FINITE ELEMENT APPROXIMATIONS AND THE DIRICHLET PROBLEM FOR SURFACES OF PRESCRIBED MEAN CURVATURE

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Abstract. We give a finite element procedure for the Dirichlet Problem corresponding to surfaces of prescribed mean curvature and prove an optimal convergence estimate in the H^1-norm.

1. H-Harmonic Maps

The numerical solution of the classical H-Plateau Problem consists of approximating disc-like surfaces with prescribed boundary curve and prescribed mean curvature H. For a detailed discussion of the algorithms and theory see [6] for the case of zero mean curvature, and [7] for the constant mean curvature case. In this paper we consider the associated H-Dirichlet problem.

Estimates for finite element approximations to solutions of general nonlinear elliptic systems are obtained in [4], using a continuity method involving L^∞ estimates for the discrete problem. Here we give a much shorter proof of the H^1 estimate, avoiding the need for L^∞ estimates and only assuming the discrete and smooth data are close in the H^1 sense. Our techniques apply to a wide class of nonlinear systems. We treat the case of a non-polygonal and non-convex boundary and give the explicit dependence on the non-degeneracy constant of the smooth solution being approximated. The arguments are prototypes of those used in [7] for treating the more difficult case of the (free boundary) H-Plateau Problem. The main tool for avoiding L^∞ norms in the present "borderline" case is the isoperimetric inequality due to Rado, see Remark 3.1.

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Throughout, $\Omega\,(\subset\,\mathbb{R}^2)$ is a bounded domain with C^2 boundary. Function spaces will consist of functions defined over Ω with values in $\mathbb{R}^3$ unless otherwise clear from context. Constants will depend on Ω and other quantities as indicated.

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By $|\cdot|_{H^1}$ is meant the H^1 seminorm, and by $\|\cdot\|_{H^1}$ the full norm. Note that by Poincaré's inequality, $|\cdot|_{H^1(\Omega)}$ is a norm on $H^1_0(\Omega)$.

For vectors $a, b, c \in \mathbb{R}^3$, the *triple product* is defined by

$$[a, b, c] = a \cdot b \times c.$$

This is invariant under cyclic permutations of a, b and c, and anti-symmetric with respect to interchanging any two. It is the volume of the parallelopiped spanned by a, b and c.

Definition 1.1. Suppose H is a real number. A function $u \in H^2(\Omega; \mathbb{R}^3)$ is *H-harmonic* with boundary data $u^0 \in H^2(\Omega; \mathbb{R}^3)$ if

$$\triangle u = 2H u_x \times u_y \qquad \text{a.e. in } \Omega \tag{1.1}$$

$$u = u^0 \qquad \text{on } \partial\Omega \tag{1.2}$$

Example 1.1. Let D be the closed unit disc in $\mathbb{R}^2$. Let

$$u^0(x, y) = (x, y, 0) \colon D \to \mathbb{R}^3$$

with $0 < H < 1$. There are two solutions of (1.1) and (1.2) obtained by mapping the unit disc D conformally, i.e. stereographically projecting from a suitable point, onto the *lower* spherical caps obtained from each of the two spheres of radius $1/H$ (mean curvature H) which contain the image of $u^0|_{\partial\Omega}$. These solutions are called *small* or *large* depending on whether their images do not, or do, contain a hemisphere. (We use this example for test computations, see Tables 1 and 2.) If $-1 < H < 0$ then one similarly obtains two solutions from the upper spherical caps. If $H = 0$ then there is exactly one solution, the map $u(x, y) = (x, y, 0) \colon D \to \mathbb{R}^3$. If $H = 1$ then one obtains a solution by mapping onto the lower hemisphere of a sphere of radius 1, and onto the upper hemisphere if $H = -1$.

Equation (1.1) is the Euler-Lagrange system associated to the *H-Dirichlet integral*

$$D_H(u) = D_H(u; \Omega) = \frac{1}{2}\int_\Omega |\nabla u|^2 + 2HV(u), \tag{1.3}$$

where

$$V(u) = V(u; \Omega) := \frac{1}{3}\int_\Omega [u, u_x, u_y] \tag{1.4}$$

can be thought of as the signed volume of the cone over the origin obtained from the image of u. In fact, direct computation and integration by parts easily gives

$$\begin{aligned} \langle D_H{}'(u), \varphi\rangle = \langle D_H{}'(u; \Omega), \varphi\rangle &:= \frac{d}{dt}\bigg|_{t=0} D_H(u + t\varphi) \\ &= \int_\Omega \nabla u \nabla\varphi + 2H\int_\Omega [\varphi, u_x, u_y] \end{aligned} \tag{1.5}$$

for $u \in C^2(\bar{\Omega};\mathbb{R}^3)$ and $\varphi \in C_0^2(\bar{\Omega};\mathbb{R}^3)$, and hence for $u \in H^1 \cap L^\infty(\Omega;\mathbb{R}^3)$ and $\varphi \in H_0^1 \cap L^\infty(\Omega;\mathbb{R}^3)$ by a limit argument, for example see [10, Remark III.1.1]. If $u \in H^1 \cap L^\infty$ is stationary for D_H, i.e.

$$\int_\Omega \nabla u \nabla \varphi + 2H \int_\Omega [\varphi, u_x, u_y] = 0 \tag{1.6}$$

for all $\varphi \in H_0^1 \cap L^\infty$, then u is said to be a *weak* solution of (1.1).

Example 1.1 is fairly typical. Arguing heuristically, the energy functional $D_H(u)$ is cubic in u and thus one expects (generically) either two or no stationary points. In the former case one expects the "smaller" solution to be a local minimum and the "larger" solution to be unstable.

Indeed, one has the following result due to the combined work of Heinz, Werner, Hildebrandt, Jäger, Wente, Brezis–Coron, Struwe and Steffen. For detailed references see Struwe [10, 11].

Theorem 1.1. *Assume $u^0 \in H^1 \cap L^\infty(\Omega;\mathbb{R}^3)$ and $H \in \mathbb{R}$ satisfy*

$$\|u^0\|_{L^\infty}|H| \le 1.$$

Then there exists $\underline{u} \in u^0 + H_0^1 \cap L^\infty$ such that

$$D_H(\underline{u}) = \min\left\{ D_H(v) : v \in u^0 + H_0^1,\ \|v\|_{L^\infty}|H| \le 1 \right\}.$$

Moreover,

$$\|\underline{u}\|_{L^\infty} \le \|u^0\|_{L^\infty} \qquad (*)$$

and $\underline{u}$ is a weak solution to (1.1) *and* (1.2)*.*

If furthermore

$$\|u^0\|_{L^\infty}|H| < 1 \qquad (**)$$

then $\underline{u}$ is the unique *local minimum of D_H in $u^0 + H_0^1 \cap L^\infty$. Moreover, $\underline{u}$ is the unique weak solution of* (1.1) *and* (1.2) *which satisfies* $(*)$. *The function $\underline{u}$ is called the* small *solution of* (1.1) *and* (1.2).

Under the same assumption $(**)$ *if $H \neq 0$ and u^0 is not constant, there is also a second weak solution $\bar{u}$ to* (1.1) *and* (1.2) *which satisfies*

$$\|\bar{u}\|_{L^\infty} > \|u^0\|_{L^\infty}.$$

Any such solution is called a large *solution to* (1.1) *and* (1.2).

If $u^0 \in H^2(\Omega,\mathbb{R}^3)$ then any weak solution to (1.1) *and* (1.2) *belongs to $H^2(\Omega,\mathbb{R}^3)$.*

Remark 1.1.

1. The large solution need not be unique, although one would expect that this is the generic situation. An example of Wente, [11, Example IV.3.7], gives a continuum of solutions for Ω the unit disc and boundary data $u^0(x,y) = (x,0,0)$. See Fig. 1 for the image of the trivial small solution and of one of the large solutions on a relatively coarse grid. Rotation of $u(\Omega)$ around the u_1-axis gives a continuum of solutions.

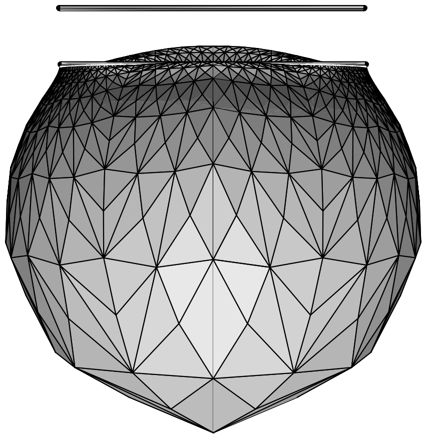

FIGURE 1. Wente's example (discrete approximations); small and one of a continuum of large solutions

2. The existence of a large solution is obtained by a mountain pass type argument, see [2] and [11, Theorem III.4.8]

Remark 1.2 (Nondegeneracy). We will be interested in approximating functions $u \in H^2(\Omega;\mathbb{R}^3)$ which are H-harmonic and nondegenerate in the sense that the second variation $D_H{}''(u)$ has no zero eigenvalues. This is always true for small solutions, see [11, Lemma III.4.7].

More precisely, for $u \in H^2(\Omega;\mathbb{R}^3)$ and $\varphi, \psi \in H^2_0(\Omega;\mathbb{R}^3)$ one first easily checks by direct computation and integration by parts that

$$D_H{}''(u)(\varphi,\psi) = D_H{}''(u;\Omega)(\varphi,\psi) := \frac{\partial^2}{\partial s\,\partial t}\bigg|_{s=t=0} D_H(u+t\varphi+s\psi)$$

$$= \int_\Omega \nabla\varphi\nabla\psi + 2H\int_\Omega [u,\varphi_x,\psi_y] + [u,\psi_x,\varphi_y] \tag{1.7}$$

$$= \int_\Omega \nabla\varphi\nabla\psi + 2H\int_\Omega [\psi,u_x,\varphi_y] + [\psi,\varphi_x,u_y], \tag{1.8}$$

see [10, Remark III.1.1] and the paragraph following (3.2). From (1.8) $D_H{}''(u)$ extends to a bounded symmetric bilinear functional on H^1_0, since

$$\int_\Omega |\nabla u|\,|\varphi|\,|\nabla\varphi| \le \|\nabla u\|_{L^4}\|\varphi\|_{L^4}\|\nabla\varphi\|_{L^2} \le c\|u\|_{H^2}\|\nabla\varphi\|^2_{L^2}$$

It follows that the inner product $|\cdot|_{H^1}$ induces a bounded self-adjoint linear operator $\nabla^2 D_H(u)\colon H^1_0 \to H^1_0$. The eigenvalues of $\nabla^2 D_H(u)$ are real, bounded below by $-\lambda_0$ (say) and have no accumulation point. Moreover, $\lambda_0 = \lambda_0(\Omega, H, \|u\|_{H^2})$, as follows from using (1.8) to estimate the Raleigh-Ritz quotient.

The *nondegeneracy constant* λ of $D_H{}''(u)$ is defined by

$$\lambda = \min\Big\{\,|\gamma| : \gamma \text{ is an eigenvalue of } \nabla^2 D_H(u)\,\Big\}$$

Then $\nabla^2 D_H(u)$ is one-one and onto iff $\lambda > 0$. Let

$$\varphi = \varphi^+ + \varphi^- \tag{1.9}$$

denote the $|\cdot|_{H^1}$ orthogonal decomposition of $\varphi \in H_0^1$ into members of the positive and negative spaces H^+ and H^- corresponding to the eigenvalues and eigenfunctions of $\nabla^2 D_H(u)$. Then

$$D_H{}''(u)(\varphi, \varphi^+ - \varphi^-) \geq \lambda |\varphi|^2_{H^1} \tag{1.10}$$

for all such φ and λ is the largest such real.

¿From the eigenfunction equation, c.f. (1.8), together with the estimate for λ_0, one obtains $\varphi^- \in H^2$ and

$$\|\varphi^-\|_{H^2} \leq \nu |\varphi^-|_{H^1} \tag{1.11}$$

where $\nu = \nu(\Omega, H, \|u\|_{H^2}, d)$ with d the dimension of H^-.

2. Discrete H-Harmonic Maps

For $h > 0$ let $\mathcal{T}_h$ be a triangulation of Ω by triangles T whose side lengths are bounded above by ch for some c independent of h and whose interior angles are bounded away from zero uniformly and independently of h. The intersection of any two different triangles is either empty, a common vertex, or a common edge.

Let

$$\Omega_h = \bigcup_{T \in \mathcal{T}_h} T.$$

Let

$$\begin{aligned} X_h &= \left\{ u_h \in C^0(\Omega_h; \mathbb{R}^3) : u_h|_T \in \mathbb{P}_1(T) \quad \forall T \in \mathcal{T}_h \right\}, \\ X_{h0} &= \{ \varphi_h \in X_h : \varphi_h|_{\partial\Omega_h} = 0 \}, \end{aligned}$$

where $\mathbb{P}_1(T)$ is the set of polynomials over T of degree at most one.

For some $\delta > 0$ and all sufficiently small h,

$$\Omega' := \{ \mathbf{x} \in \mathbb{R}^2 : d(\mathbf{x}, \Omega) < \delta \} \supset \Omega_h \cup \Omega.$$

If $u \in H^2(\Omega)$ then by the C^2 regularity of Ω there exists an extension of u to Ω', also denoted by u, such that

$$\|u\|_{H^2(\Omega')} \leq c \|u\|_{H^2(\Omega)}. \tag{2.1}$$

Definition 2.1. The *discrete H-Dirichlet integral* is defined by

$$D_H(u_h; \Omega_h) = \frac{1}{2} \int_{\Omega_h} |\nabla u_h|^2 + 2HV(u_h; \Omega_h)$$

for $u_h \in X_h$.

It follows from (1.5) and (1.7) with Ω replaced by Ω_h and a limit argument, or by direct computation and noting that boundary integrals on internal edges cancel, that

$$\langle D_H'(u_h;\Omega_h),\varphi_h\rangle = \int_{\Omega_h} \nabla u_h \nabla \varphi_h + 2H\int_{\Omega_h}[\varphi_h, u_{hx}, u_{hy}], \tag{2.2}$$

$$D_H{}''(u_h;\Omega_h)(\varphi_h,\psi_h) =$$
$$\int_{\Omega_h} \nabla\varphi_h\nabla\psi_h + 2H\int_{\Omega_h}[u_h,\varphi_{hx},\psi_{hy}]+[u_h,\psi_{hx},\varphi_{hy}], \tag{2.3}$$

for $u_h \in X_h$ and $\varphi_h, \psi_h \in X_{h0}$.

Motivated by (1.6), one has

Definition 2.2. A function $u_h \in X_h$ is *discrete H-harmonic* if

$$\int_{\Omega_h} \nabla u_h\nabla\varphi_h + 2H\int_{\Omega_h}[u_{hx}, u_{hy}, \varphi_h] = 0, \tag{2.4}$$

for all $\varphi_h \in X_{h0}$.

We will prove the following.

Theorem 2.1. *Let $u \in H^2(\Omega;\mathbb{R}^3)$ be H-harmonic and $u = u^0$ on $\partial\Omega$ where $u^0 \in H^2(\Omega;\mathbb{R}^3)$. Assume u is nondegenerate with nondegeneracy constant λ.*

Let $u_h^0 \in X_h$ and assume $\|u^0 - u_h^0\|_{H^1(\Omega_h)} \le \alpha h$.

Then there exist constants $h_0 = h_0(\|u\|_{H^2}, \|u^0\|_{H^2}, \alpha, \Omega, d, H, \lambda)$, $\varepsilon_0 = \varepsilon_0(H,\lambda)$, and $c_0 = c_0(\|u\|_{H^2}, \|u^0\|_{H^2}, \alpha, H)$ such that if $0 < h \le h_0$ then:

1. *There exists a* unique *discrete H-harmonic function u_h such that $u_h = u_h^0$ on $\partial\Omega_h$ and*
$$\|u - u_h\|_{H^1(\Omega_h)} \le \varepsilon_0;$$
2. *Moreover,*
$$\|u - u_h\|_{H^1(\Omega_h)} \le c_0\lambda^{-1}h.$$

3. Proof of Main Theorem

With u, u^0 and u_h^0 as in the main theorem *define*

$$J_h u = u_h^0 + I_h(u - u^0) \in u_h^0 + X_{h0}, \tag{3.1}$$

where I_h is the standard nodal interpolation operator.

The proof of the main theorem will use the following quantitative version of the Inverse Function Theorem with $\mathcal{X} = u_h^0 + X_{h0}$, $X = X_{h0}$, $Y = X_{h0}^*$ (the dual space of X_{h0}), $x_0 = J_h u$, $f = D_H'(\cdot\,;\Omega_h)$. The proof of the lemma follows from that in [1] pp 113–114.

Lemma 3.1. *Let $\mathcal{X}$ be an affine Banach space with Banach space X as tangent space, and let Y be a Banach space. Suppose $x_0 \in \mathcal{X}$ and*

$f \in C^1(\mathcal{X}, Y)$. Assume there are positive constants α, β, δ and ε such that

$$\begin{aligned} \|f(x_0)\|_Y &\le \delta, \\ \|f'(x_0)^{-1}\|_{L(Y,X)} &\le \alpha^{-1}, \\ \|f'(x) - f'(x_0)\|_{L(X,Y)} &\le \beta \quad \textit{for all } x \in \bar{B}_\varepsilon(x_0), \end{aligned}$$

where

$$\beta < \alpha, \quad \delta \le (\alpha - \beta)\varepsilon.$$

Then there exists a unique $x_* \in \bar{B}_\varepsilon(x_0)$ such that $f(x_*) = 0$.

Remark 3.1 (The Volume Functional). A fundamental result which is due to Wente [12] states that for any $u^0 \in H^1 \cap L^\infty(\Omega; \mathbb{R}^3)$ the functional V (and hence D_H) extends to an analytic functional on the affine space $u^0 + H^1_0(\Omega; \mathbb{R}^3)$. This is perhaps surprising, since from (1.5) and (1.7) one might expect bounds for the relevant integrals to also involve $\|\varphi\|_{L^\infty}$ and $\|u\|_{L^\infty}$ respectively.

More generally, one has the following.

For $u, v, w \in H^1 \cap L^\infty(\Omega; \mathbb{R}^3)$ *define* the trilinear functional

$$V(u, v, w) = V(u, v, w; \Omega) = \frac{1}{6}\int_\Omega [u, v_x, w_y] + [u, w_x, v_y]. \tag{3.2}$$

Note that $V(u) = V(u, u, u)$.

Assume now that *at least one* of u, v, w also belongs to $H^1_0(\Omega; \mathbb{R}^3)$. Then V is invariant under cyclic permutations of its arguments, as follows from integration by parts in the C^2 case and in general by a limit argument, see [10, Remark III.1.1.iii]. Since V is invariant under permutation of its first two arguments, it then follows it is invariant under *any* permutation of its arguments. Moreover under the same assumptions, from an argument similar to that in [10, proof of Theorem III.2.3] which uses an isoperimetric inequality due to Radó [8], one also has

$$V(u, v, w) \le c|u|_{H^1(\Omega)}|v|_{H^1(\Omega)}|w|_{H^1(\Omega)}. \tag{3.3}$$

Similar remarks and estimates apply if Ω is everywhere replaced by Ω_h.

Assume now $u, v \in H^1 \cap L^\infty(\Omega; \mathbb{R}^3)$ and $\varphi \in H^1_0 \cap L^\infty(\Omega; \mathbb{R}^3)$. It follows that

$$\langle V'(u), \varphi\rangle = 3V(u, u, \varphi) \le c|u|^2_{H^1(\Omega)}|\varphi|_{H^1(\Omega)}, \tag{3.4}$$

$$V''(u)(v, \varphi) = 6V(u, v, \varphi) \le c|u|_{H^1(\Omega)}|v|_{H^1(\Omega)}|\varphi|_{H^1(\Omega)}, \tag{3.5}$$

$$V'''(\cdot)(u, v, \varphi) = 6V(u, v, \varphi) \le c|u|_{H^1(\Omega)}|v|_{H^1(\Omega)}|\varphi|_{H^1(\Omega)}. \tag{3.6}$$

(In particular, if $u \in H^1 \cap L^\infty(\Omega; \mathbb{R}^3)$ these estimates allow one to define the integrals in (1.5) and (1.7) for arbitrary $\varphi, \psi \in H^1_0(\Omega; \mathbb{R}^3)$.) Similar results also hold if Ω is replaced by Ω_h.

For the remainder of this section, u is as in the Main Theorem. Extend u to Ω' as in (2.1) and restrict to Ω_h as necessary. Both the extension and restriction will also be denoted by u.

Lemma 3.2.

$$\|u - J_h u\|_{H^1(\Omega_h)} \le c_1 h$$

where $c_1 = c_1(\|u\|_{H^2(\Omega)}, \|u^0\|_{H^2(\Omega)}, \alpha)$.

Proof.

$$\begin{aligned}\|u - J_h u\|_{H^1(\Omega\cap\Omega_h)} &= \Big\|(u^0 - u_h^0) - (I_h(u - u^0) - (u - u^0))\Big\|_{H^1(\Omega\cap\Omega_h)} \\ &\le \|u^0 - u_h^0\|_{H^1(\Omega\cap\Omega_h)} + ch|u - u^0|_{H^2(\Omega)} \\ &\le ch,\end{aligned}$$

where $c = c(\|u\|_{H^2}, \|u^0\|_{H^2}, \alpha)$. Since

$$\|u\|_{H^1(\Omega_h\setminus\Omega)} \le ch\|u\|_{H^2(\Omega_h\setminus\Omega)} \le ch\|u\|_{H^2(\Omega)}$$

by elementary estimates and (2.1), the result follows. □

Lemma 3.3. *If* $\varphi_h \in X_{h0}$ *then*

$$|\langle D'_H(J_h u;\Omega_h), \varphi_h\rangle| \le c_2 h|\varphi_h|_{H^1(\Omega_h)},$$

where $c_2 = c_2(\|u\|_{H^2(\Omega)}, \|u^0\|_{H^2(\Omega)}, \alpha, H)$.

Proof.

$$\begin{aligned}&\langle D'_H(J_h u;\Omega_h), \varphi_h\rangle \\ &\quad= \Big(\langle D'_H(J_h u;\Omega_h), \varphi_h\rangle - \langle D'_H(u;\Omega_h), \varphi_h\rangle\Big) + \langle D'_H(u;\Omega_h), \varphi_h\rangle \\ &\quad=: A + B\end{aligned}$$

From the Taylor series expansion for $V'(\,\cdot\,;\Omega_h)$ and Remark 3.1

$$\begin{aligned}|A| &= \left|\int_{\Omega_h} \nabla(J_h u - u)\nabla\varphi_h + 2H\langle V'(J_h u;\Omega_h), \varphi_h\rangle - \langle V'(u;\Omega_h), \varphi_h\rangle\right| \\ &\le |J_h u - u|_{H^1(\Omega_h)}|\varphi_h|_{H^1(\Omega_h)} + 2|H||V''(u;\Omega_h)(J_h u - u, \varphi_h)| \\ &\quad + |H||V'''(u;\Omega_h)(J_h u - u, J_h u - u, \varphi_h)| \\ &\le |J_h u - u|_{H^1(\Omega_h)}|\varphi_h|_{H^1(\Omega_h)} \\ &\quad + c|H||u|_{H^1(\Omega_h)}|J_h u - u|_{H^1(\Omega_h)}|\varphi_h|_{H^1(\Omega_h)} \\ &\quad + c|H||J_h u - u|^2_{H^1(\Omega_h)}|\varphi_h|_{H^1(\Omega_h)} \\ &\le ch|\varphi_h|_{H^1(\Omega_h)},\end{aligned}$$

from Lemma 3.2 and (2.1), where $c = c(\|u\|_{H^2(\Omega)}, \|u^0\|_{H^2(\Omega)}, \alpha, H)$.

Also,

$$\begin{aligned}B &= \left|\int_{\Omega_h} \nabla u\nabla\varphi_h + 2H\int_{\Omega_h}[\varphi_h, u_x, u_y]\right| = \left|\int_{\Omega_h}(-\triangle u + 2Hu_x \times u_y)\cdot\varphi_h\right| \\ &= \left|\int_{\Omega_h\setminus\Omega}(-\triangle u + 2Hu_x \times u_y)\cdot\varphi_h\right| \le c\|\varphi_h\|_{L^2(\Omega_h\setminus\Omega)} \le ch\|\varphi_h\|_{H^1(\Omega_h)}\end{aligned}$$

where $c = c(\|u\|_{H^2}, H)$, as follows from (2.1), a Sobolev imbedding theorem, and elementary calculus.

The required result follows. □

Remark 3.2 (A “Discrete Eigenspace” Decomposition). If $\varphi_h \in X_{h0}$ let φ_h also denote the zero extension to $\Omega \cup \Omega_h$. Note that $\varphi_h \notin H^1_0(\Omega)$ unless Ω is convex. For this reason *define* $P : X_{h0} \to H^1_0(\Omega)$ to be the $|\cdot|_{H^1}$ projection, i.e.

$$\int_\Omega \nabla(P\varphi_h)\nabla\varphi = \int_\Omega \nabla\varphi_h \nabla\varphi$$

for all $\varphi \in H^1_0(\Omega)$.

One has

$$|\varphi_h|_{H^1(\Omega_h \setminus \Omega)} \le ch^{1/2}|\varphi_h|_{H^1(\Omega_h)} \tag{3.7}$$

$$|P\varphi_h|_{H^1(\Omega)} \le |\varphi_h|_{H^1(\Omega_h)} \tag{3.8}$$

$$|\varphi_h - P\varphi_h|_{H^1(\Omega)} \le ch^{1/2}|\varphi_h|_{H^1(\Omega_h)} \tag{3.9}$$

To see (3.7) note that $\nabla\varphi_h$ is constant on any triangle $T \in \mathcal{T}_h$ and that $|T \cap (\Omega_h \setminus \Omega)| \le ch|T|$. Inequality (3.8) is immediate, since P is just $|\cdot|_{H^1}$ orthogonal projection onto $H^1_0(\Omega)$. For (3.9) first note that $|\varphi_h - P\varphi_h|_{H^1(\Omega)} \le |\varphi_h - \varphi|_{H^1(\Omega)}$ for any $\varphi \in H^1_0(\Omega)$, by orthogonality. Now choose φ by suitably deforming φ_h in a boundary strip.

Let $(P\varphi_h)^+, (P\varphi_h)^- \in H^1_0(\Omega)$ be the components of $P\varphi_h$ as in (1.9). Note that $(P\varphi_h)^-$ is smooth, and in particular

$$\|(P\varphi_h)^-\|_{H^2(\Omega)} \le \nu|\varphi_h|_{H^1(\Omega_h)} \tag{3.10}$$

since

$$\|(P\varphi_h)^-\|_{H^2(\Omega)} \le \nu|(P\varphi_h)^-|_{H^1(\Omega)} \le \nu|P\varphi_h|_{H^1(\Omega)} \le \nu|\varphi_h|_{H^1(\Omega_h)}$$

from (1.11), (1.9) and the $|\cdot|_{H^1}$-orthogonality of $(P\varphi_h)^+$ and $(P\varphi_h)^-$, and (3.8).

Define a discrete analogue of (1.9) by

$$\begin{aligned} \varphi_h^{(-)} = I_h(P\varphi_h)^- \in X_{h0}, \quad \varphi_h^{(+)} &= \varphi_h - \varphi_h^{(-)}, \\ \varphi_h &= \varphi_h^{(+)} + \varphi_h^{(-)}. \end{aligned} \tag{3.11}$$

Taking the zero extension of $(P\varphi_h)^-$ and $(P\varphi_h)^+$ to Ω_h, and of $\varphi^{(-)}$ and $\varphi^{(+)}$ to Ω, we claim

$$\begin{aligned} |(P\varphi_h)^- - \varphi_h^{(-)}|_{H^1(\Omega\cup\Omega_h)} &\le ch|\varphi_h|_{H^1(\Omega_h)}, \\ |(P\varphi_h)^+ - \varphi_h^{(+)}|_{H^1(\Omega\cup\Omega_h)} &\le ch^{1/2}|\varphi_h|_{H^1(\Omega_h)}, \end{aligned} \tag{3.12}$$

where $c = c(\nu)$

Proof of claim.

$$|(P\varphi_h)^- - \varphi_h^{(-)}|_{H^1(\Omega_h)} \le ch|(P\varphi_h)^-|_{H^2(\Omega)} \le ch\nu|\varphi_h|_{H^1(\Omega_h)}$$

from (3.10). Also

$$|(P\varphi_h)^-|_{H^1(\Omega\setminus\Omega_h)} \le ch|(P\varphi_h)^-|_{H^2(\Omega)} \le ch\nu|\varphi_h|_{H^1(\Omega_h)}.$$

This gives the first result.

For the second,

$$
\begin{aligned}
&|(P\varphi_h)^+ - \varphi_h^{(+)}|_{H^1(\Omega_h)} \\
&\quad \le |P\varphi_h - \varphi_h|_{H^1(\Omega_h)} + |(P\varphi_h)^- - \varphi_h^{(-)}|_{H^1(\Omega_h)} \\
&\quad \le c(h^{1/2} + h\nu)|\varphi_h|_{H^1(\Omega_h)}
\end{aligned}
$$

from the first result and (3.9). On $\Omega \setminus \Omega_h$, $\varphi_h = \varphi_h^{(+)} = 0$ and so the required estimate now follows from (3.9). $\square$

We also have

$$
\begin{aligned}
|\varphi_h^{(-)}|_{H^1(\Omega_h)} &\le (1 + ch)|\varphi_h|_{H^1(\Omega_h)} \\
|\varphi_h^{(+)}|_{H^1(\Omega_h)} &\le (1 + ch^{1/2})|\varphi_h|_{H^1(\Omega_h)}
\end{aligned}
\tag{3.13}
$$

from (3.12), the orthogonal decomposition $P\varphi_h = (P\varphi_h)^- + (P\varphi_h)^+$ and (3.8).

Thus (3.11) is an "almost orthogonal" decomposition for small h.

Lemma 3.4. *If $\varphi_h \in X_{h0}$ then*

$$
D_H''(J_h u; \Omega_h)(\varphi_h, \varphi_h^{(+)} - \varphi_h^{(-)}) \ge \frac{3\lambda}{4}|\varphi_h|^2_{H^1(\Omega_h)}
$$

provided $h \le h_1$ where $h_1 = h_1(\|u\|_{H^2(\Omega)}, \|u^0\|_{H^2(\Omega)}, \alpha, \Omega, d, H, \lambda)$.

Proof. Since $V(\cdot\,;\Omega_h)$ is cubic, from (2.3)

$$
\begin{aligned}
&D_H''(J_h u; \Omega_h)(\varphi_h, \varphi_h^{(+)} - \varphi_h^{(-)}) \\
&= {D_H}''(u; \Omega_h)(\varphi_h, \varphi_h^{(+)} - \varphi_h^{(-)}) + 2HV'''(u; \Omega_h)(J_h u - u, \varphi_h, \varphi_h^{(+)} - \varphi_h^{(-)}).
\end{aligned}
$$

But

$$
\begin{aligned}
&\Big|2HV'''(u; \Omega_h)(J_h u - u, \varphi_h, \varphi_h^{(+)} - \varphi_h^{(-)})\Big| \\
&\qquad \le c|J_h u - u|_{H^1(\Omega_h)}|\varphi_h|_{H^1(\Omega_h)}|\varphi_h^{(+)} - \varphi_h^{(-)}|_{H^1(\Omega_h)} \le ch|\varphi_h|^2_{H^1(\Omega_h)}
\end{aligned}
$$

where $c = c(\|u\|_{H^2(\Omega)}, \|u^0\|_{H^2(\Omega)}, \alpha, \nu, H)$, from Remark 3.1, and also from Lemma 3.2 and (3.13).

Now

$$
{D_H}''(u; \Omega_h)(\varphi_h, \varphi_h^{(+)} - \varphi_h^{(-)}) = {D_H}''(u; \Omega)(\varphi_h, \varphi_h^{(+)} - \varphi_h^{(-)}) + E_1
$$

where

$$
|E_1| \le c(1 + \|u\|_{L^\infty})|\varphi_h|_{H^1(\Omega_h \setminus \Omega)}|\varphi_h^{(+)} - \varphi_h^{(-)}|_{H^1(\Omega_h \setminus \Omega)} \le ch|\varphi_h|^2_{H^1(\Omega_h)}
$$

with $c = c(\|u\|_{H^2(\Omega)}, \nu, H)$, from (3.7) and (3.12), since $(P\varphi_h)^+ = (P\varphi_h)^- = 0$ in $\Omega_h \setminus \Omega$.

Also

$$\begin{aligned}
&D_H{''}(u;\Omega)(\varphi_h, \varphi_h^{(+)} - \varphi_h^{(-)}) \\
&\qquad = D_H{''}(u;\Omega)(P\varphi_h, (P\varphi_h)^+ - (P\varphi_h)^-) \\
&\qquad\quad + D_H{''}(u;\Omega)(\varphi_h - P\varphi_h, (P\varphi_h)^+ - (P\varphi_h)^-) \\
&\qquad\qquad + D_H{''}(u;\Omega)(\varphi_h, (\varphi_h^{(+)} - (P\varphi_h)^+) - (\varphi_h^{(-)} - (P\varphi_h)^-)) \\
&\qquad \geq \lambda |P\varphi_h|^2_{H^1(\Omega)} + E_2 + E_3
\end{aligned}$$

from (1.10). But

$$|E_2|, |E_3| \leq c(1 + \|u\|_{L^\infty}) h^{1/2} |\varphi_h|^2_{H^1(\Omega_h)}$$

from (3.9) and (3.8), and (3.12) respectively.

It follows that

$$\begin{aligned}
D''_H(J_h u;\Omega_h)(\varphi_h, \varphi_h^{(+)} - \varphi_h^{(-)}) &\geq \lambda |P\varphi_h|^2_{H^1(\Omega)} - c h^{1/2} |\varphi_h|^2_{H^1(\Omega_h)} \\
&\geq \frac{3\lambda}{4} |\varphi_h|^2_{H^1(\Omega_h)}
\end{aligned}$$

from (3.9), for $h \leq h_1 = h_1(\|u\|_{H^2(\Omega)}, \|u^0\|_{H^2(\Omega)}, \alpha, \Omega, d, H, \lambda)$. □

Lemma 3.5. *If $v_h \in u_h^0 + X_{h0}$ and $\varphi_h, \psi_h \in X_{h0}$ then*

$$|D''_H(v_h;\Omega_h)(\varphi_h, \psi_h) - D''_H(J_h u;\Omega_h)(\varphi_h, \psi_h)| \leq \frac{\lambda}{4} |\varphi_h|_{H^1(\Omega_h)} |\psi_h|_{H^1(\Omega_h)}$$

provided $|v_h - J_h u|_{H^1(\Omega_h)} \leq \varepsilon_1$ where $\varepsilon_1 = \varepsilon_1(H, \lambda)$.

Proof. This follows from

$$\begin{aligned}
D''_H(v_h;\Omega_h)(\varphi_h, \psi_h) &- D''_H(J_h u;\Omega_h)(\varphi_h, \psi_h) \\
&= 2HV_h'''(J_h u;\Omega_h)(v_h - J_h u, \varphi_h, \psi_h),
\end{aligned}$$

(3.6) and Lemma 3.2. □

Completion of proof of Main Theorem. We use Lemma 3.1 with $\mathcal{X} = u_h^0 + X_{h0}$, $X = X_{h0}$, $Y = X^*_{h0}$, $x_0 = J_h u$, $f = D'_H(\cdot\,;\Omega_h)$. The norm on X_{h0} is $|\cdot|_{H^1(\Omega_h)}$ and on X^*_{h0} is the corresponding dual norm. Note that

$$D'_H(\cdot\,;\Omega_h) \colon u_h^0 + X_{h0} \to X^*_{h0}$$

with derivative

$$D''_H(\cdot\,;\Omega_h) \colon u_h^0 + X_{h0} \to L(X_{h0}, X^*_{h0})$$

using standard identifications.

From Lemma 3.3

$$\|D'_H(J_h u;\Omega_h)\| \leq c_2 h. \tag{3.14}$$

From Lemma 3.4, $D''_H(J_h u;\Omega_h)$ is invertible and

$$\left\| [D''_H(J_h u;\Omega_h)]^{-1} \right\| \leq \left(\frac{3\lambda}{4} \, |\varphi_h|_{H^1(\Omega_h)} \Big/ |\varphi_h^{(+)} - \varphi_h^{(-)}|_{H^1(\Omega_h)} \right)^{-1}$$

provided $h \le h_1$. But

$$\begin{aligned} |\varphi_h^{(+)} - \varphi_h^{(-)}|_{H^1(\Omega_h)} &\le |(P\varphi_h)^+ - (P\varphi_h)^-|_{H^1(\Omega_h)} + |\varphi_h^{(+)} - (P\varphi_h)^+|_{H^1(\Omega_h)} \\ &\quad + |\varphi_h^{(-)} - (P\varphi_h)^-|_{H^1(\Omega_h)} \\ &\le (1 + ch^{1/2})|\varphi_h|_{H^1(\Omega_h)} \end{aligned}$$

where $c = c(\nu)$, from (3.8) and (3.12). Hence

$$\left\|[D''_H(J_h u; \Omega_h)]^{-1}\right\| \le \left(\frac{\lambda}{2}\right)^{-1} \tag{3.15}$$

if $h \le h_3$ where $h_3 = h_3(\|u\|_{H^2(\Omega)}, \|u^0\|_{H^2(\Omega)}, \alpha, \Omega, d, H, \lambda)$.

Finally, from Lemma 3.5

$$\|D''_H(v_h; \Omega_h) - D''_H(J_h u; \Omega_h)\| \le \frac{\lambda}{4} \tag{3.16}$$

if $|v_h - J_h u|_{H^1(\Omega_h)} \le \varepsilon_1$ where $\varepsilon_1 = \varepsilon_1(H, \lambda)$.

Take $\delta = c_2 h$, $\alpha = \lambda/2$, $\beta = \lambda/4$ and $\varepsilon = \varepsilon_1$. Then from (3.14)–(3.16) the hypotheses of Lemma 3.1 are satisfied provided $h \le h_3$, $c_2 h \le \frac{\lambda}{4}\varepsilon_1$. This establishes the first (uniqueness) part of the main theorem with $\varepsilon_0 = \varepsilon_1$ and $h_0 = h_0(h_3, \varepsilon_1, \lambda, c_2) = h_0(\|u\|_{H^2}, \|u^0\|_{H^2}, \alpha, \nu, H, \lambda)$.

Taking $\delta = c_2 h$, $\alpha = \lambda/2$, $\beta = \lambda/4$ and $\varepsilon = \lambda^{-1} c_0 h$ the hypotheses of Lemma 3.1 are again satisfied from (3.14)–(3.16) provided $h \le h_3$, $c_2 h \le \frac{1}{4} c_0 h$. This establishes the second ($O(h)$ convergence) part of the main theorem with $h_0 = h_3$ and $c_0 = 4c_2$. □

4. Numerical Results

In Tables 1 and 2 we present the results of test computations for the explicitly known spherical solutions described in Example 1.1 with $H = 0.5$ and $\Omega = B_1(0)$. Denote by e_h the error between the continuous solution and the discrete solution in the chosen norm. For two successive grids with grid sizes h_1 and h_2 the experimental order of convergence is

$$eoc = \ln \frac{e_{h_1}}{e_{h_2}} \Big/ \ln \frac{h_1}{h_2}.$$

The test computations confirm the order 1 for the $H^1(\Omega)$-norm and additionally show the order 2 for the $L^2(\Omega)$-norm.

Figures 2 and 3 show computational results with $\Omega = B_1(0)$, $H = 0.5$ and boundary values $u(e^{i\phi}) = (\cos(\phi), \sin(\phi), (2 + \sqrt{3})\cos(2\phi) - 0.5\cos(6\phi))$ on a grid with 8192 triangles. For better visibility the resulting surfaces are scaled, but the boundaries of the solution surfaces are the same.

Figure 4 shows a solution for the annular domain $\Omega = \{\, x \mid 1 < |x| < 2 \,\}$ and boundary data which give knotted boundary curves.

nodes	level	h	L^2-error	L^2-eoc	H^1-error	H^1-eoc
9	2	1.0000	1.0020e-1	-	0.2607	-
25	4	0.7368	3.9040e-2	3.09	0.1822	1.17
81	6	0.4203	1.0682e-2	2.31	9.6455e-2	1.13
289	8	0.2219	2.6916e-3	2.16	4.8223e-2	1.09
1089	10	0.1137	6.6871e-4	2.08	2.3909e-2	1.05
4225	12	0.05736	1.6621e-4	2.04	1.1876e-2	1.03
16641	14	0.02893	4.1401e-5	2.02	5.9160e-3	1.01

TABLE 1. Small solution, $H = 0.5$

nodes	level	h	L^2-error	L^2-eoc	H^1-error	H^1-eoc
81	6	0.4203	1.2292	-	6.1915	-
289	8	0.2219	0.4677	1.51	2.9080	1.18
1089	10	0.1137	0.1610	1.60	1.3131	1.19
4225	12	0.05736	0.04707	1.81	0.5870	1.18
16641	14	0.02893	0.01239	1.94	0.2772	1.09

TABLE 2. Large solution, $H = 0.5$

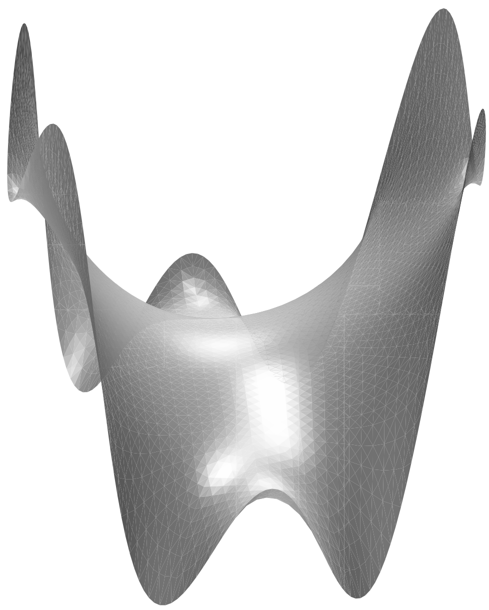

FIGURE 2. Small solution, $H = 0.5$

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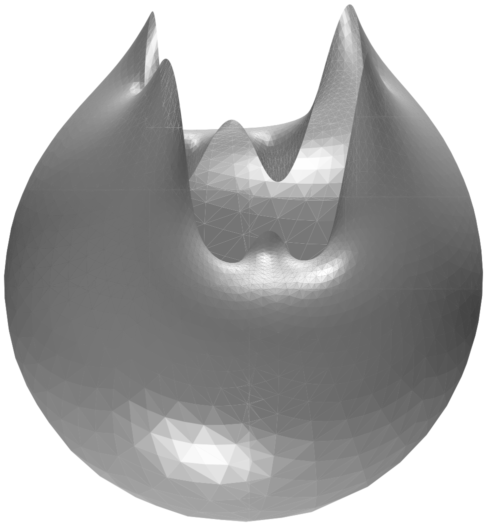

FIGURE 3. Large solution, $H = 0.5$

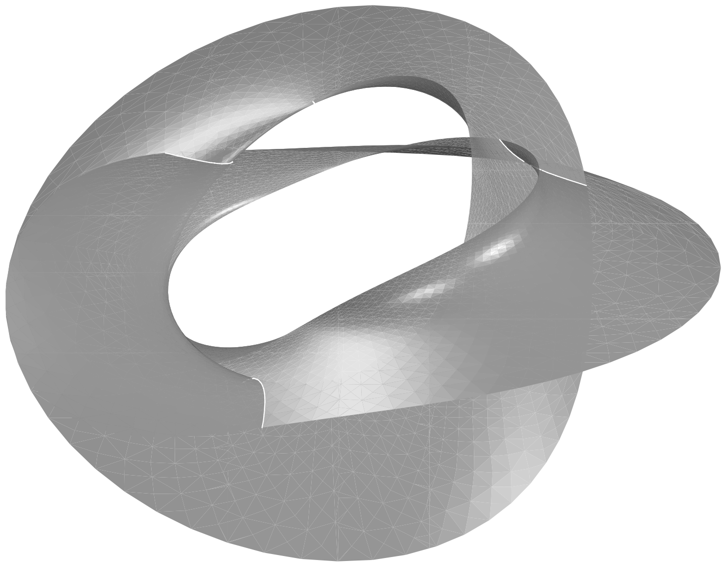

FIGURE 4. Annulus, $H = 0.5$

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