - y_i'')^2 \\ &= \tfrac{1}{2}\lambda + \tfrac{1}{2}\lambda = \lambda \end{aligned}$$

Thus $d(x,y) < \lambda$, contradiction.

2. Suppose the original sequence does not converge to x. Then for some $\varepsilon > 0$, it is not true that there exists an N for which

$$d(x_n, x) < \varepsilon \quad \text{if} \quad n \geq N$$

Hence we can find a subsequence $(x_{n'})$ such that $d(x_{n'}, x) \geq \varepsilon$ for all n' (why???)

But this subsequence does not contain a further subsequence which converges to x. Hence the original sequence does converge to x.

Problem 9.4 1. Suppose S is a closed subset of a compact set X.

Let (x_n) be any sequence from S. Since $(x_n) \subset X$, there exists a subsequence with a limit in X.

Since S is closed, this limit must be in S. Hence S is compact.

2. (a) First note that if C is a compact subset of a metric space (X,d), then C is closed in X. To see this, suppose C is not closed. Then $\exists$ a sequence $(x_n) \subset C$ so that

$$x_n \to x \notin C$$

But any subsequence must then also converge to x, which contradicts the compactness (Definition 9.3.1) of C. Hence C is closed.

Now let $\{S_\lambda\}_{\lambda \in \Lambda}$ be a collection of compact sets. Let

$$S = \bigcap_{\lambda \in \Lambda} S_\lambda .$$

Then S is closed, being an intersection of closed sets. Since $S \subset S_{\lambda_0}$ (some fixed $\lambda_0 \in \Lambda$) and since S_{λ_0} is compact, it follows now from (1) that S is also compact.

(b) Let $S = S_1 \cup \ldots \cup S_N$ where the S_i are compact. Let (x_n) be any sequence from S. Then at least one of the S_i must contain an (infinite) subsequence of (x_n).

Since S_i is compact, there must be a further subsequence (of this subsequence) which has a limit in S_i (and hence in S). This implies S is compact.

$$\bigcup_{n=1}^{\infty} [n, n+1] = [1, \infty)$$

Remarks.

(1) Do not try to prove 9.4.2(a) by saying that any sequence $(x_n) \subset S$ is also a sequence in each S_λ (correct) and saying hence there was a

subsequence with a limit in each S_λ , as S_λ was compact. The problem here is that different S_λ may give different subsequences, and hence different limits.

(2) It is incorrect to let the collection of compact sets be $\{S_1, S_2, S_3., \ldots\}$. This assumes the collection is countable.

Problem 9.5 1. Straightforward.

2. Consider the sequence

$$\begin{aligned} x^1 &= (1, 0, 0, \ldots) \\ x^2 &= (1, \tfrac{1}{2}, 0, 0, \ldots) \\ x^n &= (1, \tfrac{1}{2}, \ldots, \tfrac{1}{n}, 0, 0, \ldots) \end{aligned}$$

Then $d(x^m, x^n) = \frac{1}{n+1}$ if $m > n$. Hence (x^n) is Cauchy. But (x^n) has no limit in X.

[Informally, the limit is $(1, \frac{1}{2}, \frac{1}{3} \ldots) \notin X$. But this is not really a rigorous argument since we have no definition of convergence to elements not in X.

This argument can be made rigorous by extending X to a larger metric space, but it is probably easier to justify the above as follows.]

Take any $x \in X$ and let

$$x = (a_1, \ldots, a_N, 0, 0, \ldots).$$

Then $d(x, x^n) \geq \frac{1}{N+1}$ for any $n \geq N$. Hence $x^n \not\to x$. Since $x \in X$ was arbitrary, this means (x^n) does not converge (in X). HENCE X is not complete.

3. Let S be the set consisting of all sequences of the form

$$x^n = (0, \ldots, 0, 1, 0, 0, \ldots)$$

where x_n has 1 in the n-th position. Then S is bounded since $d(x, Q) = 1$, where Q is the sequence of all zeros.

S is closed since the distance between any 2 members of S is 1. Thus S has no limit points (i.e. all its points are isolated) - and so S is closed.

S is not compact, since the sequence

$$x^1, x^2, x^3, x^4, \ldots$$

has no convergent subsequence (reason: the distance between any 2 members of the given sequence is 1, and the same must also be true for any subsequence).

10 Limits of Functions

Problem 10.1 (1) We have

$$\begin{aligned} 0 \le \frac{x^4}{x^2+y^2} &= x^2\frac{x^2}{x^2+y^2} \\ &\le x^2 \\ &\to 0 \end{aligned}$$

as $x \to 0$. Similarly

$$\frac{y^4}{x^2+y^2} \to 0$$

as $y \to 0$. Hence

$$\frac{x^4+y^4}{x^2+y^2} \to 0$$

as $(x, y) \to (0, 0)$.

NOTE: The point is that x^4 is fourth order and so for small x is much less than x^2, and hence much less than $x^2 + y^2$.

(2) On the line $y = x$, the function equals $x^3/(x^2 + x^4)$, and so approaches 0 as $(x, y) \to (0, 0)$.

On the curve $y = \sqrt{x}$, the function equals $x^2/(2x^2)$, and so approaches $1/2$ as $(x, y) \to (0, 0)$.

Hence the limit as $(x, y) \to (0, 0)$ does not exist.

(3)

From the diagram we expect the limit to be 1.

To prove this note

$$|\mathbf{x} - \mathbf{x}_1| \le |\mathbf{x} - \mathbf{x}_2| + |\mathbf{x}_2 - \mathbf{x}_1|,$$

and so

$$\frac{|\mathbf{x} - \mathbf{x}_1|}{|\mathbf{x} - \mathbf{x}_2|} \le 1 + \frac{|\mathbf{x}_2 - \mathbf{x}_1|}{|\mathbf{x} - \mathbf{x}_2|}. \qquad (26)$$

Note that the right side approaches 1 as $|\mathbf{x}| \to \infty$.[28]

[28]Since $|\mathbf{x} - \mathbf{x}_2| \ge |\mathbf{x}| - |\mathbf{x}_2| \to \infty$.

Similarly,

$$\frac{|\mathbf{x}-\mathbf{x}_2|}{|\mathbf{x}-\mathbf{x}_1|} \leq 1 + \frac{|\mathbf{x}_1-\mathbf{x}_2|}{|\mathbf{x}-\mathbf{x}_1|},$$

and so

$$\frac{|\mathbf{x}-\mathbf{x}_1|}{|\mathbf{x}-\mathbf{x}_2|} \geq \frac{1}{1+\frac{|\mathbf{x}_1-\mathbf{x}_2|}{|\mathbf{x}-\mathbf{x}_1|}}. \tag{27}$$

The right side again approaches 1

It now follows from (26), (27) and the Comparison Theorem that the required limit is 1.

Problem 10.2 (i)

$$\begin{aligned} \lim_{\substack{(x,y)\to(0,0)\\(x,y)\in S_1}} f(x,y) &= \lim_{x\to 0} \frac{ax^3}{x^4+a^2x^2} \\ &= \lim_{x\to 0} \frac{ax}{x^2+a^2} \\ &= 0 \quad \text{evenif } a=0 \end{aligned}$$

(ii)

$$\begin{aligned} \lim_{\substack{(x,y)\to(0,0)\\(x,y)\in S_2}} f(x,y) &= \lim_{x\to 0} \frac{ax^4}{x^4+a^2x^4} \\ &= \lim_{x\to 0} \frac{a}{1+a^2} \\ &= \frac{a}{1+a^2} \end{aligned}$$

(iii)

$$\begin{aligned} \lim_{\substack{(x,y)\to(0,0)\\(x,y)\in S_3}} f(x,y) &= \lim_{x\to 0} \frac{y^4}{y^6+y^2} \\ &= \lim_{x\to 0} \frac{y^2x}{y^4+1} \\ &= 0 \end{aligned}$$

(iv) Thus $\lim_{(x,y)\to(0,0)}$ does not exist, since if it did, the various limits in (i) – (iii) would be the same.

(v) $\lim_{y\to 0} f(x,y) = 0$ is clear – just fix X and take usual limit. Thus

$$\lim_{x\to 0}(\lim_{y\to 0} f(x,y)) = 0$$

(vi) SImilarly,

$$\lim_{y\to 0}(\lim_{x\to 0} f(x,y)) = 0$$

Problem 10.3 (a) To see that f is not bounded on any open ball centred at $(0,0)$, set $y = x^3$, so that

$$f(x,y) = \frac{x^5}{x^6 + x^6} = \frac{1}{x}$$

which is clearly unbounded as $(x,y) \to (0,0)$.

(b) The restriction of f to any straight line $L \subset \mathbb{R}^2$ which does not pass through the origin is continuous on L – the function os just the ratio of two continuous functions for which the denominator does not vanish.

If L does pass through the origin, then $y = \lambda x$ on L, for some $\lambda \in \mathbb{R}$. Thus on L,

$$f(x,\lambda x) = \begin{cases} \frac{\lambda x^3}{x^6+\lambda^2 x^2} & = \quad (x,y) \neq (0,0) \\ 0 & = \quad (x,y) = (0,0) \end{cases}$$

This function is in fact continuous everywhere.

11 Continuity

Problem 11.1 1. Let $f(x) = x^3 - x$. Then f is continuous and so $f^{-1}[0, \infty)$ is closed. That is $\{x : x^3 - x \geq 0\}$ is closed. Since $[-2, 2]$ is closed, the given set is thus closed as it is the intersection of two closed sets.

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(\mathbf{x}) = |\mathbf{x}| - \mathbf{x} \cdot \mathbf{y}_0$. Then f is continuous, since we can write

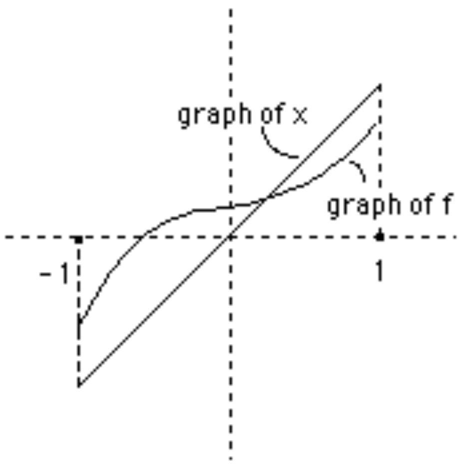
$$\begin{aligned} f(\mathbf{x}) &= f(x^1, \ldots, x^n) \\ &= \sqrt{(x^1)^2 + \cdots + (x^n)^2} - \left(x^1 \cdot y_0^1 + \cdots x^n \cdot y_0^n\right). \end{aligned}$$

Hence

$$f^{-1}[0, \infty) = \{\mathbf{x} : \mathbf{x} \cdot \mathbf{y}_0 \leq |\mathbf{x}|\}$$

is closed.

Problem 11.2 1.



Let $g(x) = f(x) - x$ for $x \in [-1, 1]$. Then $g(-1) = f(-1) + 1 \geq 0$ since $f(-1) \geq -1$. Similarly $g(1) \leq 0$. Since g is continuous, it follows from the Intermediate Value Theorem that $g(x) = 0$ for some $x \in [-1, 1]$. That is, $f(x) = x$ for some $x \in [-1, 1]$.

2. Let $f_k = (1 - 1/k)f$. Then f_k has Lipschitz constant $1 - 1/k$, and so has a fixed point x_k, say.

By compactness, on passing to a subsequence we have $x_{k'} \to x$, say, as $k' \to \infty$.

Now $x_{k'} = f_{k'}(x_{k'})$ and $x_{k'} \to x$. Thus if we can show $f_{k'}(x_{k'}) \to f(x)$, it follows $f(x) = x$ and so x is a fixed point of f. But

$$\left|f(x) - f_{k'}(x_{k'})\right| = \left|f(x) - f(x_{k'}) + \frac{1}{k'} f(x_{k'})\right|$$

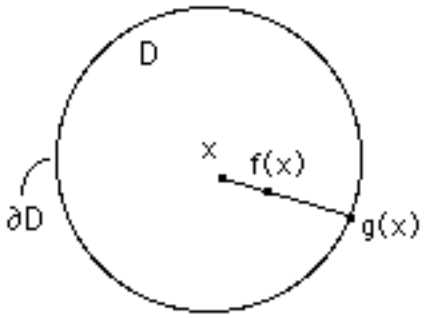
$$\begin{aligned} &\le \left|f(x) - f(x_{k'})\right| + \left|\frac{1}{k'} f(x_{k'})\right| \\ &\to 0 \end{aligned}$$

as $k' \to \infty$ (using the continuity and boundedness of f). Hence $f_{k'}(x_{k'}) \to f(x)$ as $k' \to \infty$, and this completes the proof.

3. Let f be rotation about the origin through any angle $\pi/2$, for example. Since the ray of angle θ is rotated onto the ray of angle $\theta + \pi/2$, there are no fixed points of f in the annulus A.

REMARK: We cannot prove the Brouwer Fixed Point Theorem at this stage, but it can be made plausible as follows.

Suppose there is *no* fixed point of f where $f : D \to D$ and f is continuous. For each $x \in D$ define $g(x) \in \partial D$ by taking the straight line from x to $f(x)$ and continuing it to the boundary. Let the corresponding boundary point be denoted by $g(x)$. Note that this construction is only well-defined if $f(x) \neq x$. It is not hard to write out an explicit formula for $g(x)$ and hence to show that g is continuous.



In other words, *assuming* that f has no fixed points, it follows that there exists a *continuous* map $g : D \to \partial D$. That this is not so is plausible, since our intuition is that such a *continuous* map cannot exist.

Problem 11.3 1. Theorem 7.3.4 implies that

$$x_n \to x \Rightarrow d(a, x_n) \to d(a, x)$$

i.e.

$$f(x_n) \to f(x)$$

Hence f is continuous.

2.
$$\begin{aligned} |f(x,y)| &\le |x| \qquad \text{all } (x,y), \text{ (why?)} \\ &\to \text{ as } x \to 0 \end{aligned}$$

(and hence as $(x, y) \to 0$ since $|x|$ does not even depend on y) i.e. f is continuous at $(0, 0)$.

3. Let

$$\begin{aligned} A &= \{(x,y) : x^2 \le y^3 \quad \text{and} \ \sin 2 \ge 3y\} \\ &= \{(x,y) : f(x,y) \le 0 \quad \text{and} \ g(x,y) \ge 0\} \\ (\text{where} f(x,y) &= x^2 - y^3 \text{ and } g(x,y) = \sin x - 3y.) \\ &= f^{-1}(-\infty, 0] \cup g^{-1}[0,\infty) \end{aligned}$$

Since f and g are continuous, A is thus the intersection of 2 closed sets, and so is closed.

Problem 11.4 1. If A is the given set then

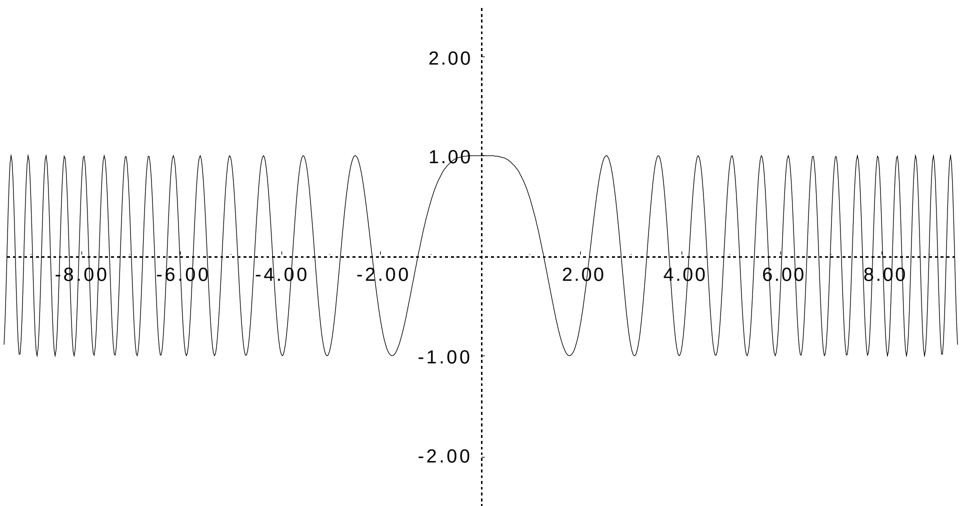
$$A = f^{-1}(-\infty), 7) \cap g^{-1}[(-\infty, 0) \cup (0, \infty)$$

where

$$\begin{aligned} f(x,y) &= x^2 - 3xy \\ g(x,y) &= \sin x \end{aligned}$$

2. One idea is to give a function f which becomes "steeper and steeper" as $x \to \infty$. For example

$$f(x) = \cos x^2 .$$



Then

$$f(x) = \begin{cases} 1 & x = \sqrt{2n\pi} \\ -1 & x = \sqrt{(2n+1)\pi} \end{cases}$$

But

$$\sqrt{(2n+1)\pi} - \sqrt{2n\pi} = \sqrt{\pi}\left(\sqrt{2n+1} - \sqrt{2n}\right) \to 0$$

(why?) as $n \to \infty$.

Hence $\nexists\ \delta > 0$ so that

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < 2$$

Hence f is <u>not</u> uniformly continuous (but it <u>is</u> continuous and bounded)

3. Since f is continuous on $[-a, a]$, which is a closed bounded interval, f must be uniformly continuous on $[-a, a]$. We can also show this directly:

$$\begin{aligned} |f(x) - f(y)| &= |x^3 - y^3| \\ &= |(x - y)(x^2 + xy + y^2)| \\ &\leq 3a^2|x - y| \quad \text{if } x, y \in [-a, a) \end{aligned}$$

Hence $|f(x) - f(y)| < \varepsilon$ if $|x - y| < \varepsilon/3a^2$ (and if x, $y \in [-a, a)$) that is f is uniformly continuous on $[-a, a]$.

To show f is not uniformly continuous on $\mathbb{R}$, we argue as in part 2.

We will find a sequence (x_n, y_n) such that

$$\underbrace{|x_n - y_n| \to 0 \text{ as } n \to \infty}_{(a)}$$

and yet

$$\overbrace{|f(x_n) - f(y_n)| = 1}^{(b)}$$

Just choose x_n so $x_n^3 = n$ and y_n so $y_n^3 = n + 1$. Then

$$|f(x_n) - f(y_n)| = 1$$

but

$$\begin{aligned} |x_n - y_n| &= |\sqrt[3]{n+1} - \sqrt[3]{n}| \\ &= \tfrac{(n+1)-n}{(n+1)^{2/3}+n^{1/3}(n+1)^{1/3}+n^{2/3}} \\ [\text{using } a^3 - b^3 &= (a - b)(a^2 + ab + b^2) \\ \text{and so } a - b &= (a^{1/3} - b^{1/3})(a^{2/3} + a^{1/3}b^{1/3} + b^{2/3})] \\ &= \tfrac{1}{(n+1)^{2/3}+n^{1/3}(n+1)^{1/3}+n^{2/3}} \\ &\to 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

From (a) and (b) we see (as in part 2) that f is not uniformly continuous in $\mathbb{R}$.

Problem 11.5 Clearly $f(x) = 0$ for $x \in A$, $f(x) = 1$ fpr $x \in ??$ and $0 < f(x) < 1$ otherwise. Moreover, $d(x, A)$ and $d(x, B)$ is continuous as a function of x, since it is in fact Lipschitz by Problem 9.2.

Thus f us continuous (being the ratio of continuous functions, where the denominator is $\neq 0$ as A and B are disjoint closed* sets).

Problem 11.6 Let

$$(G(f))(x) = c + \int_a^x K(t, f(t))dt$$

Then $t \mapsto K(t, f(t))$ is continuous; since it is obtained by composition from the continuous functions f, K and $t \mapsto t$.

(*) If $d(x, A) = 0$ then $x \in A$ (why?) and if $d(x, B) = 0$ then $x \in B$ (why?)

Hence $A \cup B \neq \phi$, contradiction.

Since the "indefinite integral" of a continuous function is continuous (last year!), we see $G(f)$ is continuous.

Hence

$$G : \mathcal{C}[a, b] \to \mathcal{C}[a, b] \tag{1}$$

We next compute on $\mathcal{C}[a, a + h]$

$$\begin{aligned} \|G(f_a) - G(f_2)\|_u &= \\ &= \sup_{x\in[a,a+h]} |G(f_1))(x) - (G(f_2))(x)| \\ &= \sup_{x\in[a,a+h]} |\textstyle\int_a^x K(t, f_1(t))dt - \int_a^x K(t, f_2(t))dt| \\ &\leq \sup_{x\in[a,a+h]} \textstyle\int_a^x |K(t, f_1(t)) - K(t, f_2(t))|dt \end{aligned}$$

[since $|\int_a^c h| \leq \int_a^c |h|$ (last year!)]

$$\begin{aligned} &\leq \textstyle\int_a^{a+h} M|f_1(t) - f_2(t)|dt \\ &\leq Mh\|f_1 - f_2\| \end{aligned}$$

(uniform metric u on $\mathcal{C}[a, a + h]$)

Thus if $a + h \leq b$, i.e. $h \leq b - a$ and $Mh < 1$, i.e. $h < 1/M$ we see G is a contraction map on $\mathcal{C}[a, a + h]$. That is,

$$\text{if } h < \min\{b - a, 1/M\}, \text{then } G \text{ is a contraction on } \mathcal{C}[a, a + h] \tag{2}$$

Since G is a contraction map in the complete metric $\mathcal{C}[a, a + h]$, it has a unique fixed point.

But u is a fixed point of G means exactly the same as saying

$$u(x) = c + \int_a^x K(t, u(t))dt$$

for all $x \in [a, a + h]$.

Thus we are finished.

Problem 11.7 (1) Let $f(x) = g(x) = x \quad \forall x \in \mathbb{R}$.

Why is $h(x) = x^2$ not uniformly continuous?

(2) Assume f and g are both uniformly continuous. Suppose $\varepsilon > 0$. Choose $\delta_1 > 0$ so that

$$d(x, y) < \delta_1 \Rightarrow |f(x) - f(y)| < \varepsilon/2$$

Choose $\delta_2 > 0$ so that

$$d(x,y) < \delta_2 \Rightarrow |g(x) - g(y)| < \varepsilon/2$$

Then

$$d(x,y) < \min\{\delta_1, \delta_2\} \Rightarrow |(f+g)(x) - (f+g)(y)| < \varepsilon$$

Hence $f+g$ is uniformly continuous.

Problem 11.8 Suppose f, g are continuous, with notation as in the Question.

(1) Take any $a \in X_1$. Take any $\varepsilon > 0$. First choose $\delta > 0$ so that

$$g[B_\delta^{X_2}(a)] \subset B_\varepsilon^{X_3}[f(a)] \ldots \text{Theorem 11.1.2(3)}$$

Next choose $\delta' > 0$ so that

$$f[B_{\delta'}^{X_1}(a)] \subset B_\delta^{X_2}(a) \ldots \text{Theorem 11.1.2(3) again}$$

Hence

$$g[f[B_{\delta'}^{X_1}(a)]] \subset g[B_\delta^{X_2}(a)]$$

i.e.

$$(g \circ f)[B_{\delta'}^{X_1}(a)] \subset g[B_\delta^{X_2}(a)]$$

These give

$$(g \circ f)[B_{\delta'}^{X_1}(a)] \subset B_\varepsilon^{X_3}[f(a)]$$

Thus $g \circ f$ is continuous at (any) $a \in X_1$ (Theorem 11.1.2(3) again) and hence $g \circ f$ is continuous.

(2) Suppose $E \subset X_3$ is open. Hence $g^{-1}[E](\subset X_2)$ is open ....Theorem 11.4.1(2).

Hence $\underbrace{f^{-1}[g^{-1}[E]]}_{(g\circ f)^{-1}[E]}(\subset X_3)$ is open .....Theorem 11.4.1(2)

Hence $g \circ f$ is continuous (Theorem 11.4.1(2))

Problem 11.9 Suppose (X,d) and (Y,p) are metric spaces and $D \subset X$ is dense.

1. Suppose $f : D \to Y$ is uniformly continuous.

If $d_n(\in D) \to x \in X$, define

$$\overline{f}(x) = \lim f(d_n)$$

(A) We need to check

(i) $\lim f(d_n)$ exists

[PROOF: by <u>uniform</u> continuity and the fact (d_n) is Cauchy, it follows $(f(d_n))_{n=1}^{\infty}$ is Cauchy]

(ii) If $d_n(\in D) \to x$ and $d'_n(\in D) \to x$ then $\lim f(d_n) = \lim f(d'_n)$

[PROOF: $d(d_n, d'_n) \to 0$, and hence $(f(d_n), f(d'_n)) \to 0$, using once again the <u>uniform</u> continuity of f]

(iii) If $x \in D$ then $\overline{f}(x) = f(x)$

[PROOF: This is just a particular case of (ii) - take one of the approximating sequences having all terms equal to x.]

(B) We next need to show that $\overline{f}$ is uniformly continuous. So suppose $\varepsilon > 0$. Choose $\delta > 0$ so that

$$d, d' \in D \text{ and } d(d, d') < \delta \Rightarrow p(f(d), f(d')) < \varepsilon$$

Now suppose x, $y \in X$ and $d(x, y) < \delta$. Choose

$$\begin{array}{lcl} d_{n_1}(\in D) & \to & x \\ d_n(\in D) & \to & y \end{array}$$

Then

$$d(d_n, d'_n) \to d(x, y)(< \delta)$$

by Theorem 7.3.4. Hence

$$d(d_n, d'_n) < \delta \quad \text{if } n \geq N \text{ (say)}$$

Hence

$$\rho(f(d_n), f(d'_n)) < \varepsilon \quad \text{if } n \geq N$$

Hence

$$\rho(\overline{f}(x), \overline{f}(y)) < \varepsilon \ldots \text{by Thm.7.3.4}$$

Hence $\overline{f}$ is uniformly continuous.

12 Uniform Convergence of Functions

Problem 12.1 Let $f_m(x) = x^m$ if $x \in [0,1]$. Let $f(x) = 0$ if $x \in [0,1)$ and let $f(1) = 1$. Then $f_m \to f$ pointwise as $m \to \infty$.

In Definition 12.1.1 consider $\epsilon = 1/2$. For each m there exist x such that

$$|f_m(x) - f(x)| \geq 1/2.$$

To see this, just choose $x \in [0,1)$ such that $x^m \geq 1/2$. Thus it is *not* the case that $f_m \to f$ uniformly.

Problem 12.2 Let

$$f_n(x) = \sum_{k=1}^{n} \frac{\sin kx}{k^2}.$$

Then $f_n(x) \to f(x)$ for all x (by the definition of f).[29]

Each f_n is continuous, being a finite sum of continuous functions. Moreover,

$$\begin{aligned} |f(x) - f_n(x)| &= \left| \sum_{k>n+1} \frac{\sin kx}{k^2} \right| \\ &\leq \sum_{k>n+1} \left| \frac{\sin kx}{k^2} \right| \\ &\leq \sum_{k>n+1} \left| \frac{1}{k^2} \right| \\ &\to 0 \end{aligned}$$

as $n \to \infty$.

Thus $f_n \to f$ *uniformly.* Hence f is the uniform limit of continuous functions, and hence is continuous by Theorem 12.3.1.

Problem 12.3 1. Consider the double sequence

$$\begin{matrix} 1 & 1 & 1 & 1 & \ldots \\ 0 & 1 & 1 & 1 & \ldots \\ 0 & 0 & 1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

Then for each n, $a_{mn} \to 0$ as $m \to \infty$. And for each m, $a_{mn} \to 1$ as $n \to \infty$. Hence $b_n = 0$ for all n and $c_m = 1$ for all m. In particular, $\lim_{m\to\infty} c_m$ and $\lim_{n\to\infty} b_n$ both exist, but are not equal.

[29] Note that the series does converge for each x, since each term in the series has absolute value $\leq 1/k^2$, and $\sum 1/k^2$ converges. See Problem 8.2.

2.(a) Suppose $\epsilon > 0$. Then there exists M such that

$$m \geq M \Rightarrow |a_{mn} - b_n| < \epsilon \ \forall n.$$

Hence, if $p, m \geq M$, then for all n

$$|a_{mn} - a_{pn}| \leq |a_{mn} - b_n| + |b_n - a_{pn}| < 2\epsilon.$$

Fixing p and m and letting $\rho \to \infty$, it follows

$$|c_m - c_p| \leq 2\epsilon \quad \text{if } p, m \geq M.$$

Hence $(c_m)_{m=1}^{\infty}$ is Cauchy.

2.(b) Note that

$$|b_n - c| \leq |b_n - a_{mn}| + |a_{mn} - c_m| + |c_m - c|. \tag{28}$$

Suppose $\epsilon > 0$.

Using uniform convergence, first choose M so

$$m \geq M \Rightarrow |b_n - a_{mn}| < \epsilon/3 \tag{29}$$

for all n. By increasing M if necessary we can also assume

$$m \geq M \Rightarrow |c_m - c| < \epsilon/3. \tag{30}$$

Next use the fact $a_{Mn} \to c_M$ as $n \to \infty$ to choose N so that

$$n \geq N \Rightarrow |a_{Mn} - c_M| < \epsilon/3. \tag{31}$$

From (28), (29), (30) and (31), it follows that

$$|b_n - c| < \epsilon,$$

if $n \geq N$. Thus $b_n \to c$, as required.

13 First Order Systems of Differential Equations

Problem 13.1 1. Assume

$$x(t) = 1 + \int_0^t [x(x))^2 ds \ldots t \in [0,1] \tag{1}$$

Then (by differentiating)

$$x'(t) = [x(t)]^2, \quad x(0) = 1 \tag{2}$$

Conversely, assuming (2), for $t \in [0,1]$ we get

$$\begin{aligned} x(t) &= x(0) + \textstyle\int_o^t x'(s)ds \\ &= x(0) + \textstyle\int_o^t [x(x)]^2 ds \end{aligned}$$

i.e. (1) holds.

<u>Summary</u> (1) and (2) are equivalent.

2. Assume

$$\begin{cases} x''(t) + x'(t) + y(t) &= 0 \\ y'(t) + y(t) + x(t) &= 0 \\ x(0) = 1 \ , \ x'(0) &= 0 \ , \ y(0) = 1 \end{cases}$$

Let $x_1(t) = x(t)$, $x_2(t) = x'(t)$, $x_3(t) = y(t)$.

Then

$$\begin{cases} x_1^1(t) &= x_2(t) \\ x_2^1(t) &= -x_2(t) - x_3(t) \\ x_3^1(t) &= -x_1(t) - x_3(t) \\ x_1(0) = 1, & x_2(0) = 0, \ x_3(0) = 1 \end{cases}$$

Conversely, assume these latter and let $x(t) = x_1(t)$, $y(t) = x_3(t)$. Then we can easily derive the first.

3.

$$\begin{aligned} |f(t_1) - f(t_2)| &= \left| \textstyle\int_a^b (K(s,t_1) - K(s,t_2))\, x(s) ds \right| \\ &\leq \textstyle\int_a^b |K(s,t_1) - K(s,t_2)| \ |x(s)| ds \end{aligned}$$

Since $x(t)$ is continuous on $[a,b]$, $x(s) \leq M$ (say) for $a \leq s \leq b$.

Suppose $\varepsilon > 0$. Since K is uniformly continuous, there exists δ so that

$$|(s_1,t_1) - (s_2,t_2)| < \delta \Rightarrow |K(s_1,t_1) - K(s_2,t_2)| < \varepsilon$$

In particular (for all s)

$$|t_1 - t_2| < \delta \Rightarrow |K(s,t_1) - K(s,t_2)| < \varepsilon$$

Thus

$$|t_1 - t_2| < \delta \Rightarrow |f(t_1) - f(t_2)| \le \int_a^b \varepsilon M ds = \varepsilon M(b-a)$$

Hence f is continuous on $[a, b]$.

4. For $x \in C[0,1]$ define the function Tx by

$$(Tx)(t) = e^t + \frac{1}{2}\int_0^1 t\cos(ts)x(s)ds$$

for $0 \le t \le 1$.

Note (i) Clearly (the function) x is a fixed point of T iff

$$x(t) = e^t + \frac{1}{2}\int_0^1 t\cos(ts)x(s)ds$$

(ii) If we apply 3 (above) with

$$K(s,t) = \frac{1}{2}t\cos(ts)$$

we see that $Tx : C[0,1]$.

Thus

$$T : C[0,1] \to C[0,1]$$

(iii) T is a contraction map in the supnorm (i.e. the uniform norm), since

$$\begin{aligned} |Tx_1(t) - Tx_2(t)| &= \tfrac{1}{2}\left|\int_0^1 t\cos(ts)(x,(s)) - x_2(s))ds\right| \\ &\le \tfrac{1}{2}\int_0^1 |t\cos(ts)| \; |x_1(s) - x_2(s)|ds \\ &\le \tfrac{1}{2}\max_{s\in[0,1]} |x_1(s) - x_2(s)| \end{aligned}$$

that is, $\|Tx_1 - Tx_2\|_u \le \frac{1}{2}\|x_1 - x_2\|_u$, so T is a contraction map (with contraction ratio 1/2).

Since $C[0,1]$ is a complete metric space, it follows that T has a unique fixed point x, say. From (i) it follows that x is a solution of the given integral equation - and is in fact the unique solution.

14 Fractals

Problem 14.1 Assume (by way of obtaining a contradiction) that $I \subset C$ where I is a non-empty open interval. Choose $x \in I$.

There are points arbitrarily close to x which do not have a ternary expansion consisting solely of 0 and 2. To see this, choose y so that its ternary expansion agrees with that of x for the first N terms (N suitably large), but so that all remaining terms are 1.

Since $y \notin C$, it follows we can choose points *not* in C but arbitrarily close to x. This contradicts the fact I is open. Hence there is no I as assumed.

Problem 14.2 To show that $G(f)$ is compact, assume $(x_i, f(x_i))$ is a sequence of points from $G(f)$. By compactness of A, $x_i' \to x \in A$ for some subsequence (x_i'). But then $f(x_i') \to f(x)$, since f is continuous.

Since $x_i' \to x\ (\in A)$ and $f(x_i') \to f(x)$, it follows $(x_i', f(x_i')) \to (x, f(x))$.[30] Hence $G(f)$ is compact.

Next assume that $f_k \to f$ uniformly on A. Assume $\epsilon > 0$. Then there exists N such that $|f(x) - f_k(x)| \leq \epsilon$ for all $k \geq N$.

Claim: $d(G(f), G(f_k)) \leq \epsilon$ if $k \geq N$.

Suppose that $(x, f(x)) \in G(f)$. Then

$$d\Big((x, f(x)), G(f_k)\Big) \leq d\Big((x, f(x)), (x, f_k(x))\Big) = |f(x) - f_k(x)| \leq \epsilon.$$

Since $(x, f(x))$ is an arbitrary point in $G(f)$, it follows that

$$G(f) \subset \Big(G(f_k)\Big)_\epsilon.$$

Similarly,

$$\Big(G(f_k)\Big)_\epsilon \subset G(f).$$

This proves the claim.

From the claim, we immediately have that

$$G(f_k) \to G(f)$$

in the Hausdorff metric sense.

[30] This uses the fact that a sequence of n-tuples ($n+1$-tuples) converges to a point iff the associated sequences of components converge to the corresponding components of the point.

15 Compactness

Problem 15.1 Let $S \subset X$, (X, d) a metric space. Suppose that S is totally bounded. Given $\epsilon > 0$, there exist $x_1, \ldots, x_n \in S$ such that for any $y \in S$, $d(y, x_j) < \epsilon/2$ for some $1 \le j \le n$. But if $z \in \overline{S}$ then there is $y \in S$, $d(z, y) < \epsilon/2$, and so $d(z, x_j) < \epsilon$ for some $1 \le j \le n$. But this says exactly that $\overline{S}$ is totally bounded.

Since any subset of a totally bounded set is totally bounded, the converse is clear.

Problem 15.2 Since $f_y(x) = f(x, y)$, the set $\mathcal{F}$ is equicontinuous iff for any $\epsilon > 0$ there is $\delta > 0$ such that for any $y \in [0, 1]$,

$$|x_1 - x_2| < \delta \Rightarrow |f(x_1, y) - f(x_2, y)| < \epsilon\,.$$

But this is immediate from the *uniform* continuity of f, which holds because f is continuous on a compact set.

Problem 15.3

1. Two examples are $f_1(x) = \sin(\frac{1}{x})$ and $f_2(x) = \frac{1}{x}$. Neither has a limit at $x = 0$, but for different reasons.

2. *Uniqueness* Suppose that $g_1, g_2 : \overline{A} \to \mathbb{R}$ are continuous and both agree with f on A. The subset of $\overline{A}$ on which they agree is a closed subset of $\overline{A}$ containing A, and so must be all of $\overline{A}$.

   Existence For $x \in \overline{A} \backslash A$, let $(x_n) \subset A$, $x_n \to x$. Then (x_n) is Cauchy, whence so is $f(x_n)$. Define

   $$g(x) = \lim_n f(x_n)\,.$$

   Essentially the same argument shows firstly that g is in fact well defined, and is continuous at every point of $\overline{A} \backslash A$, and hence on $\overline{A}$ since it agrees with f on A. (In fact g will be uniformly continuous.)

3. Suppose X, Y are metric spaces, $A \subset X$ and $f : A \to Y$ is uniformly continuous. Suppose further that Y is complete. Then f extends uniquely to a (uniformly) continuous function $g : \overline{A} \to Y$. The proof is identical to the above.