1 Introduction

A *minimal surface* is a surface for which the area is a minimum among all other surfaces with the same boundary and topology; or more generally one for which the area functional is stationary within the class of surfaces having the same boundary. The problem of investigating such surfaces is known as the *Plateau Problem*.

We prove optimal convergence results for the problem of approximating parametrised disc-like minimal surfaces by functions from certain finite dimensional spaces. The minimal surfaces may be *unstable*, and are assumed to be *non-degenerate*. In the absence of branch points this non-degeneracy condition holds for a dense open set of boundaries, i.e. for "almost all" boundaries.

The main results are simply, if somewhat informally, stated. Let Γ be a smooth curve in $I\!R^n$ and let D be the unit disc in $I\!R^2$. Let $u_0 : D \to I\!R^n$ be a non-degenerate minimal surface with boundary Γ. Without loss of generality assume u_0 is a conformal parametrisation. For each small $h > 0$ fix a sequence of points on ∂D with distance between consecutive vertices being of order h. Consider piecewise linear maps with respect to arc length $s_0 : \partial D \to I\!R^n$ which send the prescribed vertices to points on Γ. Within this finitely parametrised class of maps, consider those maps for which the (unique) harmonic extension either minimises the Dirichlet energy or more generally for which the Dirichlet energy is stationary. The harmonic extension of such a map is called a *semi-discrete minimal surface*. Semi-discrete minimal surfaces can be computed to within a prescribed degree of accuracy.

Then the main theorem, Theorem 6.3, implies there are semi-discrete minimal surfaces u_h such that

$$||u_0 - u_h||_{H^1(D)} \leq ch^{3/2}.$$

In [DH] we prove the further result

$$||u_0 - u_h||_{L^2(D)} \leq ch^{5/2}.$$

These rates of convergence are optimal with respect to the exponent of h.

Such convergence results are of the type usually proved via boundary element methods for solutions of partial differential equations, but this appears to be the first class of results of this type for geometric objects solving a highly non-linear geometric variational problem. We introduce a number of new techniques which we expect will be of use in other geometric problems.

Apart from their intrinsic interest, such results have computational significance. It follows from Theorem 6.4 that any convergent sequence of solutions of the discrete problem we consider will have as limit a conformally parametrised minimal surface u_0. Moreover, if u_0 is non-degenerate (which "generically" means that u has no branch points, and this fact can be determined by observation) and if the observed rate of convergence is at least h^α for some $\alpha > 0$ (a slower rate of convergence would not normally be numerically observable) then in fact the convergence rate of Therorem 6.3 must apply. These rates of convergence typically appear in computational experiments after a small number of grid refinements and provide very good evidence that one is indeed converging towards a non-degenerate minimal surface.

The outline of the paper is as follows. In Section 2 we review the history of the problem. In Section 3 we recall the formulation of the Plateau Problem in terms of stationary points for the Dirichlet Energy of the harmonic extension of monotone boundary maps. In Section 4 we follow [St2] and introduce the two spaces H and T of variations of boundary maps and establish some properties of the energy functional. The finite dimensional approximation spaces are introduced in Section 5 and various properties are established. The proof of the main result, Theorem 6.3, is given in Section 6. In the final Section we discuss the numerical results.

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2 Historical Remarks

Plateau's Problem always has served as a model problem for highly nonlinear problems in analysis as well as in numerical analysis. In numerical analysis the approximation of minimal surfaces which are graphs has been treated extensively. In this case one studies the minimal surface equation

$$\sum_{i=1}^{n} D_i \left(\frac{D_i u}{\sqrt{1+|Du|^2}} \right) = 0$$

on some domain under Dirichlet or Neumann boundary conditions. Perhaps the first paper was by Douglas himself [Dou2], using a finite difference approach. The paper contains both numerical examples and graphics, but, as remarked by Douglas, the appropriate techniques were not available at that time to prove the necessary convergence results (even for graphs). If the domain is convex, there are now optimal error estimates in all relevant norms for the Finite Element approximation to the solution, including L^∞-estimates. References for this problem are for example Concus [Co] and Ciarlet [Ci].

In this paper we treat the parametric minimal surface problem with prescribed boundary curve. This problem is theoretically and numerically completely different from the case of a minimal graph.

The numerically most successful approach up to now in parametric situations uses Courant's function for polygonal boundary curves. If the boundary curve is a polygon with vertices a_j ($j = 1, \ldots, n+3$) then the boundary of the unit disk D is decomposed into the segments $\gamma_j = \{e^{i\phi} : \tau_j < \phi < \tau_{j+1}\}$ for given $\tau = (\tau_1, \ldots, \tau_{n+3})$. Let us call the segments of the polygon Γ_j. One then minimises Dirichlet's integral over $X(\tau) = \{u \in H^1(D; I\!R^3) : u(\gamma_j) \subset \Gamma_j,\ j = 1, \ldots, n+3\}$. Let

$$d(\tau) = \inf_{u \in X(\tau)} \frac{1}{2} \int_D |Du|^2 = \frac{1}{2} \int_D |Du_0(\tau)|^2.$$

Due to Courant [Cou] $u_0(\tau) : \overline{D} \to I\!R^3$ is a minimal surface spanning Γ iff $\nabla d(\tau) = 0$. Because of the conformal group three fixed points on ∂D, $\tau_{n+k} = (1+k)\pi/2, k = 1, 2, 3$, correspond to three fixed vertices $a_{n+1}, a_{n+2}, a_{n+3}$ on Γ.

This idea was used by Jarausch [J] to compute approximations of minimal surfaces using a finite dimensional subspace of $X(\tau)$ consisting of Finite Elements on the unit disk D which are bilinear with respect to polar coordinates. A serious drawback of this method is that the grid on D moves according to τ and has a singularity at the origin. He proves convergence of the functional D with respect to the grid size. Wohlrab [Wo] extended this method to partially free minimal surfaces and even more general variational problems.

One of the problems with Courant's function $d(\tau)$ is that it is not smooth enough. Heinz [He] proved that a slight change in the definition of $d(\tau)$ makes it an analytic function in $\tau \in \{\tau \in I\!R^n : 0 < \tau_1 < \ldots < \tau_n < \pi\}$. Instead of $u(\gamma_j) \subset \Gamma_j$ in the definition of $X(\tau)$ one allows u to map the boundary part γ_j into the straight line $\overline{\Gamma}_j$ which contains the segment Γ_j. The resulting function $d(\tau)$ then is called Shiffman's function. Hinze used this function in [Hi1] and [Hi2] to compute minimal surfaces bounded by polygons.

Some work has been done to go the direct way to minimal surfaces, namely to minimise the area functional

$$A(u) = \int_D |u_{x_1} \wedge u_{x_2}|$$

over some discrete space. Of course any such numerical method leads to theoretical and numerical problems because of the invariance of the area functional under arbitrary diffeomorphisms. Wagner [Wa1], [Wa2] used the area functional to minimise area for polyhedra spanned by a boundary curve. The same approach was used by Steinmetz [Ste] for more complicated problems involving minimal surfaces, especially partially free minimal surfaces. See also Tsuchiya [T1; Prop. 1].

Mean curvature flow of surfaces is the gradient flow for the area functional. This is used by Dziuk [Dz] to compute stable minimal surfaces by a Finite Element Method using finite elements on surfaces. No convergence proof is given. A somewhat similar idea with infinite time step is used by Pinkall and Polthier [PP] to compute minimal surfaces and their conjugates.

Parks [P] approximates minimal surfaces by the level sets of functions of least gradient. A public-domain program "Surface Evolver", which can obtain minimisers for many discrete functionals (including the discrete area functional), has been written by Brakke [Br]. Conditions under which there is a smooth minimal surface near a discrete minimal surface have been obtained by Underwood [U]. Sullivan [S] has a max flow/min cut type algorithm which uses a polyhedral of space to obtain approximations to area minimising currents; and provides a theoretical analysis. In all these cases, in order to obtain reasonable accuracy the numerical estimates require decompositions too fine for current workstations.

Following the lines of the proof of Rado and Douglas, Tsuchiya gives an existence proof for discrete minimal surfaces in [T2,3] and a convergence proof of the discrete surfaces to a continuous solution in the $H^1(D)$-norm. This convergence can be arbitrarily slow because the author uses an indirect argument in connection with the Courant-Lebesgue Lemma and so cannot prove any order of convergence with respect to the grid size. Although the result of Tsuchiya seems to be the first complete convergence result for the approximation of minimal surfaces, it is proved for minimisers only.

A numerical method for the computation of solutions of Plateau's Problem which one could call a Boundary Element Method was proposed by Wilson in [Wi] who used the Douglas Integral.

Since the difference between Dirichlet's integral and Area always is nonnegative and $D(u) - A(u) = 0$ only for minimal surfaces u, Hutchinson [Hu] minimises this difference, which is called the *conformal energy* of the surface u. In some situations this has significant numerical advantages over minimising the Dirichlet energy. In addition arbitrary, not necessarily stable, minimal surfaces can be found in this way by a minimisation procedure.

3 Formulation of the Plateau Problem

Let Γ be a Jordan curve in $\mathbb{R}^n$ with a C^r parametrisation

$$\gamma : S^1 \to \Gamma,$$

where $r \geq 3$. Let D be the (open) unit disc in $\mathbb{R}^2$.

> We will be interested in maps $u : D \to \mathbb{R}^n$ such that $u|_{\partial D} : \partial D \to \Gamma$ is monotone and such that $A(u)$, the area of the image of u, is stationary with respect to variations in this class of maps. We need to take account of the fact that $A(u)$ is invariant under the operation of composing u with a diffeomorphism of D.
>
> Let
> $$\mathcal{D}(u) = \frac{1}{2} \int_D |Du|^2$$
> denote the *Dirichlet Energy* of u. It is well-known that a more precise reformulation of the Plateau Problem is to find those *harmonic maps* $u : D \to \mathbb{R}^n$ such that $u|_{\partial D} : \partial D \to \Gamma$ is monotone (see below) and such that u is stationary for $\mathcal{D}$ in this class. Any solution u of the reformulated Plateau Problem will be a solution of the original Plateau Problem, and moreover u will be a *conformal* map. Conversely, if u is a (smooth) solution of the original Plateau Problem then for some diffeomorphism $d : D \to D$ the map $u \circ d$ will be a solution of the reformulated Plateau Problem (and in particular $u \circ d$ will be conformal).
>
> It will be convenient to factor boundary maps through the fixed parametrisation $\gamma : S^1 \to \Gamma$. Thus we write $w = \gamma \circ s$ where $s : \partial D \to S^1$. See Figure 1.

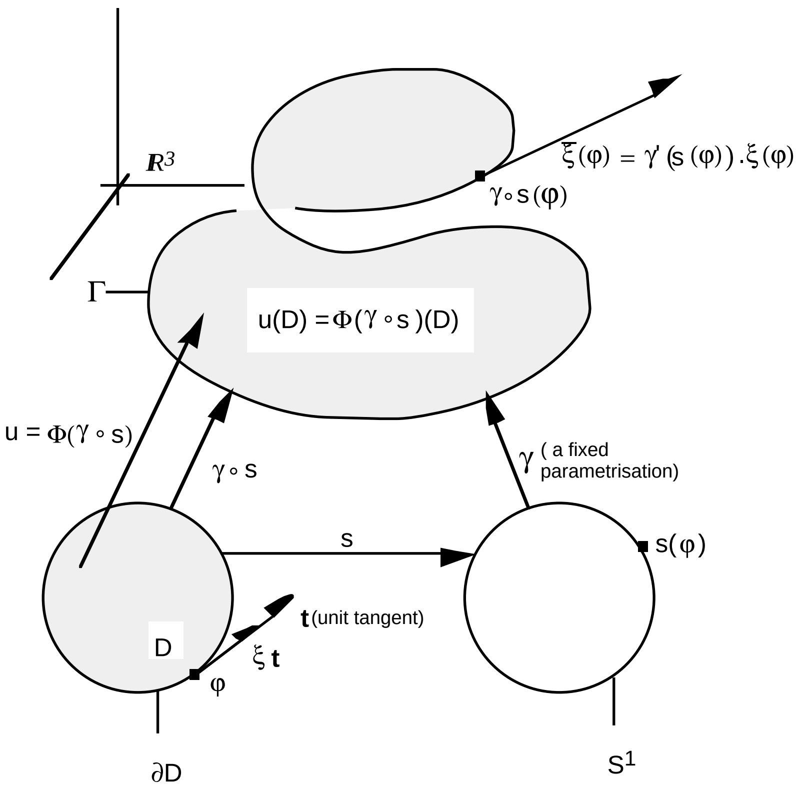

Figure 1

We now make these ideas precise. We use the notation

$$D = \{z = (x, y) \in I\!R^2 : |z| < 1\}$$

and

$$\partial D \cong S^1 = \{e^{i\phi} : 0 \le \phi < 2\pi\} \cong I\!R/2\pi \cong \{\phi : 0 \le \phi < 2\pi\}.$$

Although ∂D and S^1 are naturally isomorphic, it will convenient to consider S^1 as the domain of the fixed parametrisation γ of Γ, and to consider ∂D as the boundary of the unit disc D which is the fixed parameter domain for various parametrised surfaces spanning Γ.

A map $s \in C^0(\partial D, S^1)$ is *monotone* if s is *positively oriented* and $s^{-1}\{p\}$ is connected for all $p \in S^1$ (as ϕ moves once around ∂D, so $s(\phi)$ moves once around S^1 in the same direction, possibly pausing but never retracing its path). We similarly define the notion of a monotone map from $\partial D \to \Gamma$. There is a one-one correspondence $s \leftrightarrow \gamma \circ s$ between monotone maps in $C^0(\partial D, S^1)$ and monotone maps in $C^0(\partial D, \Gamma)$.

Let

$$\mathrm{id} : \partial D \to S^1$$

be the "identity" map given by $\mathrm{id}(\phi) = \phi$. Then id is monotone. Any monotone map $s \in C^0(\partial D, S^1)$ (or more generally, any $s \in C^0(\partial D, S^1)$ with winding number $+1$) can be written in the form

$$s = \mathrm{id} + \sigma$$

for some 2π-*periodic* function $\sigma \in C^0(\partial D, I\!R)$. Addition is performed modulo 2π and the function σ is unique up to a constant multiple of 2π. It is often more convenient to work with σ rather than s.

For $w \in C^0(\partial D, I\!R^n)$ denote by

$$\Phi(w)$$

the unique *harmonic extension* of w to D.

We can now define the *energy functional* E.

Definition 3.1 For $s \in C^0(\partial D, S^1)$ let

$$E(s) = \frac{1}{2}\int_D |D\Phi(\gamma \circ s)|^2 = \mathcal{D}(\Phi(\gamma \circ s)).$$

Thus $E(s)$ is just the Dirichlet Energy of the harmonic extension of $\gamma \circ s$. The Energy can be expressed directly in terms of the values of the function $\gamma \circ s$ by means of the *Douglas Integral*

$$E(s) = \frac{1}{16\pi}\int_{\partial D}\int_{\partial D} \frac{|(\gamma \circ s)(\phi) - (\gamma \circ s)(\phi')|^2}{\sin^2\left(\frac{\phi-\phi'}{2}\right)} d\phi\, d\phi', \tag{1}$$

c.f. [N2; §§310–311].

There is one further restriction that we need to make on the class of boundary maps. It is easily checked that the Dirichlet Energy of a function is invariant under composition of the function with any conformal diffeomorphism of its domain. If the original function is harmonic then so is the composed function. For this reason we factor out the class of Möbius transformations. The usual way to do this is to specify that $u|_{\partial D}$ satisfies a *three-point condition* $u(p_i) = q_i$ for $i = 1, 2, 3$, where the p_i are any three distinct points on ∂D and q_i are any three distinct points on Γ. But here it is more convenient to impose the following integral constraints on $s = \mathrm{id} + \sigma$:

$$\int_0^{2\pi} \sigma(\phi)\, d\phi = 0, \quad \int_0^{2\pi} \sigma(\phi) \cos\phi\, d\phi = 0, \quad \int_0^{2\pi} \sigma(\phi)\ \sin\phi\, d\phi = 0. \tag{2}$$

It can be shown by a homotopy argument that for any monotone $t \in C^0(\partial D, S^1)$ there exists $g\colon \partial D \to \partial D$, where g is the restriction of a Möbius transformation, such that $s = t \circ g = \mathrm{id} + \sigma$ satisfies (2).

We next define a class of monotone boundary maps.

Definition 3.2

$$\mathcal{M} = \{s \in C^0(\partial D, S^1) : s = \mathrm{id} + \sigma \text{ is monotone, } \sigma \text{ satisfies (2), } E(s) < \infty\}.$$

Because of the first condition in (2) the function σ in the previous definition is *uniquely* determined by s (otherwise it would only be determined up to multiples of 2π). Note that $\mathcal{M}$ is *convex* but not linear or even affine, since arbitrarily small (in the C^0 norm) deformations can destroy monotonicity.

We now formulate the Plateau Problem in the form we will use it.

Definition 3.3 The function $s \in \mathcal{M}$ is a *stationary point* for E if

$$\left.\frac{d}{dt}\right|_{t=0} E(s + t\xi) \geq 0$$

whenever $s + \xi \in \mathcal{M}$. If s is stationary for E then we say that the harmonic map $u = \Phi(\gamma \circ s)$ is a *minimal surface* or that u is a *solution of the Plateau Problem.*

In Section 4 we will imbed $\mathcal{M}$ in an affine Banach space $\mathcal{T}$ on which E extends to a C^{r-1} functional in the Fréchet sense. We will see that $s \in \mathcal{M}$ is *stationary* for E in the previous sense iff s is stationary for E in the sense of $\mathcal{T}$, i.e. iff $dE(s) = 0$ in $\mathcal{T}$.

An equivalent formulation which also shows the nonlinearity of the Plateau Problem is (c.f. [St2; Lemmas I.2.2, II.2.9]):

Definition 3.4 Suppose $u \in H^1(D; I\!R^n)$ and $u|_{\partial D}$ is a continuous monotone map onto Γ. Then u solves the Plateau Problem if u is harmonic and conformal: i.e.,

$$\triangle u = 0,$$

$$|u_x| = |u_y|\,, \quad u_x \cdot u_y = 0.$$

We finally make a few comments on properties of solutions of the Plateau Problem.

The existence of a solution was first established by Douglas [Dou] and Rado [R] by, in essence, showing the existence of a minimiser of $\mathcal{D}(u)$ subject to the three-point condition. Solutions of the Plateau problem which are not minimisers were found by Morse-Tompkins [MT] and Shiffman [S] using Morse theory methods. A simple example is Enneper's surface; see also the later discussion of numerical results. For a very clean Morse theory treatment, see [St2; Section II].

If u is a solution then it is analytic in the (open) set D, being harmonic there. If u is smooth up to the boundary then there are at most a finite number of points where $|Du| = 0$. Such points are called *branch points.* For the behaviour of u in a neighbourhood of a branch point see [St2; Definition I.5.4] and the subsequent remarks there.

The boundary behaviour of a solution u to the Plateau problem is also well understood. In particular, $u|_{\partial D}$ is a homeomorphism ([St2; Theorem I.5.3]). From results of Hildebrandt [Hil], Jäger [Ja] and J.C.C. Nitsche [N1], see also Heinz [He], if the parametrisation γ of Γ is $C^{m,\alpha}$, where $m \geq 1$ and $0 < \alpha < 1$, then $u \in C^{m,\alpha}(\overline{D}, I\!R^n)$.

In case $n = 3$, it was established by Osserman [O], with improvements due to Alt [A1,A2] and Gulliver [G], that the Douglas-Rado solutions have no interior branch points. This is not true in case $n \geq 4$. If γ is analytic (and $n = 3$) there are no boundary branch points, but this is not known in case γ is not analytic.

Good general references are the books by J.C.C. Nitsche [N2] and by Dierkes, Hildebrandt, Küster and Wohlrab [DHKW].

4 Properties of the Energy Functional

Assume that

$$\gamma \in C^r, \quad r \geq 3.$$

For given $s \in \mathcal{M}$ the class of perturbations ξ of s such that $s + \xi \in \mathcal{M}$ is not linear, since arbitrarily small (in the C^0 norm) perturbations can destroy monotonicity. For this reason it is necessary to enlarge the class $\mathcal{M}$. In fact it is necessary to consider *two* spaces $\mathcal{H}$ and $\mathcal{T}$ extending $\mathcal{M}$ (see below).

The functional E extends to a smooth functional on $\mathcal{T}$ (but not even to a C^1 functional on $\mathcal{H}$). It follows from regularity theory (Proposition 4.5) that $s_0 \in \mathcal{M}$ is stationary for E in the sense of perturbations that remain in $\mathcal{M}$ iff s_0 is stationary in the sense of $\mathcal{T}$. We will simply say that s_0 is *stationary*. On the other hand, $\mathcal{T}$ is not the correct space on which to define non-degeneracy for stationary points (c.f. the remark following Corollary 4.11), and for this reason one introduces the space $\mathcal{H}$. For stationary s_0 the bilinear function $d^2E(s_0)\colon T \times T \to I\!R$ extends to a bounded symmetric bilinear operator $d^2E(s_0)\colon H \times H \to I\!R$ (Proposition 4.4). We associate with $d^2E(s_0)$ a self-adjoint map $\nabla^2 E(s_0)\colon H \to H$. Then (Proposition 4.9) $\nabla^2 E(s_0)$ has finite dimensional negative and null eigenspaces. The natural notion of *non-degeneracy* for the stationary point s_0 is the requirement that the null space of $\nabla^2 E(s_0)$ be trivial. It follows (Corollary 4.11) that the eigenvalues of $\nabla^2 E(s_0)$ are bounded away from zero.

It is "almost always" true that s_0 is non-degenerate iff the associated minimal surface $\Phi(\gamma \circ s_0)$ has no branch points. More precisely, if s_0 is stationary and $\Phi(\gamma \circ s_0)$ has a branch point then the null space of $\nabla^2 E(s_0)$ will contain the so-called "forced Jacobi fields" and in particular s_0 will be degenerate. Conversely, if s_0 is stationary and $\Phi(\gamma \circ s_0)$ has no branch points then, at least generically (for an open dense set of boundary maps γ), s_0 is non-degenerate. For these important results see Böhme and Tromba [BT].

We follow the approach of [St1, St2]. We will use the notation

$$[\phi - \phi'] \tag{3}$$

to denote the distance in $I\!R/2\pi$ between the points $e^{i\phi}$ and $e^{i\phi'}$ on ∂D or S^1 corresponding to ϕ and ϕ' respectively.

Definition 4.1 Let $|\cdot|_{H^{1/2}}$ be the $H^{1/2}(\partial D; I\!R^m)$ semi-norm defined by

$$|\xi|^2_{H^{1/2}} = \int_{\partial D}\int_{\partial D} \frac{|\xi(\phi) - \xi(\phi')|^2}{[\phi - \phi']^2} d\phi\, d\phi'.$$

Setting $m = 1$, let H be the Hilbert space

$$H \;=\; \{\xi\colon \partial D \to I\!R : |\xi|_{H^{1/2}} < \infty \text{ and (2) is satisfied with } \sigma \text{ replaced by } \xi\}$$

with *inner product*

$$(\xi, \eta)_{H^{1/2}} = \int_{\partial D}\int_{\partial D} \frac{\Big(\xi(\phi) - \xi(\phi')\Big) \cdot \Big(\eta(\phi) - \eta(\phi')\Big)}{[\phi - \phi']^2} d\phi\, d\phi'$$

and corresponding *norm* $|\cdot|_{H^{1/2}}$ (by the first condition in (2) and the Poincaré inequality, this is a norm on H and is equivalent to the usual $H^{1/2}$ norm).

Define the norm

$$||\xi||_T = |\xi|_{H^{1/2}} + ||\xi||_{C^0},$$

and in case $m = 1$ the corresponding Banach space T by

$$T = H \cap C^0(\partial D; I\!R).$$

It follows from the Douglas integral (1) that $E(s)$ is comparable to $|\gamma \circ s|^2_{H^{1/2}(\partial D;R^n)}$.

By performing the addition $s = \mathrm{id} + \xi$ modulo 2π we define corresponding (affine) spaces of boundary maps from $\partial D \to S^1$ as follows:

Definition 4.2 The *affine* Banach spaces $\mathcal{H}$ and $\mathcal{T}$ are defined by

$$\begin{aligned} \mathcal{H} &= \{\mathrm{id} + \xi : \xi \in H\}, \\ \mathcal{T} &= \{\mathrm{id} + \xi : \xi \in T\}. \end{aligned}$$

The corresponding *metrics* on $\mathcal{H}$ and $\mathcal{T}$ are defined to be those induced from H and from T. Thus we write

$$|s - t|_{H^{1/2}} \text{ and } ||s - t||_T,$$

for the distance between s and t in the $H^{1/2}$ and T sense respectively.

Note that the tangent space of variations at any $s \in \mathcal{T}$ ($s \in \mathcal{H}$) is naturally identified with T (H) respectively.

Note also that the map $\sigma \mapsto s = \mathrm{id} + \sigma$ is one-one for $\sigma \in H$, because of the first condition in (2). More precisely, let $s_i = \mathrm{id} + \sigma_i : \partial D \to S^1$ for $i = 1, 2$. If $s_1 = s_2$ then $\sigma_1 - \sigma_2 \equiv 0$ a.e. modulo 2π. Since $|\sigma_1 - \sigma_2|_{H^{1/2}} < \infty$ it follows from the definition of $|\cdot|_{H^{1/2}}$ that $\sigma_1 - \sigma_2$ is constant a.e. From the first condition in (2) it follows $\sigma_1 - \sigma_2 \equiv 0$ a.e.

Suppose $s \in \mathcal{H}$ and $\xi \in H$. It is sometimes convenient to think of ξ as a variation of s as follows. Define

$$\overline{\xi}(\phi) = \left.\frac{d}{dt}\right|_{t=0} \Big(\gamma \circ (s + t\xi)\Big)(\phi) = \gamma'\Big(s(\phi)\Big)\xi(\phi).$$

Then for each ϕ, we consider $\overline{\xi}(\phi)$ as a tangent vector to Γ at $(\gamma \circ s)(\phi)$. By taking the harmonic extension $\Phi(\overline{\xi})$ we can think of ξ as an harmonic vector field "tangent" to the harmonic surface $\Phi(\gamma \circ s)$. See Figure 1.

From [St2; Lemma II.4.1] one has

1. $\mathcal{M} \subset \mathcal{T} \subset \mathcal{H}$ and moreover $\mathcal{M}$ is a closed convex subset of $\mathcal{T}$.

2. $\mathcal{T}$ is dense in $\mathcal{H}$.

3. If $(s_n) \subset \mathcal{M}$ and $s_n \to s$ in $\mathcal{H}$, then $s \in \mathcal{M}$ and $s_n \to s$ in $\mathcal{T}$ (so $\mathcal{M}$ is closed in $\mathcal{H}$).

We write the energy functional in the form

$$E = \mathcal{D} \circ \Phi \circ G$$

where

1. $G(s) = \gamma \circ s$ and $G : \mathcal{T} \to H^{1/2}(\partial D; I\!R^n)$ from [St2; Lemma II.2.6(i)];
2. Φ is the harmonic extension operator and $\Phi : H^{1/2}(\partial D; I\!R^n) \to H^1(D; I\!R^n)$;
3. $\mathcal{D}$ is the Dirichlet Energy Functional given by $\mathcal{D}(u) = \frac{1}{2}\int_{\partial D} |Du|^2 = \frac{1}{2}(u,u)_{H^1(D;I\!R^n)}$ and $\mathcal{D} : H^1(D; I\!R^n) \to I\!R$.

The map Φ is a bounded linear map by standard trace theory, and in particular is analytic. The map $\mathcal{D}$ is quadratic and hence also analytic. Thus the differentiability properties of E will depend on those of G, which is C^{r-1} from [St2; Lemma II.2.6] and the preceeding comments there. We will make frequent use of the following properties of E implicit in [St2]. The second Proposition will be applied to s which are stationary for E.

Proposition 4.3 *Let $s = \mathrm{id} + \sigma$. Then $E : \mathcal{T} \to I\!R$ is C^{r-1}. Moreover*

$$\begin{aligned} E(s) &\le c\,(||\gamma||_{C^1})\left(1 + |\sigma|^2_{H^{1/2}}\right), \\ |d^j E(s)(\xi_1, \ldots, \xi_j)| &\le c(||\gamma||_{C^{j+1}})(1 + |\sigma|^2_{H^{1/2}})||\xi_1||_T \cdot \ldots \cdot ||\xi_j||_T \quad 1 \le j \le r-1. \end{aligned}$$

If $\sigma \in C^0(\partial D; S^1)$ then

$$|\sigma|^2_{H^{1/2}} \le c\,(E(s) + 1),$$

where c depends on $||\gamma^{-1}||_{C^1}$ and the modulus of continuity of σ.

PROOF: The first two claims follow from the proof of [St2; Lemma II.2.3]. The third claim follows from [St2; II.2.7] and the estimates in [St2; Lemma II.2.6]. ∎

The reason for the continuity assumption on s is clear. If $s(\phi) = -\pi$ for $0 \le \phi < \pi$ and $s(\phi) = \pi$ for $\pi \le \phi < 2\pi$ then $\gamma \circ s$ is constant and $E(s) = 0$ but $s \notin \mathcal{H}$.

Proposition 4.4 *If s is C^2 then $dE(s)$ and $d^2E(s)$ extend to bounded linear and bilinear operators respectively on H, and*

$$\begin{aligned} |dE(s)(\xi)| &\le c\,(||\gamma||_{C^2}, ||s||_{C^1})\,|\xi|_{H^{1/2}}, \\ |d^2E(s)(\xi_1, \xi_2)| &\le c\,(||\gamma||_{C^2}, ||s||_{C^2})\,|\xi_1|_{H^{1/2}}|\xi_2|_{H^{1/2}}. \end{aligned}$$

PROOF: Both claims follow from the estimates in [St2; II.2.6]; the second along the lines of the proof of [St2; Lemma II.4.2]. ∎

We previously defined the notion of stationarity for the energy functional at $s \in \mathcal{M}$ for "monotone" variations (Definition 3.3). The following Proposition [St2; Proposition II.2.9] shows that this agrees with the standard definition in $\mathcal{T}$. The main point in the proof is to use the stationarity condition to first establish the regularity result $\Phi(\gamma \circ s) \in H^2(D; I\!R^n)$.

Proposition 4.5 *The function $s \in \mathcal{M}$ is stationary for E with respect to monotone variations iff s is stationary in the sense of $\mathcal{T}$, i.e. iff*

$$dE(s)(\xi) = 0 \quad \forall \xi \in T. \tag{4}$$

For our purposes we may take the condition (4) as the *definition of stationarity.*

The regularity results of [Hil], [Ja], [N1], [He] imply the regularity of stationary s.

Proposition 4.6 *If γ is $C^{r,\alpha}$ where $r \geq 1$ and $0 < \alpha < 1$, and $s \in \mathcal{M}$ is stationary for E, then s is $C^{r,\alpha}$ and $||s||_{C^{r,\alpha}}$ is controlled by $||\gamma||_{C^{r,\alpha}}$.*

Since we are assuming γ is C^3 it follows from Proposition 4.4 that $d^2E(s)$ is a (symmetric) bounded bilinear map. Using the Riesz Representation Theorem we then make the following definition.

Definition 4.7 Suppose s is a stationary point for E. The self-adjoint bounded linear map

$$\nabla^2 E(s) \colon H \to H$$

is defined by

$$\left(\nabla^2 E(s)(\xi), \eta\right)_{H^{1/2}} = d^2E(s)(\xi, \eta) \quad \forall \xi, \eta \in H.$$

In order to obtain our approximation results we will need to consider the second order behaviour of E near a stationary point s.

Definition 4.8 Suppose $s \in \mathcal{M}$ is stationary for E. Denote by

$$H = H^- \oplus H^0 \oplus H^+$$

the standard orthogonal decomposition into the negative, null and positive subspaces generated by the self-adjoint map $\nabla^2 E(s)$. For $\xi \in H$ let

$$\xi = \xi^- + \xi^0 + \xi^+,$$

where

$$\xi^- \in H^-, \quad \xi^0 \in H^0, \quad \xi^+ \in H^+.$$

Thus H^0 is the kernel of $\nabla^2 E(s)$ and H^- (H^+) is the maximal invariant subspace satisfying $\left(\nabla^2 E(s)(\xi), \xi\right) < 0$ (> 0) for all $0 \neq \xi \in H^-$ (H^+). It follows from the following Proposition and standard elliptic theory that H^-, H^0 and H^+ are generated by the eigenfunctions with negative, zero, and positive eigenvalues respectively.

Proposition 4.9 *Assume $r \geq 5$, $s \in \mathcal{M}$ is stationary, and $H = H^- \oplus H^0 \oplus H^+$ is the decomposition corresponding to $\nabla^2 E(s)$. Then*

(i) $\dim(H^- \oplus H^0) < \infty$, $H^- \oplus H^0 \subset H^{5/2}(\partial D; \mathbb{R}) \subset C^1(\partial D; \mathbb{R})$,
and moreover the injections are continuous with norm controlled by $||s||_{C^{4,\alpha}}$ and hence in particular by $||\gamma||_{C^5}$;

(ii) $||\xi^-||_{H^{5/2}},\ ||\xi^0||_{H^{5/2}} \leq c(||\gamma||_{C^5})||\xi||_{H^{1/2}}\ \forall \xi \in H$;

(iii) *The non-zero eigenvalues of* $d^2E(s)$ *are bounded away from* 0.

PROOF: For (i) see the proof of [St2; II.5.6].

For (ii) let $\tau_1, \ldots, \tau_k$ be an orthonormal basis for H^-. Then $\xi^- = \sum_{i=1}^k (\xi, \tau_i)\tau_i$, and the result for ξ^- follows from (i) and the fact the sum is finite. Similarly for ξ^0.

For (iii) We have $H/\nabla^2E(s)(H) \cong H^0$. Since $\nabla^2E(s)(H^0) = \{0\}$ and $\nabla^2E(s)(H^- \oplus H^+) \subset H^- \oplus H^+$, it then follows by a dimension argument that $\nabla^2E(s)(H^- \oplus H^+) = H^- \oplus H^+$. Also $\ker \nabla^2E(s)|_{H^- \oplus H^+} = \{0\}$. Hence $\nabla^2E(s)\colon H^- \oplus H^+ \to H^- \oplus H^+$ is bijective, and so has bounded inverse by the Open Mapping Theorem. This implies (iii). ∎

Definition 4.10 A stationary point $s \in \mathcal{M}$ is *non-degenerate* if the corresponding null eigenspace $H^0 = \{0\}$. The corresponding minimal surface $\Phi(\gamma \circ s)$ is also said to be *non-degenerate.*

The following non-degeneracy estimate will be important. It follows immediately from the previous proposition.

Corollary 4.11 *Assume* $r \geq 5$ *and* $s \in \mathcal{M}$ *is stationary. Then* s *is non-degenerate iff the eigenvalues of* $\nabla^2E(s)$ *are bounded away from* 0. *Equivalently, iff there exists* $\nu > 0$ *such that*

$$d^2E(s)(\xi, \xi^+ - \xi^-) \geq \nu|\xi|^2_{H^{1/2}} \quad \forall \xi \in H.$$

In case s is *stable*, i.e. the negative space is trivial, this is just the ellipticity estimate

$$d^2E(s)(\xi, \xi) \geq \nu|\xi|^2_{H^{1/2}} \quad \forall \xi \in H.$$

The constant ν is called the *non-degeneracy constant* for $d^2E(s)$.

Remark One cannot expect similar estimates with the T norm on the right side, despite the fact that the natural (affine) space on which to work with respect to the differentiability properties of E is $\mathcal{T}$. In fact a computation (c.f. the proof of [St2; Lemma II.4.2]) shows

$$d^2E(s)(\xi_1, \xi_2) = \int_D D\Big(\Phi(\gamma' \circ s \cdot \xi_1)\Big) \cdot D\Big(\Phi(\gamma' \circ s \cdot \xi_2)\Big) + \int_{\partial D} \frac{\partial}{\partial \nu}\Phi(\gamma \circ s) \cdot \gamma'' \circ s \cdot \xi_1 \cdot \xi_2.$$

But $H^{1/2}(\partial D; I\!R)$ does not imbed into $C^0(\partial D; I\!R)$.

5 Approximation Spaces

For $h > 0$ we define finite-dimensional spaces H_h approximating the space H and consisting of piecewise linear functions. While the norms $||.||_H$ and $||.||_T$ are not comparable, they are comparable on the spaces H_h up to a factor $|\ln h|^{1/2}$ (Proposition 5.3). Although this factor blows up as $h \to 0$, it approaches zero when multiplied by any power of h (c.f. the proof of Lemma 6.2). In Proposition 5.8 we establish approximations in H_h to the positive, null and negative eigenspaces of the bilinear operator $d^2E(s)$ at a stationary point s. This requires the finite dimensionality of the null and negative eigenspaces and the regularity of their members.

For each $h > 0$ let $\mathcal{G}_h$ be any grid on $\partial D \cong [0, 2\pi)$ such that

$$c^{-1}h < |I| < ch \qquad \forall I \text{ an interval of } \mathcal{G}_h,$$

where $|I|$ is the length of I and c is independent of h. If I is an interval, denote by

$$P_1(I)$$

the space of first order polynomials (in the arc length variable) defined over I.

Definition 5.1 Let

$$\begin{aligned} H_h \quad = \quad \Big\{ \xi_h \in C^0(\partial D, I\!\!R) : \xi_h \in P_1(I) \; \forall I \in \mathcal{G}_h, \; (2) \text{ is satisfied} \\ \text{with } \sigma \text{ there replaced by } \xi_h \Big\} \,. \end{aligned}$$

Then

$$H_h \subset T \subset H.$$

Let

$$\mathcal{H}_h = \; \{\mathrm{id}\} \; + H_h \subset \mathcal{T} \subset \mathcal{H}$$

be the corresponding finite dimensional affine space of continuous piecewise affine maps (with respect to arc length) from ∂D to S^1.

Remarks (i) It will usually be convenient to think of H_h as having its "origin" at some $s_h \in \mathcal{H}_h$. That is, H_h is a finite dimensional space of possible variations at s_h. We sometimes give H_h the inner product and norm induced from H, and sometimes the norm induced from T.

(ii) The elements $\xi_h \in H_h$ are linear with respect to arc length along arc segments in $\mathcal{G}_h$. It is sometimes convenient to regard ξ_h as a continuous 2π-periodic map $\xi_h : I\!\!R \to I\!\!R$, where ξ_h is linear on each interval of $I\!\!R$ which corresponds to some $I \in \mathcal{G}_h$. Similarly, we can regard $s_h \in \mathcal{H}_h$ as a continuous map $s_h : I\!\!R \to I\!\!R$, linear on the same intervals as above, and satisfying the condition $s_h(\phi + 2\pi) = s_h(\phi) + 2\pi$ for all $\phi \in I\!\!R$.

(iii) Let

$$\phi_1 < \phi_2 < \ldots < \phi_N$$

be the vertices of $\mathcal{G}_h$. Then for $\xi_h \in H_h$, (2) is equivalent to

$$\begin{aligned} \sum_{i=1}^{N} \frac{\xi_h(\phi_{i+1}) + \xi_h(\phi_i)}{2} \, (\phi_{i+1} - \phi_i) \quad &= \quad 0, \\ \sum_{i=1}^{N} \frac{\xi_h(\phi_{i+1}) - \xi_h(\phi_i)}{\phi_{i+1} - \phi_i} \, (\cos \phi_{i+1} - \cos \phi_i) \quad &= \quad 0, \\ \sum_{i=1}^{N} \frac{\xi_h(\phi_{i+1}) - \xi_h(\phi_i)}{\phi_{i+1} - \phi_i} \, (\sin \phi_{i+1} - \sin \phi_i) \quad &= \quad 0, \end{aligned}$$

where $\phi_{N+1} = \phi_1 + 2\pi$. In particular the normalisation conditions (2) are linear constraints on ξ_h.

The standard interpolation operator may not map T ($\mathcal{T}$) into H_h ($\mathcal{H}_h$) since the normalisation conditions (2) may not be satisfied. However, by first interpolating and then projecting, we obtain a suitable operator.

Proposition 5.2 *There is a bounded linear map*

$$I_h : T \to H_h$$

such that

$$\begin{array}{rcl} ||\xi - I_h\xi||_{H^{1/2}} & \leq & ch^{3/2}||\xi||_{H^2}, \\ ||\xi - I_h\xi||_{C^0} & \leq & ch^2||\xi||_{C^2}. \end{array}$$

Proof: The main point is to preserve the normalisation conditions.

Let

$$\overline{H}_h = \left\{\xi_h \in C^0(\partial D; I\!R) : \xi_h \in P_1(I)\ \forall I \in \mathcal{G}_h\right\}.$$

Thus $\overline{H}_h$ is defined as for H_h, but without the normalisation condition (2).

Let $\overline{I}_h : T \to \overline{H}_h$ be the interpolation operator uniquely defined by

$$\overline{I}_h\xi(\phi_i) = \xi(\phi_i),$$

for all vertices ϕ_i of $\mathcal{G}_h$. One has the following standard estimates for $\overline{I}_h$:

$$\begin{array}{rcl} ||\xi - \overline{I_h}\xi||_{L^2} & \leq & ch^2||\xi||_{H^2}, \\ ||\xi - \overline{I_h}\xi||_{H^1} & \leq & ch||\xi||_{H^2}, \\ ||\xi - \overline{I_h}\xi||_{H^{1/2}} & \leq & ch^{3/2}||\xi||_{H^2}, \\ ||\xi - \overline{I_h}\xi||_{C^0} & \leq & ch^2||\xi||_{C^2}. \end{array} \tag{5}$$

The first two estimates c.f. [C; Theorem 3.2.1, p. 132] are classical. The third inequality then follows easily by interpolation between the spaces L^2 and H^1:

$$||\xi - \overline{I_h}\xi||_{H^{1/2}} \leq ||\xi - \overline{I_h}\xi||_{L^2}^{1/2}||\xi - \overline{I_h}\xi||_{H^1}^{1/2}.$$

Note that this can easily be seen by expressing the non-integer norms in terms of Fourier coefficients. The fourth estimate is standard.

Let $P_h : H \to \overline{H}_h$ be the L^2-projection operator uniquely defined by

$$(P_h\xi, \eta_h)_{L^2} = (\xi, \eta_h)_{L^2} \quad \forall \eta_h \in \overline{H}_h.$$

Because of the optimality of the L^2-projection we have from the first equation in (5)

$$||P_h\xi - \xi||_{L^2} \leq ch^2||\xi||_{H^2}, \tag{6}$$

Let (τ_1, τ_2, τ_3) be the L^2-orthonormal set

$$\tau_1 = \frac{1}{2\pi}, \quad \tau_2 = \frac{1}{\pi}\cos\phi, \quad \tau_3 = \frac{1}{\pi}\sin\phi.$$

It follows from (6) that $(P_h\tau_i)_{i=1}^3$ is "almost" an L^2 orthonormal set, i.e.

$$1 - ch^2 \leq ||P_h\tau_i||_{L^2} \leq 1 + ch^2 \tag{7}$$

and

$$|(P_h\tau_i, P_h\tau_j)_{L^2}| \leq ch^2 \quad \text{if } i \neq j.$$

In particular, $(P_h\tau_i)_{i=1}^3$ spans a 3-dimensional space if h is sufficiently small. Since τ_i and hence $P_h\tau_i$ is L^2-orthogonal to H_h, it follows $(P_h\tau_i)_{i=1}^3$ spans the L^2 orthogonal complement of H_h in $\overline{H}_h$.

Let $(P_h^*\tau_i)_{i=1}^3$ be obtained from $(P_h\tau_i)_{i=1}^3$ by the Gram-Schmidt orthogonalisation process. Then it is easy to check that

$$|||P_h^*\tau_i - P_h\tau_i||| \leq ch^2 \max_{j=1,2,3} |||P_h\tau_i||| \tag{8}$$

for any norm $|||\cdot|||$ on $\overline{H}_h$.

For $\xi \in H$ we now define

$$I_h\xi = \overline{I}_h\xi - \sum_{i=1}^{3} \left(\overline{I}_h\xi, P_h^*\tau_i\right)_{L^2} P_h^*\tau_i.$$

It follows $(I_h\xi, P_h^*\tau_i)_{L^2} = 0$ for $i = 1, 2, 3$ and so

$$I_h : H \to H_h.$$

For $\xi \in H$ one has $(\xi, P_h\tau_i)_{L^2} = 0$ and so

$$\overline{I}_h\xi - I_h\xi = \sum_{i=1}^{3}\Big(\left(\overline{I}_h\xi, P_h^*\tau_i - P_h\tau_i\right)_{L^2} P_h^*\tau_i + \left(\overline{I}_h\xi - \xi, P_h\tau_i\right)_{L^2} P_h^*\tau_i\Big).$$

The required estimates now follow from (5), (6), (7) and (8). ∎

The following inverse type estimate will be important.

Proposition 5.3 *If $\xi_h \in H_h$ then*

$$||\xi_h||_{H^{1/2}} \leq ||\xi_h||_T \leq c|\ln h|^{1/2}\, ||\xi_h||_{H^{1/2}},$$

for $h \leq 1/2$, say. If $L : H_h \to H_h$ is a linear map, then

$$||L||_{H^{1/2}} \leq c|\ln h|^{1/2}\, ||L||_T, \quad ||L||_T \leq c|\ln h|^{1/2}\, ||L||_{H^{1/2}},$$

where $||L||_T$ and $||L||_{H^{1/2}}$ denote the corresponding operator norms.

Proof: The first inequality is trivial and the last two will follow from the second.

We first prove that for a function $u \in H^{1/2}(\partial D; I\!R)$ we have the estimate

$$\Big|\frac{1}{2\rho}\int_{\phi_0-\rho}^{\phi_0+\rho} u\Big| \leq c|\ln \rho|^{1/2} \|u\|_{H^{1/2}} \tag{9}$$

for any small $\rho > 0$ and any ϕ_0. We may assume that $\phi_0 = 0$, and by replacing u by its symmetrisation about 0 that u is an even function. Since the right hand side of the inequality contains the L^2-norm we may assume that $\int_{-\pi}^{\pi} u = 0$.

We expand u in a Fourier series

$$u(\phi) = \sum_{k=1}^{\infty} a_k \cos k\phi$$

and integrate

$$\begin{aligned}\frac{1}{2\rho}\int_{-\rho}^{\rho} u &= \sum_{k=1}^{\infty} a_k \frac{\sin k\rho}{k\rho} \\ &\leq \Big(\sum_{k=1}^{\infty} a_k^2 k\Big)^{1/2}\Big(\sum_{k=1}^{\infty}\frac{\sin^2 k\rho}{k^3\rho^2}\Big)^{1/2} \\ &= c|u|_{H^{1/2}}\rho^{-1}(\Phi(\rho))^{1/2},\end{aligned}$$

where

$$\Phi(\rho) = \sum_{k=1}^{\infty}\frac{\sin^2 k\rho}{k^3}.$$

Since $\Phi(0) = 0$, $\Phi'(0) = 0$ and

$$\Phi''(\rho) = 2\sum_{k=1}^{\infty}\frac{\cos 2k\rho}{k} = -2\ln(2\sin\rho),$$

we conclude that

$$\Phi(\rho) \leq c\rho^2|\ln\rho|.$$

This proves (9).

If $\xi_h \in H_h$ then $||\xi_h||_{C^0} = |\xi_h(\phi_0)|$ for some grid point ϕ_0. Suppose $[\phi_0 - \alpha, \phi_0]$ and $[\phi_0, \phi_0 + \beta]$ are grid elements. Then

$$|\xi_h(\phi_0)| \leq \gamma^{-1}\Big|\int_{\phi_0-\gamma}^{\phi_0+\gamma}\xi_h\Big|,$$

where $\gamma = \min\{\alpha/2, \beta/2\}$. The second inequality in the Proposition now follows from (9). ∎

Definition 5.4 The *energy functional* E_h on $\mathcal{H}_h$ is defined by

$$E_h = E|_{\mathcal{H}_h},$$

i.e. E_h is the restriction of E to $\mathcal{H}_h$.

For later purposes we require the following definitions, which are justified by the Riesz Representation Theorem.

Definition 5.5 The operator

$$\nabla E_h : \mathcal{H}_h \to H_h$$

is uniquely defined by

$$\Big(\nabla E_h(s_h), \eta_h\Big)_{H^{1/2}} = dE_h(s_h)(\eta_h) \quad \forall \eta_h \in H_h.$$

Suppose $s_h \in \mathcal{H}_h$. Then

$$\nabla^2 E_h(s_h) : H_h \to H_h$$

is uniquely defined by

$$\Big(\nabla^2 E_h(s_h)(\xi_h), \eta_h\Big)_{H^{1/2}} = d^2 E_h(s_h)(\xi_h, \eta_h) \quad \forall \eta_h \in H_h.$$

The proof of the following is immediate.

Proposition 5.6

$$d\nabla E_h(s_h) = \nabla^2 E_h(s_h) \quad \forall s_h \in \mathcal{H}_h.$$

Note that $d^2 E_h(s_h)$ is the restriction of $d^2 E(s_h)$ to $H_h \times H_h$, but $\nabla^2 E_h(s_h)$ is not necessarily the restriction of $\nabla^2 E(s_h)$ to H_h.

Definition 5.7 A function $s_h \in \mathcal{H}_h$ is called a *semi-discrete stationary point* for E if

$$dE_h(s_h)(\xi_h) = 0 \quad \forall \xi_h \in H_h.$$

The associated function $u_h = \Phi(\gamma \circ s_h)$ is called a *semi-discrete minimal surface.*

Note that we do not require that s_h is monotone. Note also that u_h is analytic in the interior of D, but of course only Hölder-continuous on $\overline{D}$.

We will be interested in the existence, uniqueness and convergence (as $h \to 0$) of semi-discrete minimal surfaces near a smooth minimal surface u_0.

Assumptions: For the remainder of this Section, assume

$$r \geq 5 \textit{ and } s \textit{ is a stationary point for } E$$

and let

$$H = H^- \oplus H^0 \oplus H^+,$$

as in Definition 4.8, where

$$\dim H^- = k, \quad \dim H^0 = d.$$

Note that Proposition 4.9 applies.

We now define a discrete approximation to this decomposition of H.

Definition 5.8 Let

$$\begin{aligned} H_h^{(-)} &= I_h(H^-), \\ H_h^{(0)} &= \text{the orthogonal complement of } H_h^{(-)} \text{ in } I_h(H^- \oplus H^0), \\ H_h^{(+)} &= \text{the orthogonal complement of } H_h^{(-)} \oplus H_h^{(0)} \text{ in } I_h(H) = H_h. \end{aligned}$$

Thus

$$H_h = H_h^{(-)} \oplus H_h^{(0)} \oplus H_h^{(+)}.$$

For $\xi_h \in H_h$ let

$$\xi_h = \xi_h^{(-)} + \xi_h^{(0)} + \xi_h^{(+)},$$

where

$$\xi_h^{(-)} \in H_h^{(-)}, \quad \xi_h^{(0)} \in H_h^{(0)}, \quad \xi_h^{(+)} \in H_h^{(+)}.$$

We think of $H_h^{(-)}$ (respectively $H_h^{(0)}$, $H_h^{(+)}$) as being discrete approximations to H^- (respectively H^0, H^+) by members of H_h. More precisely we have the following result. Note that the estimates rely on the finite dimensionality of $H^- \oplus H^0$.

Proposition 5.9 *For all sufficiently small h,*

$$\dim H_h^{(-)} = \dim H^- = k, \quad \dim H_h^{(0)} = \dim H^0 = d.$$

Moreover, if $\xi_h \in H_h$ then

$$\begin{aligned} ||\xi_h^- - \xi_h^{(-)}||_{H^{1/2}} &\leq ckh^{3/2}||\xi_h||_{H^{1/2}}, \\ ||\xi_h^0 - \xi_h^{(0)}||_{H^{1/2}} &\leq cdh^{3/2}||\xi_h||_{H^{1/2}}, \\ ||\xi_h^+ - \xi_h^{(+)}||_{H^{1/2}} &\leq c\max\{k, d\}h^{3/2}||\xi_h||_{H^{1/2}}, \end{aligned}$$

where $c = c(||\gamma||_{C^5})$.

Proof: In the following, constants c will depend on $||\gamma||_{C^5}$.

Let $\tau_1, \ldots, \tau_{k+d}$ be an orthonormal basis for $H^- \oplus H^0$. Since $\tau_1, \ldots, \tau_{k+d} \in H^2(\partial D)$ from Proposition 4.9, it follows from Proposition 5.2 that

$$\|\tau_i - I_h\tau_i\|_{H^{1/2}} \leq ch^{3/2}.$$

Hence

$$1 - ch \leq ||I_h\tau_i||_{H^{1/2}} \leq 1 + ch^{3/2}.$$

If $i \neq j$ then

$$\begin{aligned} \left|(I_h\tau_i, I_h\tau_j)_{H^{1/2}}\right| &= \left|\Big((I_h\tau_i - \tau_i) + \tau_i\ ,\ (I_h\tau_j - \tau_j) + \tau_j\Big)_{H^{1/2}}\right| \\ &\leq ch^{3/2}. \end{aligned}$$

Thus the $I_h\tau_i$ form an orthonormal basis for $H_h^{(-)} \oplus H_h^{(0)}$ to within an error at most $ch^{3/2}$.

Applying the Gram-Schmidt process to $I_h\tau_1, \dots, I_h\tau_{k+d}$ we obtain an orthonormal basis $I_h^*\tau_1, \dots, I_h^*\tau_k$ for $H_h^{(-)}$ and an orthonormal basis $I_h^*\tau_{k+1}, \dots,\ I_h^*\tau_{k+d}$ for $H_h^{(0)}$. Moreover, it is straightforward to check that

$$||I_h^*\tau_i - I_h\tau_i||_{H^{1/2}} \le ch^{3/2}, \quad |\,(I_h^*\tau_i, I_h\tau_j)_{H^{1/2}}\,| \le ch^{3/2} \quad \text{if } i \ne j.$$

It follows from Proposition 5.2 that

$$||I_h^*\tau_i - \tau_i||_{H^{1/2}} \le ch^{3/2}, \quad |\,(I_h^*\tau_i, \tau_j)_{H^{1/2}}\,| \le ch^{3/2} \quad \text{if } i \ne j.$$

Hence for $\xi_h \in H_h$,

$$\begin{aligned} ||\xi_h^- - \xi_h^{(-)}||_{H^{1/2}} &= \Big\|\sum_{i=1}^{k} \ (\xi_h, \tau_i)_{H^{1/2}}\, \tau_i - (\xi_h, I_h^*\tau_i)_{H^{1/2}}\, I_h^*\tau_i\Big\|_{H^{1/2}} \\ &= \Big\|\sum_{i=1}^{k} \ (\xi_h, \tau_i)_{H^{1/2}}\, (\tau_i - I_h^*\tau_i) + (\xi_h, \tau_i - I_h^*\tau_i)_{H^{1/2}}\, I_h^*\tau_i\Big\|_{H^{1/2}} \\ &\le \ ckh^{3/2}||\xi_h||_{H^{1/2}}. \end{aligned}$$

Similarly, $||\xi_h^0 - \xi_h^{(0)}||_{H^{1/2}} \le cdh^{3/2}||\xi_h||_{H^{1/2}}$.

Finally,

$$\begin{aligned} ||\xi_h^+ - \xi_h^{(+)}||_{H^{1/2}} &= ||(\xi_h - \xi_h^+) - (\xi_h - \xi_h^{(+)})||_{H^{1/2}} \\ &= ||(\xi_h^- + \xi_h^0) - (\xi_h^{(-)} + \xi_h^{(0)}||_{H^{1/2}} \\ &= ||(\xi_h^- - \xi_h^{(-)}) + (\xi_h^0 - \xi_h^{(0)}||_{H^{1/2}} \\ &\le c\max\{k, d\}h^{3/2}||\xi_h||_{H^{1/2}}, \end{aligned}$$

from the previous results. ∎

6 Approximation Results

The main result is Theorem 6.3. The proof is by means of a quantitative version of the Inverse Function Theorem. The necessary estimates are obtained in Theorem 6.2 using the results from Sections 4 and 5. The $H^{1/2}(\partial D)$ and $H^1(D)$ estimates are optimal with respect to the exponent of convergence. We also remark that in order to establish the estimates involving the discrete minimal surfaces u_h we need to first establish the corresponding boundary estimates for s_h. In Theorem 6.4 we prove a result which in numerical computations provides strong evidence that a discrete stationary solution is indeed close to a true (smooth) stationary solution. Finally, we finish this section with remarks about other related results.

We will apply the following quantitative form of the Inverse Function Theorem to the approximation space H_h, with $k = ch^s$ for suitable s.

Proposition 6.1 *Suppose V, W are finite dimensional normed linear spaces of equal dimension. Suppose $\zeta_0 \in U \subset V$ where U is open and suppose $G \in C^1(U; V)$.*

Suppose the following are true for some $\beta > 0$, $\lambda > 0$, and some $\alpha \in (0, \beta)$:

(a) *if* $\zeta \in B_\beta(\zeta_0)$ *then* $dG(\zeta){:}V \to W$ *is a homeomorphism and* $\|[dG(\zeta)]^{-1}\| \le \lambda^{-1}$,

(b) *if* $\zeta, \eta \in B_\beta(\zeta_0)$ *then* $||dG(\zeta) - dG(\eta)|| \le \frac{1}{2}\lambda$,

(c) $||G(\zeta_0)|| \le \frac{1}{2}\lambda k$.

Then there exists a unique $\zeta \in B_\beta(\zeta_0)$ *such that* $G(\zeta) = 0$. *Moreover,* $||\zeta - \zeta_0|| \le k$.

Proof: This follows directly from the proof of the Inverse Function Theorem in [Be; pp 113–114]. ∎

The following Lemma now verifies the hypotheses of the previous Proposition for the operator $\nabla E_h{:}\mathcal{H}_h \to H_h$ (see Definition 5.5). Note that the previous Proposition is still true for an *affine* space V. Recall also the definition of the non-degeneracy constant following Corollary 4.11.

Lemma 6.2 *Assume* $r \ge 5$. *Let* s_0 *be a non-degenerate stationary point for* E *with non-degeneracy constant* λ. *Then there exist constants* h_0^*, ϵ_0^* *and* M^* *(depending on* λ, $dim H^-$ *and* $||\gamma||_{C^5}$*) with the following properties:*

If $h \le h_0^*$, $\zeta_h, \eta_h \in H_h$ *and* $|\zeta_h|_{H^{1/2}}, |\eta_h|_{H^{1/2}} \le \epsilon_0^* |\ln h|^{-3/2}$, *then*

(i) $d\nabla E_h(I_h s_0 + \zeta_h){:}H_h \to H_h$ *is invertible and*

$$\left|\left|[d\nabla E_h(I_h s_0 + \zeta_h)]^{-1}\right|\right|_{H^{1/2}} \le 2\lambda^{-1};$$

(ii)

$$||d\nabla E_h(I_h s_0 + \zeta_h) - d\nabla E_h(I_h s_0 + \eta_h)||_{H^{1/2}} \le \frac{1}{4}\lambda;$$

(iii)

$$|\nabla E_h(I_h s_0)|_{H^{1/2}} \le \frac{M^*}{4}\lambda h^{3/2}.$$

Proof: In the following, constants c are allowed to depend on $||s_0||_{C^4}$ and hence on $||\gamma||_{C^5}$, see Proposition 4.6.

Suppose $h \le h_0^*$, $\zeta_h, \eta_h \in H_h$ and $|\zeta_h|_{H^{1/2}}, |\eta_h|_{H^{1/2}} \le \epsilon_0^* |\ln h|^{-3/2}$. We will place various restrictions on $h_0^* = h_0^*(n, ||\gamma||_{C^5})$ and $\epsilon_0^* = \epsilon_0^*(n, ||\gamma||_{C^5})$ as we proceed.

We will apply Proposition 6.1 to show that $d\nabla E_h(I_h s_0 + \eta_h)$ is invertible. Recall from Proposition 5.6 that $d\nabla E_h(I_h s_0 + \eta_h)$ is the linear operator corresponding to the bilinear form $d^2 E_h(I_h s_0 + \eta_h)$.

Suppose $\xi_h \in H_h$ satisfies $|\xi_h|_{H^{1/2}} = 1$. Then

$$\begin{aligned}
&d^2 E_h(I_h s_0 + \zeta_h)(\xi_h, -\xi_h^{(-)} + \xi_h^{(+)}) \\
&\quad = d^2 E(s_0)(\xi_h, -\xi_h^{(-)} + \xi_h^{(+)}) + \\
&\qquad\quad \left(d^2 E(I_h s_0 + \zeta_h) - d^2 E(s_0)\right)(\xi_h, -\xi_h^{(-)} + \xi_h^{(+)}) \\
&\quad = A + B.
\end{aligned}$$

But

$$\begin{aligned}
A &= d^2E(s_0)\Big(\xi_h, -\xi_h^- + \xi_h^+ + (\xi_h^- - \xi_h^{(-)}) + (\xi_h^{(+)} - \xi_h^+)\Big) \\
&= d^2E(s_0)(\xi_h, -\xi_h^- + \xi_h^+) + d^2E(s_0)(\xi_h, \xi_h^- - \xi_h^{(-)}) \\
&\qquad\qquad + d^2E(s_0)(\xi_h, \xi_h^{(+)} - \xi_h^+) \\
&\geq \lambda - ch^{3/2} \quad \text{from Propositions 4.4, 5.9} \\
&\geq \frac{3}{4}\lambda.
\end{aligned}$$

provided h_0^* is sufficiently small. Also

$$\begin{aligned}
|B| &\leq \Big\|d^2E(I_h s_0 + \zeta_h) - d^2E(s_0)\Big\|_T \, ||\xi_h||_T \, || - \xi_h^{(-)} + \xi_h^{(+)}||_T \\
&\leq c\,\|I_h s_0 + \zeta_h - s_0\|_T \; |\ln h| \quad \text{from Propositions 5.3, 4.3} \\
&\leq c\,(\|I_h s_0 - s_0\|_T + \|\zeta_h\|_T)\,|\ln h| \\
&\leq c|\ln h|\Big(h^{3/2} + |\ln h|^{1/2}|\zeta_h|_{H^{1/2}}\Big) \quad \text{from Propositions 5.2, 5.3} \\
&\leq \frac{1}{4}\lambda,
\end{aligned}$$

provided h_0^* and ϵ_0^* are sufficiently small. Thus

$$d^2E_h(I_h s_0 + \zeta_h)(\xi_h, -\xi_h^{(-)} + \xi_h^{(+)}) \geq \frac{1}{2}\lambda$$

provided h_0^* and ϵ_0^* are sufficiently small.

Result (i) now follows from algebra and the arbitrariness of ξ_h, since $|-\xi_h^{(-)} + \xi_h^{(+)}|_{H^{1/2}} = 1$.

For (ii) we note, using the fact $d\nabla E(s_h)$ is self-adjoint, that

$$\begin{aligned}
&||d\nabla E_h(I_h s_0 + \zeta_h) - d\nabla E_h(I_h s_0 + \eta_h)||_{H^{1/2}} \\
&\quad = \sup_{\substack{\xi_h \in H_h \\ |\xi_h|_{H^{1/2}}=1}} \Big(\Big(d\nabla E(I_h s_0 + \zeta_h) - d\nabla E(I_h s_0 + \eta_h)\Big)(\xi_h), \xi_h\Big)_{H^{1/2}}.
\end{aligned}$$

But if $\xi_h \in H_h$ and $|\xi_h|_{H^{1/2}} = 1$ then from Proposition 5.6

$$\begin{aligned}
&\Big(\Big(d\nabla E(I_h s_0 + \zeta_h) - d\nabla E(I_h s_0 + \eta_h)\Big)(\xi_h), \xi_h\Big)_{H^{1/2}} \\
&\quad = \Big(d^2E(I_h s_0 + \zeta_h) - d^2E(s_0)\Big)(\xi_h, \xi_h) - \\
&\qquad\quad \Big(d^2E(I_h s_0 + \eta_h) - d^2E(s_0)\Big)(\xi_h, \xi_h) \\
&\quad \leq \frac{1}{4}\lambda,
\end{aligned}$$

for h_0^*, ϵ_0^* sufficiently small, by the same argument used to estimate B. This establishes (ii).

Finally,

$$|\nabla E_h(I_h s_0)|_{H^{1/2}} = \sup_{\substack{\xi_h \in H_h \\ |\xi_h|_{H^{1/2}}=1}} dE_h(I_h s_0)(\xi_h).$$

But if $\xi_h \in H_h$ and $|\xi_h|_{H^{1/2}} = 1$ then

$$
\begin{aligned}
dE_h(I_h s_0)(\xi_h) &= dE(I_h s_0)(\xi_h) \\
&\le dE(s_0)(\xi_h) + d^2E(s_0)(I_h s_0 - s_0, \xi_h) + c\,\|I_h s_0 - s_0\|_T^2\,||\xi_h||_T \\
&\qquad \text{by Proposition 4.3} \\
&\le c|I_h s_0 - s_0|_{H^{1/2}}|\xi_h|_{H^{1/2}} + ch^3|\ln h|^{1/2}\,|\xi_h|_{H^{1/2}} \\
&\qquad \text{as } s_0 \text{ is stationary and from Propositions 4.5, 4.4, 5.2 and 5.3} \\
&\le ch^{3/2} + ch^3|\ln h|^{1/2} \qquad \text{from Proposition 5.2} \\
&\le ch^{3/2},
\end{aligned}
$$

for sufficiently small h_0^*. Now, with this last c, choose $M^* = 4c/\lambda$. This establishes (iii). ∎

Theorem 6.3 (Energy Estimate) *Assume $r \ge 5$. Let $s_0 \in \mathcal{M}$ be a non-degenerate stationary point for E with with non–degeneracy constant λ and negative subspace H^- and let $u_0 = \Phi(\gamma \circ s_0)$ be the associated minimal surface.*

Then there exist constants h_0, ϵ_0 and c, depending on λ, $dim H^-$ and $||\gamma||_{C^5}$, such that if $0 < h \le h_0$ then there is a unique semi-discrete stationary point $s_h \in \mathcal{H}_h$ such that

$$|s_0 - s_h|_{H^{1/2}(\partial D)} \le \epsilon_0|\ln h|^{-3/2}.$$

Moreover,

$$|s_h - s_0|_{H^{1/2}(\partial D)} \le ch^{3/2} \quad \textit{and} \quad ||s_h - s_0||_{C^0(\partial D)} \le ch^{3/2}|\ln h|^{1/2}.$$

Finally, if $u_h = \Phi(\gamma \circ s_h)$ is the corresponding semi-discrete minimal surface, then

$$||u_h - u_0||_{H^1(D)} \le ch^{3/2} \quad \textit{and} \quad ||u_h - u_0||_{C^0(D)} \le ch^{3/2}|\ln h|^{1/2}.$$

Proof: In the following, constants may depend on $||s_0||_{C^4}$ and hence on $||\gamma||_{C^5}$. (a) Choose h_0^*, ϵ_0^*, M^* as in the previous Lemma.

Suppose $h \le h_0^*$.

From the previous Lemma and Proposition 6.1 applied to $G = \nabla E_h : \mathcal{H}_h \to H_h$, there exists a unique semi-discrete stationary point $s_h \in \mathcal{H}_h$ such that

$$|I_h s_0 - s_h|_{H^{1/2}} \le \epsilon_0^*|\ln h|^{-3/2}. \tag{10}$$

Moreover, s_h satisfies

$$|I_h s_0 - s_h|_{H^{1/2}} \le M^* h^{3/2}. \tag{11}$$

By Proposition 5.2 choose c_1 so that $h \le h_0^*$ implies

$$|s_0 - I_h s_0|_{H^{1/2}} \le c_1 h^{3/2}.$$

Choose $\epsilon_0 = \epsilon_0^*/2$ and $M = M^* + c_3$. For later purposes choose $h_0 \le h_0^*$ so that $h \le h_0$ implies

$$c_1 h^{3/2} \le \epsilon_0|\ln h|^{-3/2}.$$

It follows that

$$\begin{aligned}|s_0 - s_h|_{H^{1/2}} &\le |s_0 - I_h s_0|_{H^{1/2}} + |I_h s_0 - s_h|_{H^{1/2}} \\ &\le Mh^{3/2},\end{aligned}$$

thus establishing one of the required estimates.

(b) It also follows that if $t_h \in \mathcal{H}_h$ and

$$|s_0 - t_h|_{H^{1/2}} \le \epsilon_0 |\ln h|^{-3/2},$$

then

$$\begin{aligned}|I_h s_0 - t_h|_{H^{1/2}} &\le |I_h s_0 - s_0|_{H^{1/2}} + |s_0 - t_h|_{H^{1/2}} \\ &\le c_1 h^{3/2} + \epsilon_0 |\ln h|^{-3/2} \\ &\le 2\epsilon_0 |\ln h|^{-3/2} = \epsilon_0^* |\ln h|^{-3/2}.\end{aligned}$$

It then follows from (10) that s_h is the *unique* semi-discrete stationary point satisfying $|s_0 - s_h|_{H^{1/2}} \le \epsilon_0 |\ln h|^{-3/2}$.

(c) It follows from (11) and Proposition 5.3 that

$$||s_h - I_h s_0||_T \le ch^{3/2} |\ln h|^{1/2}.$$

Combining this with Proposition 5.2 gives

$$||s_h - s_0||_{C^0} \le ch^{3/2} |\ln h|^{1/2}.$$

(d) The estimate for $||u_h - u_0||_{H^1(D)}$ follows from

$$|\gamma(s_h) - \gamma(s_0)|_{H^{1/2}} \le c\|\gamma\|_{C^1} |s_h - s_0|_{H^{1/2}} + c\|\gamma\|_{C^2} \|s_0\|_{C^1} \|s_h - s_0\|_{L^2}. \qquad (12)$$

and from the regularity of s_0, Proposition 4.6.

To see (12) compute

$$\begin{aligned}&\Big|\Big(\gamma(s_0(\phi)) - \gamma(s_h(\phi))\Big) - \Big(\gamma(s_0(\psi) - \gamma(s_h(\psi))\Big)\Big| \\ &\quad\le \left|\int_0^{s_h(\phi)-s_0(\phi)} \gamma'(t + s_0(\phi))\,dt - \int_0^{s_h(\psi)-s_0(\psi)} \gamma'(t + s_0(\psi))\,dt\right| \\ &\quad\le \left|\int_{s_h(\psi)-s_0(\psi)}^{s_h(\phi)-s_0(\phi)} \gamma'(t + s_0(\phi))\,dt\right| + \left|\int_0^{s_h(\psi)-s_0(\psi)} \gamma'(t + s_0(\phi)) - \gamma'(t + s_0(\psi))\,dt\right| \\ &\quad\le \Big|\Big(s_h(\phi) - s_0(\phi)\Big) - \Big(s_h(\psi) - s_0(\psi)\Big)\Big| \,\|\gamma\|_{C^1} + \|\gamma\|_{C^2} |s_0(\phi) - s_0(\psi)|\, |s_h(\psi) - s_0(\psi)|.\end{aligned}$$

Then (12) follows after dividing through by $|\phi - \psi|$, squaring and integrating with respect to ϕ and to ψ.

The final estimate for $||u_h - u_0||_{C^0(D)}$ follows from the maximum principle and the corresponding estimate for $||s_h - s_0||_{C^0}$. ∎

We will finally prove a result to the effect that convergent sequences of semi-discrete stationary points have stationary limits. Moreover, if the convergence rate is better than logarithmic to the power 3/2, then the convergence rates of Theorem 6.3 apply.

Theorem 6.4 *Suppose s_h is a sequence of semi-discrete stationary points and $||s_h - s_0||_T \to 0$ as $h \to 0$. Then s_0 is a stationary point for the Plateau Problem.*

If s_0 is monotone and non-degenerate and moreover $|\ln h|^{3/2}||s_h - s_0||_T \to 0$, then the convergence rates of Theorem 6.3 will apply.

Proof: Suppose $\xi \in T$ and $\epsilon > 0$.

Choose (e.g. by mollification) $\tilde{\xi} \in C^\infty$ such that

$$||\xi - \tilde{\xi}||_T < \epsilon/2.$$

From Proposition 5.2

$$||\tilde{\xi} - I_h\tilde{\xi}||_T \leq ch^{3/2}||\tilde{\xi}||_T \leq \epsilon/2$$

and hence

$$||\xi - I_h\tilde{\xi}||_T < \epsilon,$$

provided $h \leq h_0 = h_0(\epsilon, \xi)$.

Then from Proposition 4.3

$$\begin{aligned} |dE(s_0)(\xi)| &\leq |dE(s_0)(\xi - I_h\tilde{\xi})| + |\Big(dE(s_0) - dE(s_h)\Big)(I_h\tilde{\xi})| \\ &\leq c||\xi - I_h\tilde{\xi}||_T + c||dE(s_0) - dE(s_h)||_T||I_h\tilde{\xi}||_T \\ &= A + B. \end{aligned}$$

But $A \leq c\epsilon$ and $B \to 0$ as $h \to 0$ from Proposition 4.3. Hence s_0 is stationary for E. The last claim follows from Theorem 6.3. ∎

In [DH] we use a modification of the Aubin-Nitsche technique to show under the hypotheses of Theorem 6.3 that

$$|s_h - s_0|_{H^{-1/2}(\partial D)} \leq ch^{5/2}, \quad ||u_h - u_0||_{L^2(D)} \leq ch^{5/2}.$$

In a subsequent paper we will pursue a different approach. For each grid $\mathcal{G}_h$ on ∂D, let $\mathcal{G}_h^*$ be an extension of $\mathcal{G}_h$ to a triangulation of D. Let $D_h = \bigcup\{\kappa : \kappa \in \mathcal{G}_h^*\}$. For $s_h \in \mathcal{H}_h$ let u_h be the corresponding *discrete harmonic extension* of the interpolant of $\gamma \circ s_h$. Thus $u_h \in C^0(D_h; I\!R^n)$, u_h agrees with $\gamma \circ s_h$ at the boundary nodes of $\mathcal{G}_h$, and u_h is a polynomial of degree one on each triangle $\kappa \in \mathcal{G}_h^*$. We can compute u_h exactly up to machine error by solving a certain linear system. Define

$$E_h^*(s_h) = \mathcal{D}(u_h).$$

A single "run" of Newton's method will find a stationary point of E_h^* in $\mathcal{H}_h$ to any previously prescribed degree of accuracy.

The difficulty in analysing E_h^* is that if s_h are the corresponding stationary points then it is not clear that $dE_h^*(s_h)$ approaches zero as $h \to 0$. None-the-less, with some further techniques we obtain optimal estimates analogous to those of the previous theorem. In this case, however, if u_h is the appropriate discrete minimal surface then one has

$$||u_h - u_0||_{H^1(D)} \leq ch, \quad ||u_h - u_0||_{L^2(D)} \leq ch^2.$$

7 Implementation and Numerical Results

For the description of the implementation we shall need the following notations. We assume that $\gamma\colon [0, 2\pi) \to \Gamma$ is a fixed, 2π-periodic, smooth parametrisation of the boundary curve Γ. Let ϕ_j ($j = 1, \ldots, n$) be the grid points on $[0, 2\pi)$, $I_j = [\phi_j, \phi_{j+1}]$, $h_j = \phi_{j+1} - \phi_j$ ($j = 1, \ldots, n$) and $h = \max_{j=1,\ldots,n} h_j$ be the grid size. For piecewise linear $s(\phi)$, $\phi \in [0, 2\pi]$,

$$s(\phi) = s_j + \frac{\phi - \phi_j}{h_j}(s_{j+1} - s_j), \qquad \phi \in I_j$$

where $s_j = s(\phi_j)$ for $j = 1, \ldots, n$.

Since we cannot integrate exactly we have to choose an approximation to the Douglas Functional

$$E(s) = \frac{1}{16\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{|\gamma(s(\phi)) - \gamma(s(\phi'))|^2}{\sin^2(\frac{\phi - \phi'}{2})} d\phi \, d\phi'.$$

For this we approximate the integrand by

$$\gamma(s(\phi))\Big|_{I_j} = \gamma\Big(s_j + \frac{\phi - \phi_j}{h_j}(s_{j+1} - s_j)\Big) \cong \gamma(s_j) + \frac{\phi - \phi_j}{h_j}\Big(\gamma(s_{j+1}) - \gamma(s_j)\Big)$$

for $j = 1, \ldots, n$. We replace the functional E by the approximation

$$E(s_1, \ldots, s_n) = \frac{1}{16\pi} \sum_{j,k=1}^{n} E_{jk},$$

where E_{jk} equals

$$\int_{I_j} \int_{I_k} \frac{\Big|\Big(\gamma(s_j) + \frac{\phi - \phi_j}{h_j}(\gamma(s_{j+1}) - \gamma(s_j))\Big) - \Big(\gamma(s_k) + \frac{\phi' - \phi_k}{h_k}(\gamma(s_{k+1}) - \gamma(s_k))\Big)\Big|^2}{\sin^2\left(\frac{\phi - \phi'}{2}\right)} d\phi \, d\phi'.$$

In order to reduce the computational costs we use integration formulae for the non-singular parts E_{jk}, $|j - k| \geq 2 \pmod n$, and an approximation for the singular kernel if $|j - k| \leq 1$ $\pmod n$. The proof of the following Proposition is done by straightforward calculations using Taylor expansions.

Proposition 7.1 *For $j, k = 1, \ldots, n \pmod n$*

$$E_{jj} = 4\Big|\gamma(s_{j+1}) - \gamma(s_j)\Big|^2 \Big(1 + O(h^2)\Big),$$

$$
\begin{aligned}
E_{j,j-1} \;=\; & \Big(4\Big|\gamma(s_{j+1})-\gamma(s_j)\Big|^2\frac{h_{j-1}}{h_j}\Big(1-\frac{h_{j-1}}{h_j}\ln(1+\frac{h_j}{h_{j-1}})\Big) \\
& +4\Big|\gamma(s_j)-\gamma(s_{j-1})\Big|^2\frac{h_j}{h_{j-1}}\Big(1-\frac{h_j}{h_{j-1}}\ln(1+\frac{h_{j-1}}{h_j})\Big) \\
& +4\Big(\gamma(s_{j+1})-\gamma(s_j)\Big)\cdot\Big(\gamma(s_j)-\gamma(s_{j-1})\Big)\Big(\frac{h_{j-1}^2+h_j^2}{h_{j-1}h_j}\ln(1+\frac{h_j}{h_{j-1}})-1\Big)\Big)\cdot\Big(1+O(h^2)\Big),
\end{aligned}
$$

and if $|j-k|\geq 2 \pmod{n}$ *then*

$$
\begin{aligned}
E_{jk} \;=\; & \frac{1}{4}h_jh_k\Big(|\gamma(s_j)-\gamma(s_k)|^2(\sigma_{jk}+\sigma_{j+1,k}+\sigma_{j,k+1}+\sigma_{j+1,k+1}) \\
& +|\gamma(s_{j+1})-\gamma(s_j)|^2(\sigma_{j+1,k}+\sigma_{j+1,k+1})+|\gamma(s_{k+1})-\gamma(s_k)|^2(\sigma_{j,k+1}+\sigma_{j+1,k+1}) \\
& -2(\gamma(s_{j+1})-\gamma(s_j))\cdot(\gamma(s_{k+1})-\gamma(s_k))\sigma_{j+1,k+1} \\
& +2(\gamma(s_j)-\gamma(s_k))\cdot(\gamma(s_{j+1})-\gamma(s_j))(\sigma_{j+1,k}+\sigma_{j+1,k+1}) \\
& -2(\gamma(s_j)-\gamma(s_k))\cdot(\gamma(s_{k+1})-\gamma(s_k))(\sigma_{j,k+1}+\sigma_{j+1,k+1})\Big)\cdot\Big(1+O(h^2)\Big)
\end{aligned}
$$

where

$$
\sigma_{jk}=\frac{1}{\sin^2\left(\frac{\phi_j-\phi_k}{2}\right)}.
$$

The following Newton algorithm was used for the computation of the numerical examples. We use the previous expressions for E_{jk} without the error terms and use the corresponding expression for $E(s_1,\ldots,s_n)$. We also need to compute the first and second derivatives of E with respect to $s_1,\ldots,s_n$.

Algorithm 7.2 *Given a grid ϕ_j, $j=1,\ldots,n$, initial values $s=(s_1,\ldots,s_n)$ and parametrisation γ:*

1. *Compute the derivative of the approximate energy $E'(s)$.*
2. *If $|E'(s)|/|s|\leq\epsilon$ then stop.*
3. *Compute the second derivative of the approximate energy $E''(s)$.*
4. *Solve the linear system $E''(s)d=-E'(s)$, update the solution $s:=s+d$ and go to step 1*

Here $|s|$ is the l^2-norm of s and ϵ is a given tolerance.

The most expensive part of the algorithm is step 3 where the n^2 elements of the full matrix of the second derivatives of E are computed.

The following examples were computed on a uniform grid. It should be mentioned that in order to avoid superconvergence effects it was necessary to disturb the parametrisations of the boundary curve by a suitable diffeomorphism in such a way that the well known exact solution was not the identity. The identity map is contained in the discrete space we used.

The possibly unstable Enneper surface is given by the harmonic extension of

$$\begin{aligned}\gamma_1(\phi) &= r_0\cos\phi - r_0^3/3\cos 3\phi,\\ \gamma_2(\phi) &= r_0\sin\phi + r_0^3/3\sin 3\phi,\\ \gamma_3(\phi) &= r_0^2\cos 2\phi,\end{aligned}$$

for $\phi \in [0, 2\pi]$. It is well known that for $0 < r_0 < 1$ there is exactly one solution of Plateau‘s problem for $\Gamma = \gamma([0, 2\pi))$ and for $1 < r_0 < \sqrt{3}$ there are two minima and one unstable minimal surface bounded by Γ.

We computed the discrete analogue for $r_0 = 0.5$, $r_0 = 1.0$ and $r_0 = 1.5$ using the fixed parametrisation

$$\gamma^* = \gamma \circ \tau,\ \ \tau(s) = s + 0.1\cos 2s.$$

The error

$$e_h = \|s_0 - s_h\|$$

between the piecewise linear discrete solution s_h and the continuous solution $s = \tau^{-1} \circ \mathrm{id}$ was computed for various norms and for uniform grid sizes $h = 2\pi/n$. The experimental order of convergence *eoc* between two grid sizes h_1 and h_2 is given by

$$eoc = \ln\frac{e_{h_1}}{e_{h_2}} / \ln\frac{h_1}{h_2}.$$

As the following tables show, the numerical results confirm the asymptotic convergence in the $H^{1/2}(\partial D)$ norm predicted by the main theorem.

In the paper [DH] we prove asymptotic convergence of order $O(h^{5/2})$ in the $H^{-1/2}(\partial D)$ norm , and thence of order $O(h^2)$ in the $L^2(\partial D)$ norm. This is confirmed in the following tables for the L^2 norm. The experimental error in the $H^{-1/2}$ norm behaves like $O(h^2)$ only, due to the fact that we used an integration formula for E (and its derivatives) which restricts the order of convergence to 2. The use of a higher order quadrature would lead to a much more complicated scheme and would not change the order of convergence in the energy norm.

Stable Enneper Surface (r=0.5)						
n	$H^{-1/2}$-error	*eoc*	L^2-error	*eoc*	$H^{1/2}$-error	*eoc*
20	2.7913e-3		7.6477e-3		1.5378e-2	
40	6.6122e-4	2.08	1.7860e-3	2.10	4.2490e-3	1.86
80	1.6041e-4	2.04	4.2902e-4	2.06	1.2859e-3	1.72
160	2.9575e-5	2.44	8.3727e-5	2.36	4.0727e-4	1.66
320	4.1573e-6	2.83	1.5176e-5	2.46	1.3790e-4	1.56

Enneper Surface (r = 1.0)						
n	$H^{-1/2}$-error	*eoc*	L^2-error	*eoc*	$H^{1/2}$-error	*eoc*
20	3.4175e-3		9.7364e-3		1.9275e-2	
40	7.1311e-4	2.26	1.9620e-3	2.31	4.5275e-3	2.09
80	1.6602e-4	2.10	4.4812e-4	2.13	1.3098e-3	1.79
160	4.0129e-5	2.05	1.0722e-4	2.06	4.1841e-4	1.67
320	9.8698e-6	2.02	2.6237e-5	2.03	1.4062e-4	1.57

Unstable Enneper Surface (r=1.5),						
n	$H^{-1/2}$-error	*eoc*	L^2-error	*eoc*	$H^{1/2}$-error	*eoc*
20	3.8572e-3		1.1189e-2		2.2074e-2	
40	7.2308e-4	2.42	1.9958e-3	2.49	4.5825e-3	2.27
80	1.6672e-4	2.12	4.5050e-4	2.15	1.3128e-3	1.80
160	4.0202e-5	2.05	1.0747e-4	2.07	4.1864e-4	1.65
320	9.8783e-6	2.02	2.6266e-5	2.03	1.4064e-4	1.57

We also computed discrete solutions on some discretization levels for a case where the kernel of E'' is nontrivial. The boundary curve is given by

$$\begin{aligned}\gamma_1(\phi) &= r_0^2\cos 2\phi + r_0^4\cos 4\phi,\\ \gamma_2(\phi) &= r_0^2\sin 2\phi - r_0^4\sin 4\phi,\\ \gamma_3(\phi) &= -\frac{4\sqrt{2}}{3}r_0^3\sin 3\phi.\end{aligned}$$

The numerical results show that we have no convergence in this case.

Branch point						
n	$H^{-1/2}$-error	*eoc*	L^2-error	*eoc*	$H^{1/2}$-error	*eoc*
10	7.8308e-2		0.1109		0.1579	
20	8.5221e-2	-0.12	0.1205	-0.12	0.1706	-0.11
40	8.7044e-2	-0.04	0.1231	-0.03	0.1742	-0.03
80	8.7506e-2	-0.01	0.1238	-0.01	0.1751	-0.01
160	8.7622e-2	0.00	0.1239	0.00	0.1753	0.00

Next we consider the numerical computation of a minimal surface spanning a knotted curve. The curve is parametrized by

$$\begin{aligned}\gamma_1(\phi) &= (c_1 + c_2\cos 3\phi)\cos 2\phi,\\ \gamma_2(\phi) &= (c_1 + c_2\cos 3\phi)\sin 2\phi,\\ \gamma_3(\phi) &= -c_3\sin 3\phi,\end{aligned}$$

and we have chosen $c_1 = 1.0$, $c_2 = 0.2$, $c_3 = 0.5$ for our computations. In this case we use a solution on the very fine grid with 200 nodes as a quasi exact solution for the computation of the error. The results are as follows.

Knotted boundary curve				
n	L^2-error	*eoc*	$H^{1/2}$-error	*eoc*
33	0.1028		0.6910	
42	4.2445e-2	3.67	0.3001	3.46
56	1.6182e-2	3.35	0.1095	3.51
100	2.4769e-3	3.24	1.5841e-2	3.33

The discrete solutions are plotted in the following figures.

Fig. 2: Quasi exact solution (n=200) for the knotted boundary curve.

Fig. 3: Discrete solutions for 9,15,21,30,42 and 51 grid points.

Fig. 4: Discrete minimal surface for the knotted boundary curve.

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