V-VARIABLE FRACTALS: DIMENSION RESULTS

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Abstract

V-variable fractals, for \( V = 1, 2, 3, \ldots \), interpolate between random homogeneous fractals and random recursive fractals. We compute the almost sure Hausdorff dimension of V-variable fractals satisfying the uniform open set condition. Important roles are played by the notion of a neck, leading to spatial homogeneity at various levels of magnification, and a variant of the Furstenberg Kesten theorem for products of certain random \( V \times V \) matrices.

1. Introduction

In this paper we begin the study of analysis on V-variable fractals by computing the Hausdorff dimension, under the assumption of an open set condition. Important tools used in the proof are the notion of a neck, and a variant of the Furstenberg Kesten theorem for products of random matrices.

Fractal sets generated by a single iterated function system [IFS], and random fractal sets generated from a family of IFSs, are studied as mathematical models of disordered systems. In this latter setting the most commonly studied random fractals are random recursive fractals and random homogeneous fractals. In particular, Hausdorff, walk and spectral dimensions have been computed in special cases.

Let \( F \) be a family of IFSs operating on \( \mathbb{R}^n \), together with a probability distribution \( P \) on \( F \). For each natural number \( V \) there is a corresponding family of \( V \)-variable random fractals; these families interpolate between the family of random homogeneous fractals (\( V = 1 \)), and the family of random recursive fractals (\( V \to \infty \)), corresponding to \( F \). Each class of \( V \)-variable random fractals, with its probability distribution, has the surprising property that it can be obtained from the attractor of a single IFS operating on \( V \)-tuples of compact subsets of \( \mathbb{R}^n \).

In order to fix ideas let \( F = (\mathbb{R}^2; f_1, f_2, f_3) \) be the IFS consisting of three contraction maps on \( \mathbb{R}^2 \), each with contraction ratio \( 1/2 \), and having fixed points that are the vertices of an equilateral triangle \( T \) of unit diameter. Let \( G = (\mathbb{R}^2; f_1, f_2, f_3) \) be the IFS consisting of three contraction maps, each with contraction ratio \( 1/3 \), and having the same fixed points as the corresponding maps in \( F \). The attractors of \( F \) and \( G \) are denoted by \( S_F \) and \( S_G \) respectively, see Figure 1.

Consider \( F = \{F, G\} \), together with the probability distribution \( P = \{1/2, 1/2\} \) on \( F \). Let \( \Omega \) denote the set of labelled trees which are rooted, 3-branching, and infinite, where the label at each
node is either $F$ or $G$; see Section 2.1. A modified Sierpinski triangle $K^\omega$ can be generated from each $\omega \in \Omega$, see Figure 2.

Such fractals $K^\omega$ are examples of recursive fractals. If the nodes of $\omega$ at each fixed level are the same, but may vary from level to level, then $K^\omega$ is called a homogeneous fractal.

It will also be convenient to consider finite, level $k$, labelled trees for each $k \geq 0$; see Figure 2. Such trees correspond to level $k$ prefractals.

Fractals such as $S_F$ and $S_G$ have the following properties:

1. **Spatial self similarity**: loosely expressed, at each fixed “scale” the component parts are equivalent up to simple transformations, for example, translations in the case here.

2. **Scale self similarity**: the equivalence class at each scale is the same.

Homogeneous fractals have spatial self similarity but, generically, do not have scale self similarity. Recursive fractals generically have neither spatial nor scale self similarity. Both classes have further statistical self similarity when we impose suitable probability distributions on their construction.

Now assume that, at each level of $\omega \in \Omega$, the subtrees rooted at that level have the property that they belong to at most $V$ distinct isomorphism classes. In this case, $\omega$ is said to be a $V$-variable labelled tree, and the fractal $K^\omega$ is said to be a $V$-variable fractal. Such fractals have a form of partial spatial self similarity. Similarly, define $V$-variable finite labelled trees and $V$-variable prefractals, see Figure 2. Notice that homogeneous fractals and 1-variable fractals are the same.

Let $\Omega_V$ denote the class of $V$-variable trees $\omega$ corresponding to $F$, and let $K_V$ denote the corresponding class of $V$-variable fractals $K^\omega$.

Prefractal approximations to recursive fractals and homogeneous fractals are built by assigning labels to nodes, beginning with the root node and working upwards through the tree. If the labels of $\omega$ are chosen in an iid manner according to $P$, except that in the homogeneous case all labels at each fixed level are same, then one obtains a random recursive fractal and a random homogeneous fractal respectively. For $V$-variable fractals with $V > 1$, one cannot directly build the appropriate labelled trees and so obtain a natural probability distribution in this manner.

However, one can do something similar by building certain $V$-tuples of labelled trees as follows, see Figure 3. There are natural maps $\Phi^\alpha$ from infinite, or level $k$, $V$-tuples of labelled trees $(\omega_1', \ldots, \omega_V')$ to infinite, or level $k + 1$, $V$-tuples of labelled trees $(\omega_1', \ldots, \omega_V')$, respectively. That is,

$$ (\omega_1', \ldots, \omega_V') = \Phi^\alpha(\omega_1, \ldots, \omega_V). $$

The label at the base node of each $\omega_i'$ is determined by $\Phi^\alpha$ and is either $F$ or $G$. The three subtrees of each $\omega_i'$ rooted at level one are also determined by $\Phi^\alpha$ and taken from the set $\{\omega_1, \ldots, \omega_V\}$, possibly with repetition.
Figure 3. The map $\Phi^0$ is described by the array $a_0$. It is also described by the labels in the level 0 (bottom) boxes, and the network of lines between these boxes and the level 1 (middle) boxes. For example, row 3 of $a_0$ is $G, 2, 4, 2$. It contains the information that the third component of $\Phi^0(\omega_1, \omega_2, \omega_3, \omega_4)$ is the tree whose root node is labelled $G$, and whose three subtrees, rooted at the next level, are $\omega_2$, $\omega_4$ and $\omega_2$ respectively. This information is also provided by the facts that box 3 at level 0 contains the symbol $G$, and the three lines from this box in the order left to right connect to boxes 2, 4 and 2 at level 1. Similar remarks apply to $\Phi^1$.

The maps $\Phi^i$, or equivalently the indices $a$, are described by $V \times (M + 1)$ arrays, where $M = 3$ here, and in general $M$ bounds the number of functions in each IFS from the family $F$, see Figure 3. Let $A_V$ denote the set of all such indices $a$, corresponding to the family $F$. The cardinality of $A_V$ in the example here is $2^V (3V)^V$.

The maps $F^a$ act on $V$-tuples of compact subsets of $\mathbb{R}^n$ in an analogous manner, see Figure 4. If $a = a_0 a_1 \ldots a_k \ldots \in A_V^\infty$ then

$$ (\omega^a_1, \ldots, \omega^a_V) := \lim_{k \to \infty} \Phi^a_0 \circ \cdots \circ \Phi^a_{k-1}(\omega^0_1, \ldots, \omega^0_V), $$

$$ (K^a_1, \ldots, K^a_V) := \lim_{k \to \infty} F^a_0 \circ \cdots \circ F^a_{k-1}(K^0_1, \ldots, K^0_V). $$

The limits exist and are independent of $(\omega^0_1, \ldots, \omega^0_V)$ and $(K^0_1, \ldots, K^0_V)$ respectively. Up to level $k$, the labelled trees $(\omega^a_1, \ldots, \omega^a_V)$ depend only on $\Phi^{a_0} \circ \cdots \circ \Phi^{a_{k-1}}$, see Figure 4 where $k = 2$.

There are two ways of representing $V$-variable fractals. First, any $V$-variable fractal $K$ can be represented by a labelled tree $\omega \in \Omega$. Second, and more useful here, the fractal sets $K^a_v$ in (1.2) are also described by the infinite sequence $a$. The labelled trees $\omega^a_v$ and the fractal sets $K^a_v$ are $V$-variable, and every $V$-variable tree and fractal set can be obtained in this manner. Not only is $(\omega^0_1, \ldots, \omega^0_V)$ a $V$-tuple of $V$-variable trees, but it satisfies the stronger condition that, for each $k$, there are at most $V$ isomorphism classes of trees rooted at level $k$, chosen from all the $\omega^a_1, \ldots, \omega^a_V$ taken together. A similar remark applies to $(K^a_1, \ldots, K^a_V)$.

So far we have not required a probability distribution on the class $\Omega_V$ of $V$-variable trees or on the class $K_V$ of $V$-variable fractals. However, there is a natural probability distribution $P_V$ on $A_V$ that is inherited from $P$. This induces a probability distribution on $A_V^\infty$, thence from (1.2) on $(\Omega_V)^V \subset \Omega^V$ and $(K_V)^V$, and thence on $\Omega_V$ and $K_V$. 
prefractal for Here, on the right side, we are summing the diameters to the power $\alpha$

The vector 

$\omega = \begin{bmatrix} \omega_1 & \omega_2 & \ldots & \omega_4 \end{bmatrix}$

that now plausible that there is a unique $d$ such that $\|M^\alpha \circ \cdots \circ M^{k-1}\|$ should diverge exponentially fast to $+\infty$ a.s. for $\alpha < d$, decay exponentially fast to 0 a.s. for $\alpha > d$, and that this $d$ should be the a.s. Hausdorff dimension of $K^\alpha_1, \ldots, K^\alpha_n$. We show this is the case in the main theorem, see Sections 3 and 4. In Section 5 we compute some examples.

![Figure 4. The map $F^{n_1}$ followed by $F^{n_0}$ act on $C(\mathbb{R}^2)^4$, the space of 4-tuples of compact subsets of $\mathbb{R}^2$, in a manner which is encoded in the corresponding arrays $a_0$ and $a_1$. For example, row 3 of $a_0$ is $G, 2, 4, 2$ and contains the information that, for any $(K_1, K_2, K_3, K_4)$, component 3 of $F(K_1, K_2, K_3, K_4)$ will be $g_1(K_2) \cup g_2(K_4) \cup g_3(K_2)$. Note that $a_0$ and $a_1$ are the same as in Figure 3.](image-url)
Figure 5. The first row represents fractals \((K_1, \ldots, K_4)\), or prefractals of level \(\geq k-1\), corresponding to labelled trees \(\omega = (\omega_1, \ldots, \omega_4)\). The second row represents \((K'_1, \ldots, K'_4) = F_a(K_1, \ldots, K_4)\) with labelled trees \(\omega' = (\omega'_1, \ldots, \omega'_4)\). If the \(v\)th component of \(S(\omega, k-1, \alpha)\) is the \(k-1\) level approximation to \(H^\alpha(K_v)\) then, in the case here, 
\[
S(\omega'_3, k, \alpha) = \left(\frac{1}{3}\right)^\alpha S(\omega_2, k-1, \alpha) + 2 \left(\frac{1}{3}\right)^\alpha S(\omega_3, k-1, \alpha).
\]
Similarly, 
\[
S(\omega', k, \alpha) = M^a(\alpha)S(\omega, k-1, \alpha).
\]

Results for walk and spectral dimensions of \(V\)-variable fractals, and their properties, have been obtained in work by Uta Freiberg and Hutchinson.

We are interested in \(V\)-variable fractals for the following reasons, see [BHS08, BHS05].

1. Families of \(V\)-variable fractals, together with their probability distributions, interpolate between random homogeneous fractals and random recursive fractals.
2. The families of functions \((\Phi^a, a \in A_V, P_V)\) and \((F^a, a \in A_V, P_V)\), including the probability distribution \(P_V\) on \(A_V\), each form a standard IFS, one acting on \(\Omega^V\) and the other on \(C(\mathbb{R}^2)^V\). The attractors, projected in any of the \(V\) component directions, give the set of \(V\)-variable fractal trees and sets, together with their probability distributions.
3. The chaos game for these IFSs can be used to generate a sample of \(V\)-variable fractals, whose empirical distribution will approach the probability distribution on \(V\)-variable fractals.
4. Analogous results apply to \(V\)-variable fractal measures under weak local contractive conditions. Such conditions are natural, for example, in modelling stochastic processes where individual sample paths may be bounded, but there is no uniform bound. In this case one has an IFS operating on a non locally compact state space. But the chaos game result can be extended to this setting.
5. By taking large \(V\) the chaos game gives a fast forward process for the generation of a sample of fractals approximating random recursive fractals and their probability distribution. This is of practical interest, since normally one builds individual examples of random recursive fractals by a computationally expensive backward process.

2. Notation and Assumptions

The diameter of a set \(A\) is denoted by \(|A|\). The Hausdorff measure of \(A\) in the dimension \(\alpha\) is denoted by \(H^\alpha(A)\). The Hausdorff dimension of \(A\) is denoted by \(\dim_H(A)\).

Fix a family \(F = \{F^\lambda : \lambda \in \Lambda\}\) of IFSs acting on \(\mathbb{R}^n\), where \(F^\lambda = (\mathbb{R}^n, f_1^\lambda, \ldots, f_M^\lambda)\), each \(f_m^\lambda : \mathbb{R}^n \to \mathbb{R}^n\) with Lipschitz constant \(r_m^\lambda < 1\), and \(\Lambda\) may be infinite. Fix a probability distribution \(P\) on \(\Lambda\), or equivalently on \(F\).

Assume
\[
0 < r_{\min} \leq r_m^\lambda \leq r_{\max} < 1, \quad M := \max_{\lambda \in \Lambda} M^\lambda < \infty.
\]
We usually further assume the \( f_m^\lambda : \mathbb{R}^n \to \mathbb{R}^n \) are similitudes. That is,
\[
|f_m^\lambda(x) - f_m^\lambda(y)| = r_m^\lambda |x - y| \quad \forall x, y \in \mathbb{R}^n.
\]
In this case we will also assume that \( F \) satisfies a uniform open set condition. That is, there exists a non empty open set \( O \) such that
\[
\bigcup_{m=1}^{M_\lambda} f_m^\lambda(O) \subset O, \quad f_m^\lambda(O) \cap f_n^\lambda(O) = \emptyset \text{ if } m \neq n \text{ and } \lambda \in \Lambda.
\]
We note explicitly when (2.2) and (2.3) apply.

2.1. **Labelled trees.** We characterise the set \( \Omega \) of labelled trees \( \omega \), corresponding to the family \( F \) of IFSs, as follows:

A tree is a set \( \omega \) of finite sequences \( m = m_1 \ldots m_k \) where each \( m_j \in \{1, \ldots, M\} \), together with the empty sequence \( \emptyset \). The set \( \omega \) is closed under the operation of taking initial segments. Sequences in \( \omega \) are called nodes. We write \( m \in \omega \) if \( m \) is a node of \( \omega \).

The level \( |m| \) of the node \( m = m_1 \ldots m_k \in \omega \) is defined to be \( |m| = k \), while \( |\emptyset| = 0 \).

A labelled tree, corresponding to the family of IFSs \( F \), is a tree \( \omega \) together with a map from the set of nodes of \( \omega \) into \( \Lambda \). This map is also denoted by \( \omega \). If \( m \in \omega \) and \( \omega(m) = \lambda \) then the nodes of \( \omega \) whose immediate predecessor is \( m \) are precisely \( m_1, \ldots, mM_\lambda \). Notice that the number of edges rooted at a node equals the number of functions in the IFS given by the label at that node.

Let \( \Omega \) denote the set of all labelled trees which correspond to \( F \). We frequently refer to a labelled tree simply as a tree.

A \( V \)-variable labelled tree \( \omega \in \Omega \) is a labelled tree such that, for each level \( k \), there are at most \( V \) non isomorphic subtrees of \( \omega \) rooted at level \( k \). Let \( \Omega_V \subset \Omega \) denote the set of all \( V \)-variable labelled trees which correspond to \( F \).

2.2. **Approximating fractal sets.** For \( E \subset \mathbb{R}^n, \omega \in \Omega \) and nodes \( m = m_1 \ldots m_k \in \omega \), define
\[
E_0 = E^0 = E, \quad E_{m_1 \ldots m_k} = f_{m_1}^{\omega(0)} \circ f_{m_2}^{\omega(m_1)} \circ \cdots \circ f_{m_k}^{\omega(m_1 \ldots m_{k-1})}(E), \quad E^k = \bigcup_{|m|=k} E_{m_1 \ldots m_k}.
\]
When we need to make the dependence on \( \omega \) explicit we will write \( E_m(\omega) \) or \( E^k(\omega) \), for example.

For \( \mathcal{O} \) as in (2.3) we have
\[
\mathcal{O} \supset \mathcal{O}_m \supset \cdots \supset \mathcal{O}_{m_1 \ldots m_k} \supset \cdots, \quad \mathcal{O} \supset \mathcal{O}_m \supset \cdots \supset \mathcal{O}_{m_1 \ldots m_k} \supset \cdots, \\
\mathcal{O}_m \cap \mathcal{O}_n = \emptyset \quad \text{if } |m| = |n|, m \neq n.
\]
\[
E_{m_1} \cup E_{m_2} \cup \cdots \cup E_{m_k} \cup \cdots, \quad \mathcal{O} \cup \mathcal{O}_m \cup \cdots \cup \mathcal{O}_{m_1 \ldots m_k} \cup \cdots, \\
K^\omega := \bigcap_{k \geq 0} \mathcal{O}^k(\omega).
\]
The set \( K^\omega \) is the fractal set corresponding to \( \omega \). Even if the open set condition does not apply, we obtain the same set \( K^\omega \) by replacing \( \mathcal{O} \) by any non empty compact set \( E \) for which
\[
\bigcup_{m=1}^{M_\lambda} f_m^\lambda(E) \subset E \quad \forall \lambda \in \Lambda.
\]

The set \( K_V \) of \( V \)-variable fractals sets is defined by
\[
K_V = \{ K^\omega : \omega \in \Omega_V \}.
\]
3. The Hausdorff dimension of V-variable fractals

3.1. The maps $\Phi^a$ and $F^a$. (See the examples in Figures 3, 4, 5 and 6.) The index set $A_V$ is defined to be the set of all arrays $a$ of the form

$$a = \begin{bmatrix} I^a(1) & J^a(1, 1) & \ldots & J^a(1, M) \\ \vdots & \vdots & \ddots & \vdots \\ I^a(V) & J^a(V, 1) & \ldots & J^a(V, M) \end{bmatrix},$$

where $I^a(v) \in \Lambda$, $M$ is as in (2.1), $J^a(v,m) \in \{1, \ldots, V\}$ if $1 \leq m \leq M_{I^a(v)}$ and $J^a(v,m) = 0$ if $M_{I^a(v)} < m \leq M$. See also [BHS08, Section 5.3].

The probability distribution $P_V$ on $A_V$ is obtained by selecting all elements of $a$ independently. The elements in the first column are chosen according to $P$. The elements $J^a(v,m)$ for $1 \leq m \leq M_{I^a(v)}$ are chosen according to the uniform distribution on $\{1, \ldots, V\}$. Any remaining elements $J^a(v,m)$ for $m > M_{I^a(v)}$ are set equal to 0.

The map $\Phi^a : \Omega^V \rightarrow \Omega^V$ is defined by requiring, for any $V$-tuple $(\omega_1, \ldots, \omega_V) \in \Omega^V$, that the $v$th component of $\Phi^a(\omega_1, \ldots, \omega_V) \in \Omega^V$ is the labelled tree whose base node is labelled $F^{J^a(v)}$, and whose $m$th subtree rooted at level one is $\omega_{J^a(v,m)}$ for $1 \leq m \leq M_{I^a(v)}$. Any zeros at the end of each row of $a$ are markers so that all rows have equal length, but otherwise play no role.

Similarly, for any $V$-tuple of sets $(K_1, \ldots, K_V)$, the map $F^a : \mathcal{C}(\mathbb{R}^n) \rightarrow \mathbb{R}^n$ is defined by

$$F^a(K_1, \ldots, K_V) = \left( \bigcup_{m=1}^{M_{I^a(V)}} f_m^{J^a(1)}(K_{J^a(1,m)}), \ldots, \bigcup_{m=1}^{M_{I^a(V)}} f_m^{J^a(V)}(K_{J^a(V,m)}) \right).$$

For $a = a_0 \ldots a_{k-1} \ldots \in A_V^\infty$, assuming (2.6) for some compact $E$, we define

$$(\omega^a_1, \ldots, \omega^a_V) := \lim_{k \to \infty} \Phi^a_0 \circ \cdots \circ \Phi^a_{k-1}(\omega^0_1, \ldots, \omega^0_V),$$

$$(K^a_1, \ldots, K^a_V) := \lim_{k \to \infty} F^a_0 \circ \cdots \circ F^a_{k-1}(K^0_1, \ldots, K^0_V).$$

The limits exist, and are independent of $(\omega^0_1, \ldots, \omega^0_V)$ and $(K^0_1, \ldots, K^0_V)$, respectively.

If $v \in \{1, \ldots, V\}$ then $\omega^a_v \in \Omega_v$, and every $V$-variable labelled tree can be obtained in this manner. However, if $\Omega^*_{V^a}$ is the set of $V$-tuples of labelled trees obtained in this manner then $\Omega^*_{V^a} \subset (\Omega_V)^V$.

The probability distribution $P^\infty_V$ on $A_V^\infty$ is defined by selecting the $a_k$ in an iid manner according to $P_V$. This induces a probability distribution on $\Omega^*_{V^a} \subset (\Omega_V)^V$ via (3.3), and hence a probability distribution $P^\infty_V$ on $\Omega^*_V$ by projecting in any of the $V$ coordinate directions. Similarly one obtains a probability distribution $R^\infty_V$ on $K_V$. Both $P^\infty_V$ and $R^\infty_V$ are independent of choice of projection direction, although the initial distributions are not product distributions. See [BHS08].

3.2. Approximating Hausdorff measure. Assume the open set condition (2.3), and without loss of generality assume $|O| = 1$. (See Figure 2 and take $O$ to be the interior of the triangle $T$.)

Suppose $\omega \in \Omega$ is a labelled tree with labels from $F$. Keeping in mind (2.5), we think of the collection of sets $\{\mathcal{O}_m : m \in \omega, |m| = k\}$ as an “efficient” cover of $K^\omega$ for large $k$. In order to compute the Hausdorff dimension of $K^\omega$, we consider the following quantities for $\alpha > 0$:

$$r_\omega := |\mathcal{O}| = 1, \quad r_{m_1 \ldots m_k}(\omega) := |\mathcal{O}_{m_1 \ldots m_k}| = r_{m_1}^{(0)} \cdots r_{m_k}^{(0)} \text{ for } m_1 \ldots m_k \in \omega,$$

$$(3.4) \quad S(\omega, k, \alpha) := \sum_{\{m \in \omega : |m| = k\}} |\mathcal{O}_m|^\alpha = \sum_{\{m \in \omega : |m| = k\}} (r_m(\omega))^{\alpha}, \text{ noting } S(\omega, 0, \alpha) = 1.$$

We are interested in the behaviour of $S(\omega, k, \alpha)$ as $k \to \infty$ since, for large $k$, it is an approximation to the Hausdorff measure $H^\alpha(K^\omega)$.

3.3. Flow Matrices. (See the example in Figure 5.) As remarked in the introduction, there are two ways of representing a $V$-variable fractal $K$. One can either use the corresponding labelled tree $\omega \in \Omega$, or one can use a sequence $a \in (A_V)^\infty$ as in (3.3) which generates a $V$-tuple containing $K$ as a component. We use the latter in order to study $S(\omega, k, \alpha)$.

For $(\omega_1, \ldots, \omega_V) \in \Omega^V$ define the following vector of real numbers:

$$(3.5) \quad S((\omega_1, \ldots, \omega_V), k, \alpha) := (S(\omega_1, k, \alpha), \ldots, S(\omega_V, k, \alpha)).$$
Then $\mathbf{S}(\omega_1, \ldots, \omega_V, 0, \alpha) = \mathbf{1}$, the $V$-vector whose components all equal 1.

**Proposition 3.1.** If $(\omega_1, \ldots, \omega_V) \in \Omega^V$, and $(\omega_1', \ldots, \omega_V') = \Phi^a(\omega_1, \ldots, \omega_V)$, then

$$S(\omega'_v, k, \alpha) = \sum_{w=1}^{V} \left( \sum_{m: J^u(v, m) = w} \left( r_{m}^{v}(v) \right)^{\alpha} \right) S(\omega_w, k - 1, \alpha).$$

**Proof.** For any $\omega \in \Omega$ it follows from (3.4) that

$$r_{m_1 \ldots m_k}(\omega) = r_{m_1}^{v}(0) r_{m_2 \ldots m_k}(w^{(m_1)}),$$

where $w^{(m_1)}$ is the labelled subtree of $\omega$ rooted at node $m_1$ at level one. From the definition of $\Phi^a(\omega_1, \ldots, \omega_V)$ in Section 3.1, it follows that

$$r_{m_1 \ldots m_k}(\omega'_v) = r_{m_1}^{v} r_{m_2 \ldots m_k}(\omega_{J^u(v, m_1)}).$$

Hence,

$$S(\omega'_v, k, \alpha) = \sum_{m_1 \ldots m_k \in \omega'_v} \left( \sum_{m: J^u(v, m) = w} \left( r_{m}^{v}(v) \right)^{\alpha} \right) S(\omega_w, k - 1, \alpha).$$

Motivated by this we make the following definition.

**Definition 3.2.** The $V \times V$ flow matrix $M^a = M^a(\alpha)$ for $a \in A_V$, is defined by

$$M^a_{vw} = \sum_{m: J^u(v, m) = w} \left( r_{m}^{v}(v) \right)^{\alpha}, \quad 1 \leq v, w \leq V.$$

Flow matrices are the main book keeping tool for tracking the size of covers of $V$-variable fractals.

**Proposition 3.3.** Suppose $a \in A_V^\infty$, and $\omega = (\omega_1, \ldots, \omega_V) = \omega^a \in \Omega^V$ has address $a$ as in (3.3). Then,

$$S(\omega^a, k, \alpha) = M^{a_0} \circ \cdots \circ M^{a_{k-1}} \mathbf{1}, \quad \text{i.e.} \quad S(\omega^a, k, \alpha) = \sum_{w=1}^{V} (M^{a_0} \circ \cdots \circ M^{a_{k-1}})_{vw}$$

for $1 \leq v \leq V$.

**Proof.** From Proposition 3.1 and Definition 3.2, for any $\omega \in \Omega^V$,

$$S(\Phi^a(\omega), k, \alpha) = M^a S(\omega, k - 1, \alpha).$$

Hence,

$$S(\Phi^a \circ \cdots \circ \Phi^{a_{k-1}}(\omega), k, \alpha) = M^{a_0} \circ \cdots \circ M^{a_{k-1}} S(\omega, 0, \alpha) = M^{a_0} \circ \cdots \circ M^{a_{k-1}} \mathbf{1}.$$

Since $\omega^a$ is of the form $\Phi^a \circ \cdots \circ \Phi^{a_{k-1}}(\omega)$ for some $\omega$, the result follows. \(\square\)

For fixed $a = a_0 \ldots a_{k-1}$ and $\omega$ as in Proposition 3.3, we often write

$$(3.6) \quad S_v(k, \alpha) = S(\omega_v, k, \alpha).$$

Note that $S_v(k, \alpha)$ depends only on $a_1, \ldots, a_{k-1}$, and not on $a_j$ for $j \geq k$. 

3.4. Computing the Hausdorff dimension. If $A$ is a matrix we define the norm $\|A\| := \sum_{v,w} |A_{vw}|$. It is easily checked that this norm is submultiplicative, $\|AB\| \leq \|A\| \cdot \|B\|$. 

**MAIN THEOREM.** Fix $F = \{F^\lambda : \lambda \in \Lambda\}$, a probability distribution $P$ on $\Lambda$, an integer $V \geq 1$ and a real number $\alpha > 0$. Under the assumption (2.1),

$$\gamma(\alpha) := \lim_{k \to \infty} \frac{1}{k} \log \|M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha)\| = \lim_{k \to \infty} \frac{1}{k} \log \|M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha)\| \ a.s.$$ 

In particular, the second limit is independent of $a = a_0 \ldots a_{k-1} \ldots$ for $P^\infty_V$ a.e. $a \in A_V^\infty$. The function $\gamma(\alpha)$ is monotonically decreasing in $\alpha$, and there is a unique $d$ such that $\gamma(d) = 0$.

Assuming (2.2), and the open set condition (2.3), $\dim_H(K^\omega) = d$ for $\rho_V$ a.e. $\omega \in \Omega_V$.

This theorem is proved in Section 4. For an example see Section 5.

**Remark 3.4.** The limit $\gamma(\alpha)$ is sometimes called a “Lyapunov exponent”, since $\|M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha)\|$ grows like $e^{k\gamma(\alpha)}$ as $k \to \infty$.

The fact $\gamma(\alpha)$ exists and is independent of $a$, for a.e. $a \in A_V^\infty$, is a consequence of the version of the Furstenberg Kesten theorem [FK60] in [Coh88, Theorem C, p72]. Individual terms $\frac{1}{k} \log(M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha))_{uv}$ will not normally converge as $k \to \infty$. In particular, for fixed $v$ and $w$, $(M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha))_{vw} = 0$ infinitely often a.s. In fact, if $J^{a_{k-1}}(u,m) \neq w$ for all $u$ and $m$, which happens with positive probability, then all entries in the $w$ column of $M^{a_{k-1}}(\alpha)$ are zero, and hence all entries in the $v$ column of $M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha)$ are also zero. However, $\lim_{k \to \infty} \frac{1}{k} \log S_v(k,\alpha)$ exists a.s. and equals $\gamma(\alpha)$, for every $v \in \{1, \ldots, V\}$, see Lemma 4.4.

**Remark 3.5.** Assuming the open set condition it follows, for random homogeneous fractals $K^\omega$ (the $V = 1$ case), that

$$\dim_H(K^\omega) \text{ is the unique } \alpha \text{ for which } E_\lambda \log \sum_{m=1}^{M_\lambda} (r_m^\lambda)\alpha = 0, \text{ for a.e. } \omega.$$ 

See [Ham92] for the direct computation of the dimension in some particular cases. For random recursive fractals $K^\omega$, the “$V \to \infty$” case,

$$\dim_H(K) \text{ is the unique } \alpha \text{ for which } E_\lambda \sum_{m=1}^{M_\lambda} (r_m^\lambda)\alpha = 1, \text{ for a.e. } \omega.$$ 

See [Fal86, MW86, Gra87, Gra91].

4. PROOF OF THE MAIN THEOREM

The proof is broken into a number of lemmas.

**Assumptions.** We continue the assumptions from the beginning of Section 2, and the notation from Sections 2 and 3. The integer $V \geq 1$ is fixed and $\alpha$ is non negative.

Define

$$R_{\min}(\alpha) = \inf_{\lambda} \sum_{m=1}^{M_\lambda} (r_m^\lambda)\alpha, \quad R_{\max}(\alpha) = \sup_{\lambda} \sum_{m=1}^{M_\lambda} (r_m^\lambda)\alpha.$$ 

The sequence $a = a_0 \ldots a_{k-1} \ldots \in A_V$ is chosen according to $P^\infty_V$ as in Section 3.1. The $V$-tuple of fractal sets corresponding to $a$ is denoted $K^a$, and the corresponding $V$-tuple of labelled trees is $\omega^a$. Let

$$K^a = (K_1^a, \ldots, K_V^a) = (K_1, \ldots, K_V),$$

$$\omega^a = (\omega_1^a, \ldots, \omega_V^a) = (\omega_1, \ldots, \omega_V).$$

From the next lemma, it follows there is a unique $d$ such that as $k \to \infty$ the product $\|M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha)\|$ grows exponentially fast to $\infty$ if $\alpha < d$, and decays exponentially fast to $0$ if $\alpha > d$. This does not imply convergence to a non zero limit or boundedness of $\|M^{a_0}(d) \cdot \ldots \cdot M^{a_{k-1}}(d)\|$, but it does imply that any infinite growth, or decay to zero, should be slower than exponential.
Lemma 4.1. The limit
\[ \lim_{k \to \infty} \frac{1}{k} \log \| M^{a_0} \cdot \ldots \cdot M^{a_k-1} \| =: \gamma(a) \]
exists. In addition,
\[ \lim_{k \to \infty} \frac{1}{k} \log \| M^{a_0} \cdot \ldots \cdot M^{a_k-1} \| = \gamma(a) \text{ a.s.,} \]
and in particular is a.s. independent of \( a \). Moreover,
\[ \log R_{\min}(a) \leq \gamma(a) \leq \log R_{\max}(a). \]
The function \( \gamma \) is strictly decreasing, Lipschitz, has derivative in the interval \([\log r_{\min}, \log r_{\max}]\), \( \gamma(0) > 0 \) and \( \gamma(a) \to -\infty \) as \( a \to \infty \). In particular, there is a unique \( d \) such that \( \gamma(d) = 0 \).

Proof. The first two claims in the lemma hold for some \( \gamma(a) \) with \( -\infty \leq \gamma(a) < \infty \). This follows from the version of the Furstenberg Kesten theorem in [Coh88, Theorem C p72], since the \( a_k \) are chosen in an iid manner.

Suppose \( A \) and \( B \) are square matrices of the same size with non-negative entries. Assume
\[ \alpha_1 \leq \sum_j A_{ij} \leq \alpha_2, \quad \beta_1 \leq \sum_j B_{ij} \leq \beta_2, \text{ for all } i. \]
Let \( C = AB \). Since \( \sum_j C_{ij} = \sum_k A_{ik} \left( \sum_j B_{kj} \right) \),
\[ \alpha_1 \beta_1 \leq \sum_j C_{ij} \leq \alpha_2 \beta_2 \text{ for all } i. \]

In particular, from Definition 3.2 and (4.1),
\[ R^k_{\min}(\alpha) \leq \sum_w (M^{a_0} \cdot \ldots \cdot M^{a_{k-1}}(\alpha))_{vw} \leq R^k_{\max}(\alpha) \text{ for all } v, \]
and so
\[ VR^k_{\min}(\alpha) \leq \| M^{a_0} \cdot \ldots \cdot M^{a_{k-1}}(\alpha) \| \leq VR^k_{\max}(\alpha). \]
Taking logs of both sides, and letting \( k \to \infty \), gives the third claim in the lemma.

From Definition 3.2, if \( 0 \leq \alpha < \beta \),
\[ r^k_{\min}(\| M^{a_0} \cdot \ldots \cdot M^{a_{k-1}}(\beta) \| \leq \| M^{a_0} \cdot \ldots \cdot M^{a_{k-1}}(\beta) \| \leq r^k_{\max}(\| M^{a_0} \cdot \ldots \cdot M^{a_{k-1}}(\alpha) \|). \]
Taking logs and letting \( k \to \infty \),
\[ (\beta - \alpha) \log r_{\min} \leq \gamma(\beta) - \gamma(\alpha) \leq (\beta - \alpha) \log r_{\max}. \]
Hence \( \gamma \) is Lipschitz, differentiable a.e., monotonically decreasing to \( -\infty \), and has derivative in the range \( \log r_{\min} \to \log r_{\max} \).

Since \( \gamma(0) \in [\log R_{\min}(0), \log R_{\max}(0)] = [\log(\min M_{\Lambda}), \log M = \log(\max M_{\Lambda})] \),
it follows that \( \gamma(0) > 0 \), and so there exists a unique \( d \) such that \( \gamma(d) = 0 \).

In the next following lemmas, the notion of a “neck” \( a \in A_V \) will play an important role, see Figure 6.

Definition 4.2. The element \( a \in A_V \) is a neck if all \( J^a(v, m) \) are equal for \( v \in \{1, \ldots, V\} \) and if \( m \in \{1, \ldots, M(r(v))\} \).
A neck occurs at level \( k \) in \( a = a_0 \ldots a_{k-1} \ldots \in A^\infty_V \), or more simply we say that \( k \) is a neck, if \( a_{k-1} \) is a neck.

Remark 4.3. An element \( a \) chosen according to \( P_V \) is a neck with probability at least \( V^{1-MV} \). It follows that necks in a sequence \( a \in A^\infty_V \) occur infinitely often a.s.
If a neck occurs in \( a \) at level \( k \) then all subtrees of \((\omega^a_1, \ldots, \omega^a_V)\) rooted at level \( k \) are equal. That is, if \( |m| = |n'| = k \) and \( v, v' \in \{1, \ldots, V\} \), then \( mn \in \omega_v \) iff \( m'n' \in \omega_{v'} \), and in this case \( \omega_v(mn) = \omega_{v'}(m'n') \).
Figure 6. A neck $a$ with IFSs $F$, $G$ and $H$, and with $J^a(v,m) = 2$ or $0$ for all $v$ and $m$.

For the following lemma recall that $S_v(k,\alpha)$ is the sum of the elements in the $v$th row of $M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}}(\alpha)$. From (3.4) we have that $S(\omega,k,\alpha)$, for large $k$, is an approximation to the Hausdorff measure $\mathcal{H}^\alpha(K^\omega)$. See also Remark 3.4.

**Lemma 4.4.** For $a \in A_\infty^V$ and $\omega = \omega_a = (\omega_1, \ldots, \omega_V)$,

$$\lim_{k \to \infty} \frac{1}{k} \log S_v(k,\alpha) = \gamma(\alpha) \text{ a.s.}$$

**(Proof.** Let $S_v(k,\alpha) = S(\omega_v,k,\alpha)$, as in (3.6).

Suppose the address sequence $a$ has a neck at level $p$, where $J^a_p(v, m) = u$ for all $v \in \{1, \ldots, V\}$ and $m \in \{1, \ldots, M^{a_{p-1}(v)}\}$. It follows that all columns of $M^{a_{p-1}}$ are zero, except for the $u$th column, and hence the same is true for $M^{a_0} \circ \ldots \circ M^{a_{p-1}}$.

Suppose $A$ and $B$ are $V \times V$ matrices, such that all columns of $A$ are zero except for the $u$th column, which we denote by $a$. Let $b$ be the $u$th row of $B$. Then

$$AB = [b_1a b_2a \ldots b_Va].$$

It follows that

$$\sum_w (AB)_{uw} = a_v \sum_w b_w = \left(\sum_w A_{vw}\right) \sum_w b_w.$$ 

In particular, the second factor is independent of $v$.

Apply this to

$$S_v(k,\alpha) = \sum_w (M^{a_0} \circ \ldots \circ M^{a_{k-1}})_{vw},$$

see Proposition 3.3 and (3.6), with

$$A = M^{a_0} \circ \ldots \circ M^{a_{p-1}}, \quad B = M^{a_p} \circ \ldots \circ M^{a_{k-1}}.$$ 

It follows that

$$S_v(k,\alpha) = S_v(p,\alpha) g(k,\alpha),$$

where $g(k,\alpha)$ is independent of $v$, and where $S_v(p,\alpha) > 0$ for all $v$.

From (4.4), summing over $v$,

$$\|M^{a_0} \circ \ldots \circ M^{a_{k-1}}\| = \sum_v S_v(k,\alpha) = g(k,\alpha) \sum_v S_v(p,\alpha).$$

Hence from Lemma 4.1,

$$\gamma(\alpha) = \lim_{k \to \infty} \frac{1}{k} \log \|M^{a_0}(\alpha) \cdot \ldots \cdot M^{a_{k-1}(\alpha)}\| = \lim_{k \to \infty} \frac{1}{k} g(k,\alpha) \text{ a.s.}$$

Going back to (4.5), since $S_v(p,\alpha) > 0$, it follows that

$$\lim_{k \to \infty} \frac{1}{k} \log S_v(k,\alpha) = \gamma(\alpha) \text{ a.s.} \qed$$
Lemma 4.5. If $\alpha > d$ with $d$ as in Lemma 4.1, then $\mathcal{H}^\alpha(K_v^0) = 0$ for $v \in \{1, \ldots, V\}$ and for a.e. $a$. In particular, $\dim_H(K_v^a) \leq d$.

Proof. We usually drop the reference to $a$ and write $K_v$ for $K_v^a$, and $\omega_v$ for $\omega_v^a$.

Let $E$ be any set such that $K_1 \cup \cdots \cup K_V \subset E$, and without loss of generality suppose $|E| = 1$. For $v \in \{1, \ldots, V\}$ and $m = m_1 \ldots m_k \in \omega_v$, let

$$E_{v;m_1 \ldots m_k} = f_{m_1}^{\omega_v(0)} \circ f_{m_2}^{\omega_v(m_1)} \circ \cdots \circ f_{m_k}^{\omega_v(m_1 \ldots m_{k-1})}(E),$$

as in (2.4). Then

$$K_v \subset \bigcup_{\{m \in \omega_v : |m| = k\}} E_{v;m},$$

and

$$S_v(k, \alpha) = \sum_{\{m \in \omega : |m| = k\}} |E_{v;m}|^\alpha,$$

as in (3.6) and (3.4).

Since $\gamma(\alpha) < 0$, it follows from (4.3) that

$$\lim_{k \to \infty} S_v(k, \alpha) = \lim_{k \to \infty} e^{k(\frac{1}{\alpha} \log S_v(k, \alpha))} = 0 \text{ a.s.}$$

Hence $\mathcal{H}^\alpha(K_v) = 0$ a.s. \hfill \Box

Lemma 4.6. Assume $F$ satisfies the open set condition. If $\alpha < d$, where $d$ is as in Lemma 4.1, then $\mathcal{H}^\alpha(K_v^a) > 0$ a.s. for $1 \leq v \leq V$. In particular, $\dim_H(K_v^a) \geq d$ a.s.

Proof. Suppose $\alpha < d$. As before, $K_v = K_v^a$ and $\omega_v = \omega_v^a$.

For a.e. $a$ and each $1 \leq v \leq V$, we construct a unit mass measure $\mu$ on $K_v$, such that for some $c$,

$$\mu(B_v(x)) \leq c r^x \text{ if } r > 0, \ x \in \mathbb{R}^n.$$  

It then follows by the mass distribution principle [Fal03, p60] that $\mathcal{H}^\alpha(K_v) > 0$, and so $\dim(K_v) \geq d$.

A. Properties of Necks. For $a \in \mathcal{A}_v^\alpha$ and $k \geq 0$, let $n^a(k) = n(k)$ denote the first level $\geq k$ at which a neck occurs.

Then we claim

$$\forall \epsilon > 0 \exists N > 0 \text{ such that } \forall k(n(k) - k \leq N + \epsilon k) \text{ a.s.},$$

where $N$ will depend on $a$.

To see this, fix $\epsilon > 0$ and $k > 0$, and let

$$E_k = \{a : n^a(k) - k > \epsilon k\}.$$ It follows from Remark 4.3 that

$$P_v^\alpha(E_k) \leq (1 - V^{-1-MV})^k.$$ Since $\sum_{k \geq 1} P_v^\alpha(E_k) < \infty$, it follows from the Borel Cantelli lemma that, with probability one, $E_k$ occurs for only finitely many $k$, and so $n(k) - k \leq \epsilon k$ for all $k$ sufficiently large. Hence for some $N$ depending on $a$ and $\epsilon$, $n(k) - k \leq N + \epsilon k$ for all $k$. This proves (4.8).

B. Construction of $\nu$ and $\mu$. Suppose $a$ is as in (4.8) and consider the tree $\omega_v^a = (\omega_1, \ldots, \omega_V)$. For fixed $v$, a unit measure $\nu$ will first be constructed on the set $\tilde{\omega_v}$ of infinite paths through $\omega_v$.

For $m \in \omega_v$, the corresponding cylinder set, a subset of $\tilde{\omega_v}$, is defined by

$$[m] = \{p \in \tilde{\omega_v} : m \leq p\},$$

where $m \leq p$ means that $m$ is an initial segment of $p$.

The weight function $w$ is defined on cylinder sets by

$$w([m]) = r_{\nu_v,m}^\alpha := \left( r_{\nu_v}^{\omega_v(0)} \circ \cdots \circ r_{\nu_v}^{\omega_v(m_1 \ldots m_k)} \right)^\alpha.$$
We define a unit mass measure $\nu$ on $\tilde{\omega}_v$ by setting, if $m \in \omega_v$ and $|m| = k$ is a neck,

$$\nu([m]) = \frac{w([m])}{\sum \{w([m']) : |m'| = k\}} = \frac{r_{v,m}^\alpha}{\sum \{r_{v,m'}^\alpha : m' \in \omega_v, |m'| = k\}}.$$  

The expression for $\nu([m])$, in case $|m|$ is not a neck, can be found in (4.15).

In order to show this does define a (unit mass) measure on $\tilde{\omega}_v$, first recall that $\alpha$ has necks of arbitrarily large size. We will prove that if $k \leq j$ are both necks, $m \in \omega_v$, and $|m| = k$, then $\nu$ satisfies the consistency condition

$$\nu([m]) = \sum \{\nu([n]) : m \leq n \in \omega_v, |n| = j\}.$$  

Here, and elsewhere, $m \leq n$ means $m$ is an initial segment of the finite sequence $n$.

Note that $[m] = \bigcup \{[n] : m \leq n \in \omega_v, |n| = j\}$, and this is a union of disjoint sets. It follows from (4.11) that $\nu$ extends to a unit mass measure on the $\sigma$-algebra of subsets of $\tilde{\omega}_v$ generated by the cylinder sets $[m]$ for which $|m|$ is a neck. This is just the $\sigma$-algebra generated by all cylinder sets, i.e. the class of Borel sets.

In order to prove (4.11), note that if $m \in \omega_v$, $|m| = k$ where $k$ is a neck for $\alpha$, and $ms \in \omega_v$, then it follows from (4.9) and Remark 4.3 that

$$r_{v,ms}^\alpha = \theta^\alpha(s) r_{v,m}^\alpha,$$

where $\theta(s)$ does not depend on either $m$ or $v$. Suppose now that $j \geq k$. Then from (4.9) and (4.12),

$$\sum \{w([n]) : m \leq n \in \omega_v, |n| = j\} = \sum \{r_{v,m}^\alpha \theta^\alpha(s) : ms \in \omega_v, |s| = j - k\}$$

$$= \lambda(k, j, \alpha) r_{v,m}^\alpha \lambda(k, j, \alpha) w([m]),$$

where $\lambda$ does not depend on $m$ or $v$. Replacing $m$ by $m'$ and $n$ by $n'$, and summing also over $m'$,

$$\sum \{w([m']) : m' \in \omega_v, |m'| = j\} = \lambda(k, j, \alpha) \sum \{w([m']) : m' \in \omega_v, |m'| = k\}.$$  

Dividing (4.13) by (4.14) and using (4.10) gives (4.11).

If $m \in \omega_v$ with $|m| = k$, not necessarily a neck, and $j \geq n(k)$, then

$$\nu([m]) = \frac{\sum \{\nu([n]) : m \leq n \in \omega_v, |n| = n(k)\}}{\sum \{\nu([n']) : m' \in \omega_v, |n'| = n(k)\}} = \frac{r_{v,m}'^\alpha}{\sum \{r_{v,m'}^\alpha : m' \in \omega_v, |m'| = n(k)\}}.$$  

(4.15)

using (4.12) in the final equality.

Define the map $\tilde{T} : \tilde{\omega}_v \rightarrow K_v$ by

$$\tilde{T}(p_1p_2\ldots p_k\ldots) = \lim_{k \rightarrow \infty} f_{p_1}^{\omega_v}(\theta) \circ f_{p_2}^{\omega_v}(p_1) \circ \ldots \circ f_{p_k}^{\omega_v}(p_{k-1})(x_0) \in K_v,$$

and note that the limit does not depend on $x_0$.

The measure $\nu$ on $\tilde{\omega}_v$ projects to the unit mass measure $\mu$ on $K_v$, defined

$$\mu(A) = \nu\{p \in \tilde{\omega}_v : \tilde{T}(p) \in A\}$$

for $A$ a Borel subset of $K_v$.

C. An upper estimate for $\nu$. Again assume $\alpha$ satisfies (4.8), and consider the corresponding tree $\omega^\alpha = (\omega_1, \ldots, \omega^\alpha)$. Fix $v$. We show for $m = m_1 \ldots m_k \in \omega^\alpha_v$ that

$$\nu([m]) \leq c_1 r_{v,m}^\alpha,$$

for some constant $c_1$ depending on $\alpha$ but not on $m$.

From (4.15),

$$\nu([m]) = \frac{\sum \{r_{v,m}^\alpha : m \leq n \in \omega_v, |n| = n(k)\}}{\sum \{r_{v,m'}^\alpha : m' \in \omega_v, |m'| = n(k)\}} \leq \frac{r_{v,m}^\alpha M^{n(k) - k}}{S_v(n(k), \alpha)}.$$
To establish this inequality use (4.9) with \( m \) there replaced by \( n \), note that each \( r_v^\alpha \leq 1 \) and the branching number of \( \omega_v \) is bounded by \( M \), and use the expression in (3.4) for \( S_v(n(k), \alpha) = S(\omega_v, n(k), \alpha) \).

From (4.8), for any \( \epsilon > 0 \), there exists \( N(\epsilon) \) such that

\[
M^{n(k) - k} \leq M^{N(\epsilon)} M^{k\epsilon}
\]

for all \( k \).

From (4.3), since \( \alpha < 0 \) and so \( \gamma(\alpha) > 0 \), there exists \( c_2 \in \mathbb{R} \) such that for all \( j \),

\[
\log S_v(j, \alpha) \geq c_2 + \frac{j}{2} \gamma(\alpha).
\]

Hence,

\[
S_v(n(k), \alpha) \geq c_3 e^{\frac{1}{2} n(k) \gamma(\alpha)} \geq c_3 e^{\frac{1}{2} k \gamma(\alpha)}
\]

for some \( c_3 > 0 \) and all \( k \).

Choose \( \epsilon \leq \frac{1}{2} \gamma(\alpha)/\log M \) so that \( e^{\frac{1}{2} k \gamma(\alpha)} \geq M^{\epsilon k} \), and then choose \( N = N(\epsilon) \). Dividing (4.19) by (4.20), and using (4.18), gives (4.17) with \( c_1 = M^N/c_3 \).

D. The estimate for \( \mu \). Fix \( a \in \mathcal{A}_v^\infty \) satisfying (4.8), in which case (4.17) holds. Assume the open set condition (2.3) holds with the open set \( \mathcal{O} \).

Fix \( x \in \mathbb{R}^n \) and \( r > 0 \). With \( \mu \) the measure on \( K_v \) as in (4.16), we show by a standard argument, see [Hut81, p737] or [Fal03, p131], that

\[
\mu(B_r(x)) \leq cr^\alpha.
\]

Here \( c \) is independent of \( x \) and \( r \), and \( B_r(x) \) is the open ball of radius \( r \) centred at \( x \).

First note that, for each infinite sequence \( p = m_1 m_2 \ldots m_k \ldots \in \tilde{\omega}_v = \tilde{\omega}_v^a \), there is a least \( k \) such that

\[
r_{\min} r \leq r_{v;m_1 \ldots m_k} < r.
\]

Let \( Q(r) = Q^a(r) \) be the set of all such \( m = m_1 \ldots m_k \). The sets \( \mathcal{O}_{v,m} \) for \( m \in Q(r) \) are disjoint from (2.5) and the definition of \( Q(r) \), although the \( |m| \) are not necessarily equal. Let \( Q(x, r) = Q^a(x, r) \) be the set of \( m \in Q(r) \) such that \( \mathcal{O}_{v,m} \) meets \( B_r(x) \).

Choose \( a_1 \) and \( a_2 \) so that \( \mathcal{O} \) contains an open ball of radius \( a_1 \), and is contained in an open ball of radius \( a_2 \). Then the sets \( \mathcal{O}_{v,m} \) each contain a ball of radius \( a_1 r_{v,m} \), and hence of radius \( a_1 r_{\min} \), and they are contained in a ball of radius \( a_2 r_{v,m} \), and hence of radius \( a_2 r \). It follows by a volume comparison that if \( q = q(x, r) \) is the cardinality of \( Q(x, r) \), since the inner balls are disjoint and are subsets of \( B_{(1+2a_2)r} \), that

\[
q(a_1 r_{\min} r)^n \leq (1 + 2a_2)^n r^n,
\]

and so \( q \) is bounded independently of \( x \) and \( r \).

Hence,

\[
\mu(B_r(x)) = \mu(B_r(x) \cap K_v) = \nu(\{ p : \tau(p) \in B_r(x) \cap K_v \})
\]

\[
\leq \nu \left( \bigcup \{ [m] : m \in Q(x, r) \} \right) = \sum \{ \nu([m]) : m \in Q(x, r) \}
\]

\[
\leq c_1 q \sum \{ r_{v,m}^\alpha : m \in Q(x, r) \} \leq c_1 q r^\alpha,
\]

using the definition of \( Q(x, r) \), the disjointedness of the \( m \in Q(x, r) \subset Q(r) \), the estimate (4.17) and (4.22). This establishes (4.7) and hence the lemma. \( \square \)
Figure 7. Graphs of $\gamma_V(\alpha) = \gamma(\alpha)$ for $V = 1, 2, 5$ respectively from left to right. Here $F = \{F, G\}$ and $P = \{1/2, 1/2\}$ as in the introduction.

5. Examples

For the model problem in the introduction, with $F$ and $G$ each chosen with probability $1/2$, it follows from (3.7) that, for random homogeneous Sierpinski triangles, the dimension is $d(1) = 2\log 3/(\log 2+\log 3) \approx 1.226$. For the corresponding random recursive case, from (3.8) the dimension is the solution $d$ of $\frac{1}{2} 3 \left(\frac{1}{2}\right)^d + \frac{1}{2} 3 \left(\frac{1}{3}\right)^d = 1$, i.e. $d(\infty) \approx 1.262$. For $V > 1$ we used Maple 10 to compute the values of $\gamma_V(\alpha)$ shown in Figure 7. These values have error at most .001 at the 95% confidence level, and from this one obtains the dimensions $d(2) \approx 1.241$, $d(5) \approx 1.252$. The computed graphs for $V > 1$ are concave up, although this does not show on the scale used.

References


