Higher integrability of the gradient and dimension of the singular set for minimisers of the Mumford–Shah functional

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1 Introduction

In this paper we investigate the regularity properties of minimisers u of the Mumford–Shah functional

$$\int_\Omega |\nabla u|^2 + \alpha(u-g)^2\,dx + \beta\mathcal{H}^{N-1}(S_u\cap\Omega)$$

or, more generally, of quasi-minimisers of the main part $\int_\Omega |\nabla u|^2\,dx + \beta\mathcal{H}^{N-1}(S_u\cap\Omega)$ of the functional. Here g is bounded and measurable, $\alpha\ge 0$, $\beta>0$, u is an SBV function and S_u is the discontinuity set of u (see [6] for a discussion of the Mumford-Shah functional). As in [4, 5, 6], we are not making any restriction on the number N of dimensions of the ambient space.

Let us define the scaled Dirichlet energy $\mathcal{D}(x,\varrho)$ and the mean flatness $\mathcal{A}(x,\varrho)$ by

$$\text{(1.1)}\quad \mathcal{D}(x,\varrho)=\frac{1}{\varrho^{N-1}}\int_{B_\varrho(x)}|\nabla u|^2\,dy,\qquad \mathcal{A}(x,\varrho)=\frac{1}{\varrho^{N+1}}\min_T\int_{S_u\cap B_\varrho(x)}\mathrm{dist}^2(y,T)\,d\mathcal{H}^{N-1}$$

where the above minimum is taken over all the affine $(N-1)$-planes T. In [5] (see also [6]) we proved the existence of a relatively closed and $\mathcal{H}^{N-1}$-negligible singular set $\Sigma(u)\subset\overline{S}_u$ such that $\overline{S}_u\setminus\Sigma(u)$ is a $C^{1,1/4}$ hypersurface. Moreover, we proved that there exists an absolute constant $\varepsilon_0>0$ such that

$$\Sigma(u)=\left\{x\in\overline{S}_u:\ \liminf_{\varrho\downarrow 0}\mathcal{D}(x,\varrho)+\mathcal{A}(x,\varrho)>\varepsilon_0\right\}.$$

Therefore, for any small ball centered at $x\in\Sigma(u)$, either the scaled Dirichlet energy or the flatness are sufficiently large. Clearly we may split the singular set $\Sigma(u)$ in three parts: points where the Dirichlet energy tends to 0, points where the flatness tends to 0 and points where neither of them tends to 0. Notice that in the case $N=2$ the analysis in [19] suggests that the first set corresponds to the so called "triple junctions" (or "propellers", according to the terminology of [10]), the second set corresponds to "crack tips" and the third set is empty. In general we may expect that the Hausdorff dimension of $\Sigma(u)$ is at most $N-2$; this result is still open even in the two-dimensional case and, in our opinion, is the main open problem in the regularity theory of the Mumford–Shah functional (see [5, 8, 10, 17] for partial results).

In this paper we make one step in this direction proving that the first set, i.e.

$$\Sigma'=\left\{x\in\Sigma(u):\ \lim_{\varrho\downarrow 0}\varrho^{1-N}\int_{B_\varrho(x)}|\nabla u|^2\,dy=0\right\}$$

has Hausdorff dimension at most $N-2$ (see Theorem 5.6). As a consequence, we are able to prove in Corollary 5.7 that

$$\text{(1.2)}\qquad \mathcal{H}\text{−}\dim(\Sigma(u))\le\max\{N-2,N-p/2\}$$

provided $|\nabla u|\in L^p_{\mathrm{loc}}(\Omega)$ for some $p>2$.

E. De Giorgi conjectured in [14] that $|\nabla u|$ is locally p-summable for some $p\in(2,4)$ and this conjecture is still open; notice that $p=4$ is exactly the critical exponent leading to the optimal estimate on the Hausdorff dimension of $\Sigma(u)$ in (1.2) and that the crack tip local minimiser (see [9]), defined in polar coordinates by

$$u(r,\theta)=\sqrt{\frac{2\beta r}{\pi}}\sin(\theta/2)\ ,$$

satisfies $|\nabla u|\in L^p_{\mathrm{loc}}(\mathrm{I\!R}^N)$ for any $p<4$.

Our proof of the estimate of the Hausdorff dimension is based on a blow-up analysis of the properties of S_u near points $x \in \Sigma'$: we prove that limit points S of the rescaled sets $(S_u - x)/\varrho$ as $\varrho \downarrow 0$ are local minimisers of the area functional. Since we are not dealing here with boundaries or oriented sets, the local minimality must be properly understood: a concept perfectly tailored to our purposes is Almgren's minimality, saying that

$$\mathcal{H}^{N-1}(S \cap B_R) \leq \mathcal{H}^{N-1}\left(\varphi(S \cap B_R)\right) \tag{1.3}$$

whenever $R > 0$, $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ is a Lipschitz map and $\{x : \varphi(x) \neq x\} \subset\subset B_R$. The regularity theory for Almgren's area minimising sets provides us with the desired estimate.

In order to check (1.3) for the blown up discontinuity set S_u the main source of technical difficulties is the fact that the admissible maps φ need not be one to one (and exactly for this reason the regularity theory for Almgren's minimising sets is stronger, compared to Allard's regularity theory, see Theorem 4.1 and Theorem 4.3). Therefore, in §2 we examine more closely the behaviour of BV or SBV maps under Lipschitz change of coordinates, not necessarily one to one. §3 is devoted to the proof of a delicate approximation theorem; using this result one can check the minimality property (1.3) using only a special class of maps φ of the form $\Phi \circ \gamma \circ \Psi^{-1}$, with Φ, Φ^{-1}, Ψ, Ψ^{-1} close in C^1 norm to the identity and $\nabla\gamma$ piecewise constant (see Theorem 3.1 for the precise statement). In §4 we recall the main facts on Almgren's area minimising sets and in §5 we prove the asymptotic area minimality of the jump set S_u at points $x \in \Sigma'$. Finally, in §6 we indicate other heuristic reasons suggesting (in two dimensions) that the gradient of any minimiser is in $L^p_{\rm loc}$ for any $p < 4$. This higher integrability property seems to be related to a conjecture of Brennan stating that a conformal map from any bounded open set of the plane into the unit disk has gradient in L^p for any $p < 4$ (see [20, Chap.8]).

2 On the behaviour of BV maps under Lipschitz changes of coordinates

In this section we discuss the following problem: given $u \in BV_{\rm loc}(\Omega)$ and a proper one to one orientation preserving Lipschitz map $\varphi : \Omega \to \Omega'$, we want to relate the distributional derivative of $u \circ \varphi^{-1}$ with the distributional derivative of u. More generally, if either φ is not one to one or φ is not orientation preserving, we may define the *push forward* of u through φ by

$$\varphi_{\#}u(y) := \sum_{x \in \varphi^{-1}(y)} u(x) \mathrm{sign}\left(\det(\nabla\varphi(x))\right)$$

if $y \in \varphi(\Omega)$ and $\varphi_{\#}u(y) = 0$ if $y \in \Omega' \setminus \varphi(\Omega)$, and study its differentiability properties. This map is well defined almost everywhere in Ω', since the image of the set of points where either $\nabla\varphi$ is not defined or $\nabla\varphi$ is singular is $\mathcal{L}^N$-negligible. Moreover, the area formula shows that $\varphi_{\#}u$ is the unique $w \in L^1_{\rm loc}(\Omega')$ such that

$$\int_{\Omega'} w\phi \, dy = \int_{\Omega} u\phi(\varphi) \det\nabla\varphi \, dx \qquad \forall \phi \in C_c^\infty(\Omega').$$

The following result is well known (see for instance [6, Theorem 3.16] for a proof). In the language of the theory of currents, which identifies locally BV functions with locally normal currents, it means that the push-forward operator induced by φ maps locally normal currents to locally normal currents.

Theorem 2.1 *Let Ω, Ω' be open subsets of $\mathbb{R}^N$, let $\varphi : \Omega \to \Omega'$ be a proper Lipschitz function and $u \in BV_{\mathrm{loc}}(\Omega)$. Then $\varphi_\# u$ belongs to $BV_{\mathrm{loc}}(\Omega')$ and satisfies*

$$|D(\varphi_\# u)|(B) \le [\mathrm{Lip}(\varphi)]^{N-1} \, |Du|(\varphi^{-1}(B)) \tag{2.1}$$

for any Borel set $B \subset \Omega'$.

In our setting we are interested in understanding whether additional properties of u, as for instance $u \in SBV$ or $u \in W^{1,1}$, are preserved by the push forward operator.

If φ has a Lipschitz inverse it is easy to see that the SBV (or Sobolev) property is preserved. In fact, since φ^{-1} maps $\mathcal{L}^N$-negligible sets into $\mathcal{L}^N$-negligible sets, (2.1) still holds with the singular part of derivatives. Hence, as the measure $|D^s u|$ is concentrated on S_u, $|D^s(\varphi_\# u)|$ is concentrated on $\varphi(S_u)$. Since this set has σ-finite $\mathcal{H}^{N-1}$-measure, and since the Cantor part of the derivative does not see any set with σ-finite $\mathcal{H}^{N-1}$-measure, it follows that $\varphi_\# u \in SBV$ (and also, as a byproduct, that $\mathcal{H}^{N-1}$-almost all of $S_{\varphi_\#} u$ is contained in $\varphi(S_u)$).

However, since we will be dealing with minimality in the Almgren sense, we are forced to consider deformation maps φ which are not one to one. Quite surprisingly, in [6, Section 3.1] it is shown that in this generality no SBV or Sobolev property is preserved by the $\varphi_\#$ operator: indeed, *any* $w \in BV_{\mathrm{loc}}(\mathbb{R})$ can be represented as $\varphi_\# u$ for suitable Lipschitz maps φ, u. Though the extension of this negative result to higher dimensions seems to be a very hard problem, we are therefore led to make additional assumptions on φ.

Our first result is concerned with the approximate differential of $\varphi_\# u$; we prove that L^p integrability of the approximate differential is preserved if the multiplicity function $\mathrm{card}(\varphi^{-1}(y))$ is essentially bounded and the essential supremum

$$c_p(\varphi) := \operatorname{ess\,sup} \left\{ \|(\nabla\varphi(x))^{-1}\|^p |\det(\nabla\varphi(x))| : \ \det(\nabla(\varphi(x)) \neq 0 \right\} \tag{2.2}$$

is finite. Notice that $c_1(\varphi)$ is always finite, since it can be estimated with a constant multiple of $[\mathrm{Lip}(\varphi)]^{N-1}$. Notice also that $c_p(\varphi) < \infty$ if φ is one to one and φ^{-1} is a Lipschitz function.

Theorem 2.2 *Let Ω, Ω' be open subsets of $\mathbb{R}^N$, let $\varphi : \Omega \to \Omega'$ be a Lipschitz function and $u \in BV_{\mathrm{loc}}(\Omega)$. Then the approximate differential of $\varphi_\# u$ is given almost everywhere in $\varphi(\Omega)$ by*

$$\sum_{x \in \varphi^{-1}(y)} \nabla u(x)(\nabla\varphi(x))^{-1} \mathrm{sign}\left(\det(\nabla\varphi(x))\right). \tag{2.3}$$

Moreover, if $\mathrm{card}(\varphi^{-1}(y)) \le k$ for $\mathcal{L}^N$-almost every y, we have

$$\int_B |\nabla(\varphi_\# u)|^p \, dy \le c_p(\varphi) k^{p-1} \int_{\varphi^{-1}(B)} |\nabla u|^p \, dx \tag{2.4}$$

for any Borel set $B \subset \mathbb{R}^N$.

Proof. The proof can be easily achieved in the case when $\varphi \in C^1(\Omega, \Omega')$, using the local invertibility theorem. The general case can be obtained by a Lusin-type approximation of φ by C^1 and equi-Lipschitz functions (see Theorem 3.6 below).

□

Eventually we want to find conditions ensuring that $\varphi_{\#}u \in SBV_{\rm loc}(\mathbb{R}^N)$ whenever $u \in SBV_{\rm loc}(\mathbb{R}^N)$. A sufficient one is given in the following theorem.

Theorem 2.3 *Assume that $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ is Lipschitz, piecewise affine and proper. Then $\varphi_{\#}$ maps $SBV_{\rm loc}(\mathbb{R}^N)$ in $SBV_{\rm loc}(\mathbb{R}^N)$. Moreover, if the rank of $\nabla\varphi$ is either N or is strictly less than $N-1$ in any open region where $\nabla\varphi$ is constant, we have*

$$\mathcal{H}^{N-1}(S_{\varphi_{\#}u} \setminus \varphi(S_u)) = 0.$$

Proof. Let $(P_i)_{i\in I}$ be the open regions where $\nabla\varphi$ is constant and has rank N and let $(Q_j)_{j\in J}$ be the remaining open regions where $\nabla\varphi$ is constant and its rank is strictly less than N. We define

$$R := \mathbb{R}^N \setminus \left(\bigcup_{i\in I} P_i \cup \bigcup_{j\in J} Q_j \right).$$

The set $\Gamma = \varphi(R \cup \cup_j Q_j) \cup \varphi(S_u)$ has σ-finite $\mathcal{H}^{N-1}$-measure, because $\mathcal{H}^{N-1}(\varphi(Q_j))$ is σ-finite for any $j \in J$. Let B be a Lebesgue negligible Borel set on which $D^c(\varphi_{\#}u)$, the Cantor part of the derivative $\varphi_{\#}u$, is concentrated. Since $\varphi^{-1}(B) \cap P_i$ is Lebesgue negligible for any $i \in I$, by (2.1) we get

$$\begin{aligned} |D^c(\varphi_{\#}u)|(\mathbb{R}^N) &= |D^c(\varphi_{\#}u)|(B \setminus \Gamma) \le C \sum_{i\in I} |Du|(P_i \cap \varphi^{-1}(B) \setminus S_u) \\ &\le \sum_{i\in I} \int_{P_i\cap\varphi^{-1}(B)} |\nabla u|\, dx = 0, \end{aligned}$$

therefore $D^c\varphi_{\#}u = 0$ and $\varphi_{\#}u \in SBV_{\rm loc}(\mathbb{R}^N)$.

Under the stronger assumption on the rank of $\nabla\varphi$ the set $\varphi(\cup_j Q_j)$ is $\mathcal{H}^{N-1}$-negligible. Taking into account the fact that $|Du|$ is zero on any Borel set σ-finite with respect to $\mathcal{H}^{N-1}$ and disjoint with S_u we get

$$\begin{aligned} |D(\varphi_{\#}u)|(S_{\varphi_{\#}u} \setminus \varphi(S_u)) &= |D(\varphi_{\#}u)|(S_{\varphi_{\#}u} \setminus \varphi(S_u \cup \bigcup_{j\in J} Q_j)) \\ &\le C \left[\sum_{i\in I} \int_{P_i\cap\varphi^{-1}(S_{\varphi_{\#}u})} |\nabla u|\, dx + |Du|(R \setminus S_u) \right] = 0. \end{aligned}$$

Since

$$|Dv|(A \cap S_v) = \int_{A\cap S_v} |v^+ - v^-|\, d\mathcal{H}^{N-1}$$

for any Borel set A and any $v \in BV_{\rm loc}(\mathbb{R}^n)$, choosing $v = \varphi_{\#}u$ and $A = S_{\varphi_{\#}u} \setminus \varphi(S_u)$ we infer that $\mathcal{H}^{N-1}(A) = 0$. $\square$

Beside the piecewise affine functions satisfying the assumptions of Theorem 2.3 there are other useful Lipschitz functions φ such that $\varphi_{\#}$ maps SBV into SBV, namely those considered in the next lemma. Notice that the map φ constructed in the lemma squashes a whole neighbourhood of a Lipschitz graph Γ over the graph itself. In the sequel we denote by C_R the cylinder $B_R^{N-1} \times (-3R, 3R)$, where B_R^{N-1} is the $(N-1)$-dimensional ball $\{z = (z_1, \ldots, z_{N-1}) : |z| < R\}$.

Lemma 2.4 (Deformation) *There exists a constant C_0 depending only on the dimension N such that if $g : B_R^{N-1} \to \mathbb{R}$ is a Lipschitz function with $\mathrm{Lip}_{|B_R^{N-1}}(g) \le 1$, $g = 0$ on ∂B_R^{N-1} and $\varepsilon \in (0, 1/2)$, then there exists a Lipschitz map $\varphi : C_R \to C_R$ such that*

$$\mathrm{Lip}(\varphi) \le C_0, \qquad \varphi(x) = x \ \text{ on } \partial C_R, \qquad \varphi(U_{\varepsilon,R}) \subset \Gamma_g \cap (B^{N-1}_{R(1-\varepsilon)} \times \mathbb{R}),$$

where $U_{\varepsilon,R} = \{(z,t) : z \in B^{N-1}_{R(1-\varepsilon)},\ |g(z) - t| \le 2\varepsilon R\}$ and Γ_g is the graph of g over B_R^{N-1}. Moreover φ has the property that if $u \in SBV(C_R)$, then $\varphi_\# u \in SBV(C_R)$, $\varphi_\# u$ has the same trace of u on ∂C_R and

$$\int_{C_R} |\nabla(\varphi_\# u)|^2 \, dx \le C_0 \int_{C_R \setminus U_{\varepsilon,R}} |\nabla u|^2 \, dx \,, \qquad S_{\varphi_\# u} \setminus \Gamma_g \subset \varphi(S_u \setminus U_{\varepsilon,R}) \,.$$

Proof. Let us fix $0 < \varepsilon < 1/2$ and define two functions $g^+, g^- : B_R^{N-1} \to \mathbb{R}$ setting

$$g^+(z) = \begin{cases} g(z) + 2\varepsilon R & \text{if } \ z \in B^{N-1}_{R(1-\varepsilon)} \\ g(z) + 2\mathrm{dist}(z, \partial B_R^{N-1}) & \text{if } \ z \in B_R^{N-1} \setminus B^{N-1}_{R(1-\varepsilon)} \,, \end{cases}$$

$$g^-(z) = \begin{cases} g(z) - 2\varepsilon R & \text{if } \ z \in B^{N-1}_{R(1-\varepsilon)} \\ g(z) - 2\mathrm{dist}(z, \partial B_R^{N-1}) & \text{if } \ z \in B_R^{N-1} \setminus B^{N-1}_{R(1-\varepsilon)} \,. \end{cases}$$

Clearly $\mathrm{Lip}_{|B_R^{N-1}}(g^+), \mathrm{Lip}_{|B_R^{N-1}}(g^-) \le 3$; moreover since $\sup |g(z)| \le R$ we have $\sup g^+(z) \le 2R$, $\inf g^-(z) \ge -2R$. Let us now define $\varphi : C_R \to C_R$ as follows

$$\varphi(z,t) = \begin{cases} (z, 3R + (3 - t/R)(g(z) - R - g^+(z)) & \text{if } \ 2R \le t < 3R \\ (z, t + g(z) - g^+(z)) & \text{if } \ g^+(z) \le t \le 2R \\ (z, g(z)) & \text{if } \ g^-(z) \le t \le g^+(z) \\ (z, t + g(z) - g^-(z)) & \text{if } \ -2R \le t \le g^-(z) \\ (z, -3R + (3 + t/R)(g(z) + R - g^-(z)) & \text{if } \ -3R < t \le -2R \,. \end{cases}$$

It is easy to check that $\varphi(x) = x$ if $x \in \partial C_R$ and that $\varphi : W_R \to C_R \setminus \Gamma_g$ is invertible, where $W_R = \{(z,t) \in C_R : t > g^+(z) \text{ or } t < g^-(z)\}$, and that $\varphi(C_R \setminus W_R) = \Gamma_g$, thus in particular $\varphi(U_{\varepsilon,R}) \subset \Gamma_g$. Notice also that φ is proper and that since the Lipschitz constants of g^+, g^- are less than 3, the derivatives of φ and $(\varphi_{|W_R})^{-1}$ can be estimated by an absolute constant independent of R. Therefore if $u \in SBV(C_R)$ from Theorem 2.1 it follows that $\varphi_\# u \in BV(C_R)$. Moreover since for all $y \in C_R \setminus \Gamma_g$ $\mathrm{card}(\varphi^{-1}(y)) = 1$ from (2.4) we have

$$\int_{C_R} |\nabla(\varphi_\# u)|^2 \, dy = \int_{C_R \setminus \Gamma_g} |\nabla(\varphi_\# u)|^2 \, dx \le C_0 \int_{C_R \setminus U_{\varepsilon,R}} |\nabla u|^2 \, dx \,,$$

where C_0 is a constant depending only on N and on the bounds on the derivatives of φ and $(\varphi_{|W_R})^{-1}$, hence ultimately only on the dimension N. Also, since $\varphi(W_R) = C_R \setminus \Gamma_g$ and $\varphi_{|W_R}$ is invertible, $\varphi_\# u \in SBV(C_R \setminus \Gamma_g)$ (see the observations made after Theorem 2.1). Finally, the Cantor part of $D(\varphi_\# u)$ cannot be concentrated on Γ_g, hence we may conclude that $\varphi_\# u \in SBV(C_R)$ and that $S_{\varphi_\# u} \subset \Gamma_g \cup \varphi(S_u)$ and thus $S_{\varphi_\# u} \setminus \Gamma_g \subset \varphi(S_u) \setminus \Gamma_g \subset \varphi(S_u \setminus U_{\varepsilon,R})$. □

3 Approximation in area of the Lipschitz image of a rectifiable set

In many situations one would like to approximate the $\mathcal{H}^{N-1}$-measure of the Lipschitz image $M = \varphi(S)$ of an $\mathcal{H}^{N-1}$-rectifiable set S by approximating φ (using one of the many available classical constructions) with a sequence of piecewise affine Lipschitz maps φ_h. However in general one may only expect, by the lower semicontinuity of the area functional, that $\mathcal{H}^{N-1}(M) \leq \liminf_h \mathcal{H}^{N-1}(\varphi_h(S))$, the inequality being possibly strict.
In this section we study the problem of approximating the $\mathcal{H}^{N-1}$-measure of the Lipschitz image $\varphi(S)$ of a rectifiable set. Namely, we show that the measure of $\varphi(S)$ can be approximated by the measure of sets of the type $(\Phi \circ \psi \circ \Psi^{-1})(S)$, where ψ is a piecewise affine map whose Lipschitz constant is controlled by the Lipschitz constant of φ and Φ, Ψ are suitable diffeomorphisms arbitrarily close to the identity map. Our approximation result is stated in Theorem 3.1 and it is used in Section 5 to study the properties of certain singular points of the jump set of the minimisers of the Mumford–Shah functional. We think that the approximation provided by this result is interesting in itself and could be useful for other applications to geometric measure theory; for this reason we dedicate a separate section to it.

Theorem 3.1 *Let $S \subset B_R$ be an $\mathcal{H}^{N-1}$-rectifiable set and let $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ be a Lipschitz map such that $\varphi(x) = x$ for all $x \notin B_R$ and $\varphi(B_R) \subset B_R$. For any $\varepsilon > 0$ there exist two diffeomorphisms $\Phi, \Psi : \mathbb{R}^N \to \mathbb{R}^N$ and a piecewise affine function $\gamma : \mathbb{R}^N \to \mathbb{R}^N$ such that*

$$\mathcal{H}^{N-1}((\Phi \circ \gamma \circ \Psi^{-1})(S)) < \mathcal{H}^{N-1}(\varphi(S)) + \varepsilon\,.$$

Moreover the maps Φ, Ψ and γ coincide with the identity map outside the ball B_R, the Lipschitz constants of Φ, Ψ, Φ^{-1} and Ψ^{-1} are less than $1 + \varepsilon$ and $\mathrm{Lip}(\gamma) < c\mathrm{Lip}(\varphi) + \varepsilon$ for some constant c depending only on N and R. Also, γ can be chosen so that $\mathcal{H}^{N-1}(\Psi^{-1}(S) \cap D) < \varepsilon$, where D is the discontinuity set of $\nabla\gamma$, and such that $\det\nabla\gamma \neq 0$ in each open set where $\nabla\gamma$ is constant.

The proof of the theorem makes use of the following result, saying roughly speaking that any rectifiable set can be covered, apart from a set of small measure, with a smooth compact manifold which is arbitrarily close to a polyhedron. The proof of the result can be achieved by standard covering arguments, arguing for instance as in [16, Theorem 4.2.19]), where an analogous property is proved for integral currents.

Theorem 3.2 *Let $S \subset B_R$ be an $\mathcal{H}^{N-1}$-rectifiable set. For any $\varepsilon > 0$ there exist a polyhedron $K = \cup_{i=1}^M K_i \subset B_R$, where each K_i is a closed $(N-1)$-cube, $K_i \cap K_j = \emptyset$ if $i \neq j$, and a diffeomorphism $\Psi : \mathbb{R}^N \to \mathbb{R}^N$ such that $\Psi(x) = x$ if $x \notin B_R$, $\mathrm{Lip}(\Psi), \mathrm{Lip}(\Psi^{-1}) \leq 1 + \varepsilon$, and*

$$\mathcal{H}^{N-1}(S \triangle \Psi(K)) < \varepsilon\ .$$

In order prove Theorem 3.1 we start with the case when the rectifiable set S is indeed a polyhedron K. The next lemma deals with this simpler situation. However the lemma, as a first step in the proof of the approximation result, provides a piecewise affine map ψ which is only defined on K and not on all $\mathbb{R}^N$. The extension of ψ to a piecewise affine map defined on the whole $\mathbb{R}^N$ is then given by the subsequent Lemma 3.4.
In the sequel, whenever $\varphi : S \to \mathbb{R}^N$ is a Lipschitz map and S is a countably $\mathcal{H}^{N-1}$-rectifiable set, we denote the differential of φ at x by $d^S\varphi_x$. We recall (see for instance [21]) that $d^S\varphi_x$ is a linear map from the approximate tangent plane π_x^S to $\mathbb{R}^N$ and that it is defined at $\mathcal{H}^{N-1}$-a.e. point x of S. The corresponding Jacobian is denoted by $\mathbf{J}_{N-1}d^S\varphi_x$.

Lemma 3.3 *Let $K = \cup_{i=1}^M K_i \subset B_R$ be a polyhedron such that each K_i is a closed $(N-1)$-cube contained in the affine $(N-1)$-plane S_i, $K_i \cap K_j = \emptyset$ if $i \neq j$, and let $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ be a C^1 Lipschitz map such that $\varphi(K) \subset B_R$. For any $\varepsilon > 0$ there exist a piecewise affine map $\psi : T \to \mathbb{R}^N$ such that $\mathrm{Lip}_{|T}(\psi) \leq \mathrm{Lip}(\varphi) + \varepsilon$ and a diffeomorphism $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ such that $\Phi(x) = x$ if $x \notin B_R$, $\mathrm{Lip}(\Phi), \mathrm{Lip}(\Phi^{-1}) \leq (1+\varepsilon)$, with the property that $\|\Phi \circ \psi - \varphi\|_{L^\infty(T)} < \varepsilon$ and*

$$\mathcal{H}^{N-1}\left((\Phi \circ \psi)(K)\right) < \mathcal{H}^{N-1}(\varphi(K)) + \varepsilon \ . \tag{3.1}$$

Moreover $T = \cup_{i=1}^M T_i$, where the sets T_i are pairwise disjoint, $K_i \subset T_i \subset S_i \cap B_R$ and T_i is the union of a finite number of $(N-1)$-simplexes $T_{i,j}$ with pairwise disjoint interiors such that for all i, j, $d^{T_{i,j}}\psi$ is a constant matrix of rank $N-1$.

Proof. Step 1. Let us denote by K^r the set of points $x \in K$ such that $\mathbf{J}_{N-1} d^K \varphi_x < r$. Using the local invertibility theorem it is easy to check that for any $r > 0$ there exists $M_r \in \mathbb{N}$ such that $\mathrm{card}\left(\varphi^{-1}(y) \cap K \setminus K^r\right) \leq M_r$ for all $y \in \mathbb{R}^N$.
To prove this claim let us first notice that $\mathrm{card}\left(\varphi^{-1}(y) \cap K \setminus K^r\right) < \infty$. In fact if this is not true there exists a sequence (x_h) in $K \setminus K^r$ such that $x_h \neq x_k$ if $h \neq k$, $\varphi(x_h) = y$ for all h, $x_h \to x$. Let $i \in \{1, \ldots, M\}$ be such that $x \in K_i$ and let S_i be the affine $(N-1)$-plane containing K_i. Since $\mathbf{J}_{N-1} d^{S_i} \varphi_x \geq r$, there exists a neighbourhood U of x such that $\varphi_{|U \cap S_i}$ is a diffeomorphism and this contradicts the fact that in $U \cap K_i$ there exist infinitely many points x_h such that $\varphi(x_h) = y$.
Let us assume now that there exists a sequence (y_h) such that $\mathrm{card}\left(\varphi^{-1}(y_h) \cap K \setminus K^r\right) \to \infty$ and let us suppose, with no loss of generality, that $y_h \to y$. Let us set $m = \mathrm{card}\left(\varphi^{-1}(y) \cap K \setminus K^r\right)$. We can then construct $m+1$ sequences $(x_h^1), \ldots, (x_h^{m+1})$ such that for h large enough $x_h^i \neq x_h^j$ if $i \neq j$, $\varphi(x_h^i) = y_h$ for all $i = 1, \ldots, m+1$. Again, with no loss of generality we may assume that for each i, $x_h^i \to x^i \in \varphi^{-1}(y) \cap K \setminus K^r$. Thus at least two of these points x^i must coincide and, to fix the ideas, let us assume that $x^1 = x^2 = x$. As before, we get a contradiction since there exists a neighbourhood U of x such that $\varphi_{|U \cap K}$ is injective, but at the same time for h large the distinct points x_h^1, x_h^2 belong to U and $\varphi(x_h^1) = \varphi(x_h^2)$.
Step 2. We now construct the diffeomorphism Φ and the set T where the function ψ is going to be defined. To this aim, let us fix $0 < \varepsilon < 1$ and apply Theorem 3.2 to the $\mathcal{H}^{N-1}$-rectifiable set $\varphi(K)$, thus getting a diffeomorphism $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ and an open polyhedron P such that $\mathrm{Lip}(\Phi), \mathrm{Lip}(\Phi^{-1}) \leq 1 + \varepsilon$, $\Phi(x) = x$ for $x \notin B_R$ and

$$\mathcal{H}^{N-1}(\varphi(K) \triangle \Phi(P)) < \frac{\varepsilon^2}{M_\varepsilon} \ , \tag{3.2}$$

where M_ε is defined as in Step 1. Notice that we may always assume that $P = \cup_{i=1}^n P_i$, where each P_i is an open $(N-1)$-cube with $\mathrm{dist}(P_i, P_j) > 0$ if $i \neq j$. Let us set $\tilde{\psi} = \Phi^{-1} \circ \varphi$ and $L = \tilde{\psi}(K) \setminus P$, which is a compact subset of $\mathbb{R}^N$. From the area formula, using (3.2), we have

$$\begin{aligned} \mathcal{H}^{N-1}\left(\tilde{\psi}^{-1}(L) \cap K \setminus K^\varepsilon\right) &\leq \frac{1}{\varepsilon} \int_{\tilde{\psi}^{-1}(L) \cap K \setminus K^\varepsilon} \mathbf{J}_{N-1} d^K \varphi_x \, d\mathcal{H}^{N-1} \\ &\leq \frac{1}{\varepsilon} \int_{\varphi(\tilde{\psi}^{-1}(L) \cap K \setminus K^\varepsilon)} \mathrm{card}\left(\varphi^{-1}(y) \cap K \setminus K^\varepsilon\right) \, d\mathcal{H}^{N-1} \\ &\leq \frac{1}{\varepsilon} M_\varepsilon \mathcal{H}^{N-1}(\Phi(L)) < \varepsilon \ . \end{aligned} \tag{3.3}$$

Let us denote by $\Sigma = \{x \in \mathbb{R}^{N-1} : x = t_1 e_1 + \ldots + t_{N-1} e_{N-1}, \sum t_j \leq 1, t_j \geq 0 \, \forall j\}$ the standard $(N-1)$-simplex, and let $p : \Sigma \to \mathbb{R}^N$ be a piecewise affine function such that $p(x) = 0$ for all

$x \in \partial\Sigma$ and $\mathbf{J}_{N-1}d^\Sigma p_x > 0$ in every open region where $d^\Sigma p$ is constant. For any $i = 1, \ldots, M$ let us cover each face K_i with a mesh of simplexes congruent to Σ, having pairwise disjoint interiors. For any $h \geq 1$ and any i each simplex of the covering of K_i can be subdivided in a standard way in $2^{h(N-1)}$ simplexes $T^i_{j,h}$ of side $1/2^h$. If $T^i_{j,h} = x^i_{j,h} + (1/2^h)\Sigma$, we shall denote by $p^i_{j,h}$ the function obtained by rescaling p in $T^i_{j,h}$, i.e. $p^i_{j,h}(x) = 2^{-h}p(2^h(x - x^i_{j,h}))$ for all $x \in T^i_{j,h}$. Notice that there exists $\tilde{h}$ such that if $h \geq \tilde{h}$ the following relations hold:

$$T^i_{j,h} \cap K^\varepsilon \neq \emptyset \quad \Longrightarrow \quad T^i_{j,h} \subset \{x \in S_i : \mathbf{J}_{N-1}d^{S_i}\varphi_x < 2\varepsilon\}\,; \tag{3.4}$$

$$T^i_{j,h} \cap K \neq \emptyset \quad \Longrightarrow \quad \mathrm{diam}(T^i_{j,h}) \leq \frac{1}{3}\min_{l \neq m}\mathrm{dist}(K_l, K_m) \ \text{ and } \ T^i_{j,h} \subset B_R\,; \tag{3.5}$$

$$\mathcal{H}^{N-1}(B) < \varepsilon\,, \ \text{ where } \ B = \cup\{T^i_{j,h} : T^i_{j,h} \cap \partial K \neq \emptyset, \, i = 1, \ldots, M\}\,. \tag{3.6}$$

Given $h_0 \geq \tilde{h}$ we denote by C the union of all those T^i_{j,h_0} such that $T^i_{j,h_0} \cap \tilde{\psi}^{-1}(L) \cap K \setminus K^\varepsilon \neq \emptyset$. From (3.3) it is clear that h_0 can be chosen sufficiently large so that $\mathcal{H}^{N-1}(C) < \varepsilon$. With such a choice of h_0 let us denote by D the union of all the T^i_{j,h_0} having not empty intersection with K^ε and not contained in C. Then let us denote by G the union of those T^i_{j,h_0} such that $T^i_{j,h_0} \cap K \neq \emptyset$ and which are not contained in C nor in D. Notice that from (3.6) it follows that if $\tilde{G}$ denotes the union of those simplexes T^i_{j,h_0} contained in $G \cap K$, then $\mathcal{H}^{N-1}(G \setminus \tilde{G}) < \varepsilon$. Notice also that $\tilde{G} \subset K \setminus \tilde{\psi}^{-1}(L)$, hence $\tilde{\psi}(\tilde{G}) \subset P$. Finally let us set $T = C \cup D \cup G$ and for any i let us denote by T_i the union, running over j, of those T^i_{j,h_0} contained in T. From (3.5) it follows immediately that every T_i is contained in B_R and that the sets T_i are pairwise disjoint.

Step 3. We now define ψ with the required properties. For any $h \geq h_0$ let us denote by $\tilde{\psi}_h : T \to \mathbb{R}^N$ the piecewise affine function coinciding with $\tilde{\psi}$ on the vertices of any $T^i_{j,h}$. Then $\tilde{\psi}_h \to \tilde{\psi}$ and $d^T\tilde{\psi}_h \to d^T\tilde{\psi}$ uniformly on T. Therefore, since $\tilde{G} \subset K \setminus K^\varepsilon$, for h sufficiently large $\mathbf{J}_{N-1}d^T(\tilde{\psi}_h)_x > 0$ for all $x \in \tilde{G}$. Moreover, since $\tilde{\psi}(\tilde{G})$ is a compact subset of P, then $\tilde{\psi}(\tilde{G}) = \cup_{l=1}^n H_l$, where $H_l \subset P_l$ is compact for any l. Thus, given $\sigma > 0$, for any l there exists $A_l \supset H_l$, relatively open in P_l such that $\mathcal{H}^{N-1}(\cup_{l=1}^n(A_l \setminus H_l)) < \sigma$. Let us recall that the faces P_l of P are at a positive distance one from the other. Thus for h large $\tilde{\psi}_h(\tilde{G}) \subset P$ and therefore the uniform convergence of $\tilde{\psi}_h \to \tilde{\psi}$ implies that for h large we have $\tilde{\psi}_h(\tilde{G}) \subset \cup_{l=1}^n A_l$. From the arbitrariness of σ we then get that

$$\limsup_{h\to\infty}\mathcal{H}^{N-1}(\tilde{\psi}_h(\tilde{G})) \leq \mathcal{H}^{N-1}(\tilde{\psi}(\tilde{G}))\ .$$

Let us fix $h_1 \geq h_0$ so that $\mathrm{Lip}_{|T}(\tilde{\psi}_{h_1}) < \mathrm{Lip}(\tilde{\psi}) + \varepsilon$, $\mathbf{J}_{N-1}d^T(\tilde{\psi}_{h_1})_x < \mathbf{J}_{N-1}d^T\tilde{\psi}_x + \varepsilon$ for all $x \in T$, $\max_T|\Phi\circ\tilde{\psi}_{h_1} - \varphi| \leq \mathrm{Lip}(\Phi)\max_T|\tilde{\psi}_{h_1} - \tilde{\psi}| < \varepsilon$ and $\mathcal{H}^{N-1}(\tilde{\psi}_{h_1}(\tilde{G})) \leq \mathcal{H}^{N-1}(\tilde{\psi}(\tilde{G})) + \varepsilon$. With such a choice of h_1 we define a piecewise affine function $\psi : T \to \mathbb{R}^N$ setting $\psi(x) = \tilde{\psi}_{h_1}(x)$ if $x \in \tilde{G}$ or $x \in T^i_{j,h_1}$ for some $T^i_{j,h_1} \subset T \setminus \tilde{G}$ where $\mathbf{J}_{N-1}d^T(\tilde{\psi}_{h_1})_x > 0$. If $T^i_{j,h_1} \subset T \setminus \tilde{G}$ is such that in T^i_{j,h_1} the constant matrix $d^T\tilde{\psi}_{h_1}$ has rank strictly less than $N-1$, we set $\psi(x) = \tilde{\psi}_{h_1}(x) + \tau p^i_{j,h_1}(x)$ for all $x \in T^i_{j,h_1}$, where $\tau > 0$ is chosen small enough so that the Lipschitz constant in T of the resulting function remains strictly less that $\mathrm{Lip}(\tilde{\psi}) + \varepsilon$, $\max_T|\Phi\circ\psi - \varphi| < \varepsilon$ and $0 < \mathbf{J}_{N-1}d^T(\tilde{\psi}_{h_1} + \tau p^i_{j,h_1}) < \varepsilon$ in T^i_{j,h_1}. This choice of τ is clearly possible since this Jacobian is constant on each of the finite open regions of T^i_{j,h_1} where $d^Tp^i_{j,h_1}$ is constant and in each of these regions is a polynomial of degree $N-1$ in the variable τ. To conclude the proof it remains to estimate the measure of $(\Phi\circ\psi)(K)$. From our construction of T we then get

$$\mathcal{H}^{N-1}(\Phi(\psi(K))) \leq (1+\varepsilon)^{N-1}\mathcal{H}^{N-1}(\psi(K)) \tag{3.7}$$

$$\leq (1+\varepsilon)^{N-1}\Big[\mathcal{H}^{N-1}(\psi(C)) + \mathcal{H}^{N-1}(\psi(D\cap K))$$
$$+\mathcal{H}^{N-1}(\psi(G\setminus\tilde{G})) + \mathcal{H}^{N-1}(\psi(\tilde{G}))\Big] \ .$$

Recall that $\mathcal{H}^{N-1}(C) < \varepsilon$, hence $\mathcal{H}^{N-1}(\psi(C)) \leq c\varepsilon$ where the constant c depends only on N and $\mathrm{Lip}(\varphi)$. Similarly, $\mathcal{H}^{N-1}(\psi(G\setminus\tilde{G})) \leq c\varepsilon$, while from the area formula and (3.4) we have

$$\mathcal{H}^{N-1}(\psi(D\cap K)) \leq \int_{D\cap K} \mathbf{J}_{N-1} d^T\psi_x \, d\mathcal{H}^{N-1} \leq \int_{D\cap K} (\mathbf{J}_{N-1} d^T\tilde{\psi}_x + \varepsilon)\, d\mathcal{H}^{N-1} \leq c\varepsilon\mathcal{H}^{N-1}(K)\,,$$

where c depends only on N. Therefore, recalling that $\tilde{\psi}(\tilde{G}) \subset P$, from (3.7) and (3.2) we have

$$\mathcal{H}^{N-1}(\Phi(\psi(K))) \leq (1+\varepsilon)^{N-1}\left[c\varepsilon + \mathcal{H}^{N-1}(\tilde{\psi}(\tilde{G}))\right] \leq \mathcal{H}^{N-1}(\varphi(K)) + \tilde{c}\varepsilon \ ,$$

where the constant $\tilde{c}$ depends only on N, $\mathrm{Lip}(\varphi)$, $\mathcal{H}^{N-1}(K)$. Hence the result follows. □

We can now construct a piecewise affine extension to $\mathbb{R}^N$ of the function ψ obtained in the previous lemma.

Lemma 3.4 *Under the same assumptions of Lemma 3.3 and if $\varphi(x) = x$ for $x \notin B_R$, for any $\varepsilon > 0$ there exists a piecewise affine map $\gamma : \mathbb{R}^N \to \mathbb{R}^N$ such that $\gamma(x) = x$ if $x \notin B_R$, $\mathrm{Lip}(\gamma) \leq \mathrm{Lip}(\varphi) + \varepsilon$, $\det\nabla\gamma \neq 0$ in each open set where $\nabla\gamma$ is constant and there exists a diffeomorphism $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ such that $\Phi(x) = x$ if $x \notin B_R$, $\mathrm{Lip}(\Phi), \mathrm{Lip}(\Phi^{-1}) \leq 1+\varepsilon$ such that*

$$\mathcal{H}^{N-1}\left((\Phi\circ\gamma)(K)\right) < \mathcal{H}^{N-1}(\varphi(K)) + \varepsilon \ .$$

Moreover $\mathcal{H}^{N-1}(K\cap D) = 0$, where D is the discontinuity set of $\nabla\gamma$.

Proof. Let us fix $0<\varepsilon<1, 0<\sigma<\varepsilon\wedge R$ such that $3\sigma<\mathrm{dist}(K,\partial B_R)$ and $2\sigma<\min_{i\neq j}\mathrm{dist}(K_i,K_j)$. Let us apply Lemma 3.3 with ε replaced by $\sigma\varepsilon$ and notice that from the proof of the lemma it is clear that we may always assume that $\mathrm{dist}(T,\partial B_R) > 3\sigma$ and that $\mathrm{dist}(T_i,T_j) > 2\sigma$ whenever $i \neq j$. For any $i \in \{1,\dots,M\}$ let us denote by N_i the number of the $(N-1)$-simplexes $T_{i,j}$ where the function ψ is affine. Let us extend ψ near each T_i. To this aim let us fix i and in order to simplify the notation let us assume that the affine $(N-1)$-plane S_i containing T_i is the coordinate plane $\{x_N = 0\}$. For any $j = 1,\dots,N_i$ let us denote by $E^+_{i,j}$ and $E^-_{i,j}$ the closed pyramids of height ϱ (to be chosen later) and basis $T_{i,j}$ contained respectively in the half spaces $\{x_N \geq 0\}$ and $\{x_N \leq 0\}$. We extend ψ to the set $E_i = \cup_{j=1}^{N_i}(E^+_{i,j}\cup E^-_{i,j})$ setting for all $x \in E_i$ $\psi(x) = \psi(x',0) + \alpha x_N$, where $x' = (x_1,\dots,x_{N-1})$ and $\alpha \in \mathbb{R}^N$ is to be chosen. Notice that above definition of E_i implies that if ϱ is chosen small enough then $\mathrm{dist}(E_i,E_j) > 0$ when $i \neq j$ and $\mathrm{dist}(E,\partial B_R) > 2\sigma$, where $E = \cup_{i=1}^M E_i$. Notice that since $\mathbf{J}_{N-1}d^{T_{i,j}}\psi > 0$ for all i and j we may always choose α arbitrarily small in norm and such that $\det\nabla\psi \neq 0$ in all the sets $E^+_{i,j}$ and $E^-_{i,j}$. Thus, we choose α and ϱ so that we have also

$$\mathrm{Lip}_{|E}(\psi) < \mathrm{Lip}(\varphi) + 2\varepsilon, \qquad \|\Phi\circ\psi - \varphi\|_{L^\infty(E)} < 2\sigma\varepsilon\,. \tag{3.8}$$

Let us now set

$$\overline{\psi}(x) = \begin{cases} \psi(x) & \text{if } x \in E \\ x & \text{if } x \in \mathbb{R}^N \setminus B_{R-\sigma^2}\,, \end{cases}$$

$F = E \cup \mathbb{R}^N \setminus B_{R-\sigma^2}$ and let us estimate $\mathrm{Lip}_{|F}(\psi)$. To this aim, by the first inequality in (3.8) it is enough to consider $|\overline{\psi}(x) - \overline{\psi}(y)|$ with $x \in E$ and $y \in B_R \setminus B_{R-\sigma^2}$. Given two such vectors, recalling that $\mathrm{dist}(E,\partial B_R) > 2\sigma$ and hence $|x-y| > \sigma$, from the second inequality in (3.8) we get

$$|\overline{\psi}(x) - \overline{\psi}(y)| \ = \ |\psi(x) - y| \leq |\psi(x) - \Phi^{-1}(\varphi(x))| + |\Phi^{-1}(\varphi(x)) - \Phi^{-1}(\varphi(y))|$$

$$
\begin{aligned}
&+|\Phi^{-1}(\varphi(y)) - \Phi^{-1}(\varphi(Ry/|y|))| + |Ry/|y| - y| \\
\leq\ & \mathrm{Lip}(\Phi^{-1})|\Phi(\psi(x)) - \varphi(x)| + \mathrm{Lip}(\Phi^{-1})\mathrm{Lip}(\varphi)|x-y| + c(R - |y|) \\
\leq\ & 2(1+\varepsilon)\sigma\varepsilon + (1+\varepsilon)\mathrm{Lip}(\varphi)|x-y| + c\sigma^2 \\
\leq\ & (\mathrm{Lip}(\varphi) + \tilde{c}\varepsilon)\,|x-y|\,,
\end{aligned}
$$

where $\tilde{c}$ depends only on $\mathrm{Lip}(\varphi)$. To conclude the proof we may extend $\overline{\psi}$, thanks to Kirszbraun's theorem (see [16, 2.10.43]), to a Lipschitz map from $\mathbb{R}^N$ to $\mathbb{R}^N$, still denoted by $\overline{\psi}$, with Lipschitz constant in $\mathbb{R}^N$ equal to $\mathrm{Lip}_{|F}(\overline{\psi})$. Notice that $\nabla\overline{\psi}$ is continuous in the interior of each set $E^+_{i,j} \cup E^-_{i,j}$ and hence the intersection of the discontinuity set D of $\nabla\overline{\psi}$ with K is contained in the union of the $(N-2)$-dimensional faces of the sets $T_{i,j}$. Therefore $\mathcal{H}^{N-1}(D \cap K) = 0$. Finally, let us fix a finite union of congruent cubes Q such that $B_{R-\sigma^2} \subset\subset Q \subset\subset B_R$ and let us approximate $\overline{\psi}$ on $Q \setminus E$ with a piecewise affine map $\overline{\gamma}$ such that $\mathrm{Lip}_{|Q\setminus E}(\overline{\gamma}) < \mathrm{Lip}_{|Q\setminus E}(\overline{\psi}) + \varepsilon$, $\det\nabla\overline{\gamma} \neq 0$ in each open subset of $Q \setminus E$ where $\nabla\overline{\gamma}$ is constant and $\overline{\gamma} = \overline{\psi}$ on $\partial(Q \setminus E)$. The map γ is then obtained setting $\gamma(x) = \overline{\psi}(x)$ if $x \in E \cup (\mathbb{R}^N \setminus Q)$ and $\gamma(x) = \overline{\gamma}(x)$ if $x \in Q \setminus E$. □

We can pass now to the proof of Theorem 3.1. This proof makes use of Lemma 3.4 and of a suitable version of the Whitney extension theorem given at the end of this section.

Proof of Theorem 3.1. Let us fix $0 < \varepsilon < 1$. Since S is an $\mathcal{H}^{N-1}$-rectifiable set, there exist finitely many, pairwise disjoint, compact subsets of S, $H_1, \dots, H_m$ such that $\mathcal{H}^{N-1}(S \setminus \cup_{i=1}^m H_i) < \varepsilon$. Moreover we may always assume that each H_i is contained in the graph of a C^1 function $g_i : U_i \to \pi_i^\perp$, where U_i is an open subset of a suitable $(N-1)$-plane π_i, and that $\mathrm{Lip}_{|U_i}(\phi_i) < 1 + \varepsilon$, where $\phi_i : U_i \to \mathbb{R}^N$ is the map $\phi_i(z) = (z, g_i(z))$. Since $\varphi \circ \phi_i$ is a Lipschitz continuous map from Theorem 3.6 it follows that for any i there exists a compact set $C_i \subset \pi_i(H_i)$ such that $\varphi \circ \phi_i$ coincides on C_i with the restriction of a C^1 map $\tilde{\varphi}_i : U_i \to \mathbb{R}^N$. Moreover the sets C_i can be chosen so that $\mathcal{H}^{N-1}(\bigcup_{i=1}^m (H_i \setminus \phi_i(C_i)) < \varepsilon$. Let us now apply the Whitney Extension Theorem 3.5 to the maps f and κ defined on $C = \cup_{i=1}^m \phi_i(C_i) \cup (\mathbb{R}^N \setminus B_R)$ setting $f = \varphi$ on this set and $\kappa(x) = I$ if $x \notin B_R$, $k(x) = \nabla(\tilde{\varphi}_i \circ \pi_i)(x)$ if $x \in \phi_i(C_i)$. Notice that the assumptions of Theorem 3.5 are clearly satisfied and that since $\mathrm{Lip}_{|U_i}(\phi_i) < 1 + \varepsilon$ one immediately gets that both $\sup\{|\kappa(x)| : x \in C\}$ and $\sup\{|R(x,y)| : x \neq y,\ x, y \in C\}$ are controlled by $c\mathrm{Lip}(\varphi)$, where c is a constant depending only on the dimension N. Thus we get a C^1 map $\tilde{\varphi} : \mathbb{R}^N \to \mathbb{R}^N$ such that $\tilde{\varphi}(x) = \varphi(x)$ on C and $\mathrm{Lip}(\tilde{\varphi}) \leq c(N,R)\mathrm{Lip}(\varphi)$. Moreover, since $\cup_{i=1}^m \phi_i(C_i) \subset\subset B_R$ and $\tilde{\varphi}(x) = x$ when $x \notin B_R$, by enlarging a little the Lipschitz constant of $\tilde{\varphi}$ we may always assume that $\tilde{\varphi}(B_R) \subset B_R$. Thus we have

$$
\mathcal{H}^{N-1}(\tilde{\varphi}(S)) = \mathcal{H}^{N-1}\Big(\varphi\big(\bigcup_{i=1}^m \phi_i(C_i)\big)\Big) + \mathcal{H}^{N-1}\Big(\tilde{\varphi}\big(S \setminus \bigcup_{i=1}^m \phi_i(C_i)\big)\Big) \leq \mathcal{H}^{N-1}(\varphi(S)) + c\varepsilon\,, \tag{3.9}
$$

where c depends only on $\mathrm{Lip}(\varphi)$, N and R. Let us now apply Theorem 3.2 to S, thus getting a polyhedron K and a diffeomorphism Ψ such that $\Psi(x) = x$ for all $x \notin B_R$ and $\mathcal{H}^{N-1}(S \triangle \Psi(K)) < \varepsilon$. Then we apply Lemma 3.4 to the polyhedron K and to the function $\tilde{\varphi} \circ \Psi$. Thus, we get a piecewise affine map $\gamma : \mathbb{R}^N \to \mathbb{R}^N$ such that $\gamma(x) = x$ if $x \notin B_R$, $\mathrm{Lip}(\gamma) \leq c(N,R)\mathrm{Lip}(\varphi) + \varepsilon$, $\det\nabla\gamma \neq 0$ in each open set where $\nabla\gamma$ is constant, and a diffeomorphism Φ such that $\mathcal{H}^{N-1}(\Phi \circ \gamma)(K)) < \mathcal{H}^{N-1}((\tilde{\varphi} \circ \Psi)(K)) + \varepsilon$. Therefore, using (3.9), we obtain

$$
\begin{aligned}
\mathcal{H}^{N-1}\big((\Phi \circ \gamma \circ \Psi^{-1})(S)\big) &\leq \mathcal{H}^{N-1}\big((\Phi \circ \gamma \circ \Psi^{-1})(S \setminus \Psi(K))\big) + \mathcal{H}^{N-1}\big((\Phi \circ \gamma)(K)\big) \\
&\leq c\varepsilon + \mathcal{H}^{N-1}\big((\tilde{\varphi} \circ \Psi)(K)\big) + \varepsilon \\
&\leq \mathcal{H}^{N-1}(\tilde{\varphi}(S)) + \mathcal{H}^{N-1}\big(\tilde{\varphi}(\Psi(K) \setminus S)\big) + c'\varepsilon \leq \mathcal{H}^{N-1}(\tilde{\varphi}(S)) + \tilde{c}\varepsilon,
\end{aligned}
$$

where the constant $\tilde{c}$ depends only on $\mathrm{Lip}(\varphi)$, N and R. To conclude the proof let us remark that if D is the discontinuity set of $\nabla\gamma$, since by Lemma 3.4 $\mathcal{H}^{N-1}(K \cap D) = 0$ and $\mathrm{Lip}(\Psi^{-1}) < 1+\varepsilon$, we have

$$\mathcal{H}^{N-1}\big(\Psi^{-1}(S) \cap D\big) = \mathcal{H}^{N-1}\big((\Psi^{-1}(S) \setminus K) \cap D\big) \le (1+\varepsilon)^{N-1}\varepsilon\,.$$

□

The next result gives a suitable version of the classical Whitney extension theorem (see for instance [16]), giving sharp conditions ensuring the existence of a C^1 extension of a function defined on a closed set. Here we show that beside the existence of such extension one may also prove a precise estimate of the L^∞ norm of its gradient. For this reason we recall how the extension is obtained, but we limit ourselves to prove only the gradient bound.

Theorem 3.5 (Whitney extension) *There exists a constant $C_0(N)$ such that if $C \subset \mathbb{R}^N$ is a closed set and $f : C \to \mathbb{R}$, $\kappa : C \to \mathbb{R}^N$ are two continuous maps such that for any compact set K contained in C*

$$\lim_{\delta \downarrow 0} \sup\{|R(x,y)| :\, x, y \in K, 0 < |y-x| < \delta\} = 0\,,$$

where for all $x, y \in C$, $x \neq y$

$$R(x,y) = \frac{f(y) - f(x) - \langle \kappa(x), y-x\rangle}{|y-x|}\,,$$

then there exists a function $\tilde{f} \in C^1(\mathbb{R}^N)$ such that $\tilde{f} = f$, $\nabla\tilde{f} = \kappa$ on C and such that for all $r > 0$

$$\sup_{x \in I_r(C)} |\nabla\tilde{f}(x)| \le C_0(1+r)\Big[\sup_{x\in C}|\kappa(x)| + \sup_{x \neq y}|R(x,y)|\Big]\,, \tag{3.10}$$

where $I_r(C) = \{x :\, \mathrm{dist}(x,C) < r\}$.

Proof. Let us recall how the construction of the extension $\tilde{f}$ works. For the proof that $\tilde{f} = f$ and $\nabla\tilde{f} = \kappa$ on C we refer to [15] (see the proof given in Section 6.5 of that book).
Let us set $U = \mathbb{R}^N \setminus C$ and for all $x \in U$

$$r(x) = \frac{1}{20}\min\{1, \mathrm{dist}(x,C)\}\,.$$

By the Vitali covering theorem there exists a countable set $S \subset U$ such that $U = \cup_{s\in S} B_{5r(s)}(s)$ and the balls $B_{r(s)}(s)$ are pairwise disjoint. For each $x \in U$ let us define

$$S_x = \{s \in S :\, B_{10r(x)}(x) \cap B_{10r(s)}(s) \neq \emptyset\}\,.$$

Then (see [15]) it can be easily shown that

$$\mathrm{card}(S_x) \le 129^N, \qquad \frac{1}{3} \le \frac{r(x)}{r(s)} \le 3 \qquad \forall x \in U,\ s \in S_x\,. \tag{3.11}$$

Let us now fix $\eta \in C^\infty(\mathbb{R})$ such that $0 \le \eta(t) \le 1$, $\eta(t) = 1$ if $t \le 1$, $\eta(t) = 0$ for $t \ge 2$ and for all $s \in S$ let us define

$$u_s(x) = \eta\left(\frac{|x-s|}{5r(s)}\right)\,.$$

Notice that $u_s(y) = 0$ if $y \in B_{10r(x)}(x)$ and $s \notin S_x$, therefore, using (3.11) we get

$$|Du_s(x)| \le \frac{C_0}{r(x)} \qquad \forall x \in U,\ s \in S_x\,. \tag{3.12}$$

Let us also set $\sigma(x) = \sum_{s\in S} u_s(x)$ for all $x \in U$. Then $\sigma(x) \ge 1$ for all x. Thus we set for all s

$$v_s(x) = \frac{u_s(x)}{\sigma(x)} \qquad \forall x \in U\,.$$

Finally for any $x \in U$ let us denote by $\tau(x)$ a point such that

$$|x - \tau(x)| = \mathrm{dist}(x, C)\,.$$

Following [15] we define $\tilde f : \mathrm{I\!R}^N \to \mathrm{I\!R}$ setting

$$\tilde f(x) = \begin{cases} f(x) & \text{if } x \in C \\ \displaystyle\sum_{s\in S} v_s(x)\,[f(\tau(s)) + \langle \kappa(\tau(s)), x - \tau(s)\rangle] & \text{if } x \in U\,. \end{cases}$$

For the proof that $\tilde f \in C^1(\mathrm{I\!R}^N)$ we refer to [15] limiting ourselves to show the estimate (3.10). To this aim let us fix $x \in U$. Since $v_s \equiv 0$ in $B_{10r(x)}(x)$ if $s \notin S_x$ and $\sum_{s\in S_x} \nabla v_s(x) = 0$, we have

$$\nabla \tilde f(x) = \sum_{s\in S_x} \left\{ [f(\tau(s)) - f(\tau(x)) + \langle \kappa(\tau(s)), x - \tau(s)\rangle]\,\nabla v_s(x) + v_s(x)\kappa(\tau(s)) \right\}\,.$$

Setting $M = \sup_{x\in C} |\kappa(x)| + \sup_{x\neq y} |R(x,y)|$, from the first inequality in (3.11) and from (3.12) we have

$$\begin{aligned} |\nabla \tilde f(x)| &\le c(N)M\Big[\sum_{s\in S_x} (|\tau(s) - \tau(x)| + |x - \tau(x)|)|\nabla v_s(x)| + 1\Big] \\ &\le c(N)M\Big[\sum_{s\in S_x} \frac{|s - x| + |x - \tau(x)|}{r(x)} + 1\Big]\,. \end{aligned}$$

Since $B_{10r(x)}(x) \cap B_{10r(s)}(s) \neq \emptyset$ if $s \in S_x$, from the second inequality in (3.11) we have that

$$|s - x| \le 10r(x) + 10r(s) \le 40r(x)\,.$$

Therefore, recalling the definition of $r(x)$ we may conclude that

$$|\nabla \tilde f(x)| \le c(N)M(1 + \max\{1, \mathrm{dist}(x, C)\})$$

and from this inequality the result follows. □

The classical Whitney theorem is used to show a Lusin type property of the Lipschitz functions, i.e. that any Lipschitz function f coincides with a C^1 function $\tilde f$ outside a set of small measure. From Theorem 3.5 also this classical result can be improved showing that indeed $\tilde f$ can be constructed in such a way that its Lipschitz constant remains smaller than $C_1(N)\mathrm{Lip}(f)$, where C_1 is a constant depending only on the dimension N. The proof is a simple consequence of the a.e. differentiability of Lipschitz functions and of Egorov theorem.

Theorem 3.6 *There exists a constant $C_1(N)$ such that for any function $f \in Lip(\mathrm{I\!R}^N)$ and for any $\varepsilon > 0$, there exists $\tilde f \in C^1(\mathrm{I\!R}^N)$ such that*

$$\mathcal{L}^N\big(\{x : \tilde f(x) \neq f(x)\}\big) < \varepsilon$$

and $\mathrm{Lip}(\tilde f) \le C_1(N)\mathrm{Lip}(f)$.

4 Almgren area minimising sets

In this section we recall some basic facts on sets minimising the area functional with respect to local deformations, not necessarily one to one. This minimality property is referred to as $(\mathbf{M}, 0, \infty)$-minimality in Almgren's seminal paper [2].

Let S be a countably $\mathcal{H}^{N-1}$-rectifiable set with locally finite $\mathcal{H}^{N-1}$-measure. We say that S is an *Almgren area minimiser* if

$$\mathcal{H}^{N-1}(S \cap B_R) \le \mathcal{H}^{N-1}\left(\varphi(S \cap B_R)\right) \tag{4.1}$$

whenever $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ is a Lipschitz map, $R > 0$ and $\{x \in \mathbb{R}^N : \varphi(x) \neq x\} \subset\subset B_R$.

Theorem II.3(12)-(13) of [2] (see also [13]) implies the density bounds

$$c\varrho^{N-1} \le \mathcal{H}^{N-1}\left(S \cap B_\varrho(x)\right) \le d\varrho^{N-1} \qquad \forall x \in \mathrm{supp}\mathcal{H}^{N-1} \llcorner S,\ \varrho > 0 \tag{4.2}$$

for suitable dimensional constants c, $d > 0$. In particular, denoting by S' the support of $\mathcal{H}^{N-1} \llcorner S$, we have

$$\mathcal{H}^{N-1}(S \Delta S') = 0\,.$$

For this reason in the following we shall always assume, possibly modifying S in a $\mathcal{H}^{N-1}$-negligible set, that $S = \mathrm{supp}\mathcal{H}^{N-1} \llcorner S$.

Choosing one to one deformations $\varphi_\varepsilon(x) = x + \varepsilon\phi(x)$ it is easy to check that any area minimiser is stationary for the area functional, i.e.

$$\int_S \mathrm{div}^S \phi \, d\mathcal{H}^{N-1} = 0 \qquad \forall \phi \in C_0(\mathbb{R}^N; \mathbb{R}^N)\,.$$

We first state a compactness property of Almgren minimising sets.

Theorem 4.1 *Let S_h be Almgren area minimisers and let $x \in \bigcap_h S_h$. Then*

(i) the family S_h is relatively compact with respect to the convergence of the associated varifolds as $h \to \infty$;

(ii) any limit point of the varifolds associated to S_h is the varifold associated to a suitable Almgren area minimising set C;

(iii) if $x = 0$ and $S_h = S/\varrho_h$, where $\varrho_h \downarrow 0$ and S is an Almgren area minimiser, then any limit point is an Almgren area minimising cone C.

Proof. The proof of this theorem is analogous to the one of Proposition 5.3 and Theorem 5.4 in the next section, and actually simpler (since only surface energies are involved). For this reason we only briefly indicate the main ingredients of the proof.
(i) Denoting by V_h the rectifiable varifolds associated to S_h (i.e. measures in $G = \mathbb{R}^N \times \mathbf{G}_{N-1}$, where $\mathbf{G}_{N-1}$ is the set of unoriented $(N-1)$-subspaces of $\mathbb{R}^N$), by (4.2) we get

$$V_h(B_R \times \mathbf{G}_{N-1}) = \mathcal{H}^{N-1}(S_h \cap B_R) \le dR^{N-1} \qquad \forall R > 0\,.$$

Hence, the family (V_h) has limits points as $h \to \infty$.
(ii) Let $V = \lim_j V_{h_j}$ with $h_j \to \infty$. By the general theory of rectifiable varifolds (see for instance [1, 21]), we know that V is a stationary rectifiable varifold induced by a countably $\mathcal{H}^{N-1}$-rectifiable set C and a multiplicity function θ. Moreover, the upper semicontinuity of the multiplicity function (see [21], Theorem 42.7) implies that $\theta \ge 1$ $\mathcal{H}^{N-1}$-a.e. on C.

It remains to show that $\theta \leq 1$ $\mathcal{H}^{N-1}$-a.e. on C and that C is an Almgren area minimiser. To this aim let us remark that from the density bound (4.2) we may deduce (see the proof of Proposition 7.4 in [6]) the following *height bound*: if S is an Almgren area minimiser, π is any $(N-1)$-plane, $\varrho > 0$ and x is any point in $\mathbb{R}^N$, then

$$\sup_{y\in S\cap B_\varrho(x)} |\pi^\perp(y-x)|^{N+1} \leq c(N) \int_{S\cap B_{2\varrho}(x)} |\pi^\perp(y-x)|^2 \, d\mathcal{H}^{N-1}_y \,, \tag{4.3}$$

where $c(N)$ is a constant depending only on the dimension N. Let us fix now $x \in C$ such that there exists the approximate tangent plane $\pi_x = \pi$ at x to C and let us assume, with no loss of generality that $x = 0$. Applying (4.3) to the sets S_h and arguing as in the Step 2 of Proposition 5.3, we get that for any $\varepsilon > 0$ there exists ϱ_ε such that if $\varrho < \varrho_\varepsilon$

$$\limsup_{h\to\infty} \sup_{S_h\cap \overline{B}_\varrho} |\pi^\perp y| < c_0(N)\varrho\varepsilon \,. \tag{4.4}$$

Let us now fix $\varepsilon > 0$ such that $c_0(N)\varepsilon < 1$ and $\varrho < \varrho_\varepsilon$ and let us denote by $\tilde{\varphi}$, the function defined on $(\mathbb{R}^N \setminus B_\varrho) \cup F_{\varrho,\varepsilon}$, where $F_{\varrho,\varepsilon} = \{y \in \overline{B}_{\varrho(1-\sqrt{\varepsilon})} : |\pi^\perp y| < c_0(N)\varrho(1-\sqrt{\varepsilon})\varepsilon\}$, setting

$$\tilde{\varphi}(y) = y \quad \text{if } y \notin B_\varrho \qquad\qquad \tilde{\varphi}(y) = \pi(y) \quad \text{if } y \in F_{\varrho,\varepsilon} \,.$$

Notice that if $y_1 \notin B_\varrho$ and $y_2 \in F_{\varrho,\varepsilon}$ then

$$\begin{aligned} |\tilde{\varphi}(y_1) - \tilde{\varphi}(y_2)| &= |y_1 - \pi(y_2)| \leq |y_1 - y_2| + |\pi^\perp(y_2)| \\ &\leq |y_1 - y_2| + c_0(N)\varrho\varepsilon \leq (1 + c_0(N)\sqrt{\varepsilon})|y_1 - y_2| \,, \end{aligned}$$

hence the Lipschitz constant of $\tilde{\varphi}$ is less than or equal to $1 + c_0(N)\sqrt{\varepsilon}$. Hence we may use Kirszbraun's theorem (see [16, 2.10.43]) to extend $\tilde{\varphi}$ to a function $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ with the same Lipschitz constant of $\tilde{\varphi}$. From the minimality of the sets S_h we then have, using (4.4) and (4.2),

$$\begin{aligned} \mathcal{H}^{N-1}(C \cap B_\varrho) &= \lim_{h\to\infty} \mathcal{H}^{N-1}(S_h \cap B_\varrho) \leq \liminf_{h\to\infty} \mathcal{H}^{N-1}(\varphi(S_h \cap B_\varrho)) \\ &\leq \limsup_{h\to\infty} \mathcal{H}^{N-1}(\tilde{\varphi}(S_h \cap \overline{B}_{\varrho(1-\sqrt{\varepsilon})})) + \limsup_{h\to\infty} \mathcal{H}^{N-1}(\varphi(S_h \cap B_\varrho \setminus \overline{B}_{\varrho(1-\sqrt{\varepsilon})})) \\ &\leq \omega_{N-1}\varrho^{N-1} + (\mathrm{Lip}(\varphi))^{N-1} \lim_{h\to\infty} \mathcal{H}^{N-1}(S_h \cap B_\varrho \setminus \overline{B}_{\varrho(1-\sqrt{\varepsilon})}) \\ &\leq \omega_{N-1}\varrho^{N-1} + (1 + c_1(N)\sqrt{\varepsilon})^{N-1}\mathcal{H}^{N-1}(C \cap B_\varrho \setminus \overline{B}_{\varrho(1-\sqrt{\varepsilon})}) \\ &\leq \omega_{N-1}\varrho^{N-1} + \mathcal{H}^{N-1}(C \cap B_\varrho \setminus \overline{B}_{\varrho(1-\sqrt{\varepsilon})}) + c_2(N)\sqrt{\varepsilon}\varrho^{N-1} \end{aligned}$$

and thus

$$\mathcal{H}^{N-1}(C \cap B_{\varrho(1-\sqrt{\varepsilon})}) \leq \omega_{N-1}\varrho^{N-1} + c_2(N)\sqrt{\varepsilon}\varrho^{N-1} \,.$$

From this inequality the estimate $\vartheta(x) \leq 1$ immediately follows, letting first $\varrho \to 0$ then $\varepsilon \to 0$. Notice that, since $\vartheta(x) = 1$ for $\mathcal{H}^{N-1}$-a.e. $x \in C$ and the varifold V induced by C is stationary, we have

$$\int_C \mathrm{div}^C \eta \, d\mathcal{H}^{N-1} = 0 \qquad\qquad \forall \eta \in C^1_0(\mathbb{R}^N; \mathbb{R}^N) \,.$$

Therefore Allard's regularity theorem for stationary varifolds (see [1], [21]) implies that there exists a closed set $\Sigma(C)$, with $\mathcal{H}^{N-1}(\Sigma(C)) = 0$ such that $C \setminus \Sigma(C)$ is a C^1 hypersurface.
To prove that C is an Almgren area minimizer let us take a Lipschitz map $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ such that $\{x \in \mathbb{R}^N : \varphi(x) \neq x\} \subset\subset B_R$ for some $R > 0$ and let us fix $\varepsilon \in (0, 1/2)$. For any $x \in C \setminus \Sigma(C)$

let us denote by π_x the (classical) tangent plane to C at x and by ϱ_x a radius such that if $\varrho < \varrho_x$ then $C \cap C_\varrho(x)$ is the graph over $x + \pi_x$ of a C^1 function g_x with Lipschitz constant less than ε, where

$$C_\varrho(x) = \{y \in \mathbb{R}^N : |\pi_x(y-x)| < \varrho, |\pi_x^\perp(y-x)| < 3\varrho\},$$

and moreover

$$\limsup_{h\to\infty} \sup_{S_h \cap \overline{C}_\varrho(x)} |\pi_x^\perp(y-x)| < \varepsilon^2 \varrho \tag{4.5}$$

(see (4.4) above). By a standard argument, based on an extension of the Besicovitch–Vitali covering theorem to cylinders (see for instance [18, Theorem 5.11]), we may find a finite number of these cylinders $C_{\varrho_i}(x_i)$, $i = 1, \dots, m$, pairwise disjoint and such that $\mathcal{H}^{N-1}((C \setminus \bigcup_{i=1}^m C_{\varrho_i}(x_i)) \cap B_R) < \varepsilon$ and $\mathcal{H}^{N-1}(C \cap \partial C_{\varrho_i}(x)) = 0$. Therefore we have

$$\lim_{h\to\infty} \mathcal{H}^{N-1}((S_h \setminus \bigcup_{i=1}^m C_{\varrho_i}(x_i)) \cap B_R) < \varepsilon\,. \tag{4.6}$$

Since the Lipschitz constant of the functions g_{x_i} is less than ε, we can easily construct a Lipschitz function $\tilde{g}_i$ defined on $(x_i + \pi_{x_i}) \cap B_{\varrho_i}(x_i)$, with Lipschitz constant less than 1, and such that $\tilde{g}_i(z) = g_{x_i}(z)$ for all $z \in (x_i + \pi_{x_i}) \cap B_{\varrho_i(1-\varepsilon)}(x_i)$ and $\tilde{g}_i(z) = 0$ on $(x_i + \pi_{x_i}) \cap \partial B_{\varrho_i}(x_i)$; clearly, $\sup|\tilde{g}_i(z)| \le \varepsilon\varrho_i$. Let us now apply the deformation Lemma 2.4 to each cylinder $C_{\varrho_i}(x_i)$ and to the corresponding function $\tilde{g}_i$. Thus for all $i = 1, \dots, m$ we get a Lipschitz map $\psi_i : C_{\varrho_i}(x_i) \to C_{\varrho_i}(x_i)$ such that

$$\mathrm{Lip}(\psi_i) \le C_0, \qquad \psi_i(x) = x \text{ on } \partial C_{\varrho_i}(x_i), \qquad \psi_i(U_i) \subset C \cap C_{\varrho_i}(x_i)\,, \tag{4.7}$$

where $U_i = \{x \in \mathbb{R}^N : |\pi_{x_i}(x-x_i)| < \varrho_i(1-\varepsilon), |\tilde{g}_i(x_i + \pi_{x_i}(x-x_i)) - \pi_{x_i}^\perp(x-x_i)| \le 2\varepsilon\varrho_i\}$. Notice that from (4.7) we get that for any h sufficiently large

$$\psi_i(S_h \cap C_{\varrho_i}(x_i)) \subset \psi_i(S_h \cap C_{\varrho_i}(x_i) \setminus U_i) \cup (C \cap C_{\varrho_i}(x_i))\,. \tag{4.8}$$

Let us now define $\psi : \mathbb{R}^N \to \mathbb{R}^N$ setting $\psi(x) = x$ if $x \notin \cup_{i=1}^m C_{\varrho_i}(x_i))$, $\psi(x) = \psi_i(x)$ if $x \in C_{\varrho_i}(x_i)$ for some $i = 1, \dots, m$ and notice that from (4.7) it follows that $\mathrm{Lip}(\psi) \le \max\{1, C_0\}$. Then, using the minimality of the sets S_h we have

$$\begin{aligned}\mathcal{H}^{N-1}(C \cap B_R) &\le \liminf_{h\to\infty} \mathcal{H}^{N-1}(S_h \cap B_R) \le \liminf_{h\to\infty} \mathcal{H}^{N-1}\left((\varphi\circ\psi)(S_h \cap B_R)\right) \\ &\le \limsup_{h\to\infty} \mathcal{H}^{N-1}\left((\varphi\circ\psi)((S_h \setminus \cup_{i=1}^m C_{\varrho_i}(x_i)) \cap B_R\right) \\ &\qquad + \limsup_{h\to\infty} \mathcal{H}^{N-1}\left((\varphi\circ\psi)(S_h \cap \cup_{i=1}^m C_{\varrho_i}(x_i))\right)\,.\end{aligned}$$

Therefore, from (4.6) and (4.8), we get

$$\begin{aligned}\mathcal{H}^{N-1}(C \cap B_R) &\le c(N)[\mathrm{Lip}(\varphi)]^{N-1}\varepsilon + \limsup_{h\to\infty} \mathcal{H}^{N-1}\left(\varphi(\cup_{i=1}^m \psi_i(S_h \cap C_{\varrho_i}(x_i) \setminus U_i)\right) \\ &\qquad + \mathcal{H}^{N-1}\left(\varphi(\cup_{i=1}^m (C \cap C_{\varrho_i}(x_i)))\right) \\ &\le c(N)[\mathrm{Lip}(\varphi)]^{N-1}\Big(\varepsilon + \sum_{i=1}^m \mathcal{H}^{N-1}(C \cap C_{\varrho_i}(x_i) \setminus U_i)\Big) + \mathcal{H}^{N-1}\left(\varphi(C \cap B_R)\right)\,.\end{aligned}$$

Since $C \cap C_{\varrho_i}(x_i) \setminus U_i$ coincides with the graph of g_{x_i} on $(x_i + \pi_{x_i}) \cap (B_{\varrho_i}(x_i) \setminus B_{\varrho_i(1-\varepsilon)}(x_i))$,

$$\mathcal{H}^{N-1}(C \cap C_{\varrho_i}(x_i) \setminus U_i) \le c(N)\varepsilon\varrho_i^{N-1} \le c(N)\varepsilon\mathcal{H}^{N-1}(C \cap C_{\varrho_i}(x_i))\,, \tag{4.9}$$

and thus

$$\mathcal{H}^{N-1}(C \cap B_R) \leq c(N)[\mathrm{Lip}(\varphi)]^{N-1}\varepsilon\Big(1 + \mathcal{H}^{N-1}(C \cap B_R)\Big) + \mathcal{H}^{N-1}\left(\varphi(C \cap B_R)\right) .$$

From this inequality the minimality of C immediately follows letting $\varepsilon \downarrow 0$.
(iii) This is a consequence of (i) and of the monotonicity formula (see [21], Corollary 42.6). □

The *singular set* $\Sigma(S)$ of an Almgren area minimising set is the $\mathcal{H}^{N-1}$-negligible set of all points $x \in S$ where the approximate tangent plane to S at x does not exist. Allard's regularity theory for stationary varifolds (see [1], [21]) implies that $\Sigma(S)$ is a relatively closed subset of S and that $S \setminus \Sigma(S)$ is a smooth hypersurface. The crucial ingredient in Allard's proof is the so-called *tilt–excess*, defined by

$$\mathcal{T}(S, x, \varrho) = \min_{\pi \in \mathbf{G}_{N-1}} \varrho^{1-N} \int_{S \cap B_\varrho(x)} \|\pi_y - \pi\|^2 \, d\mathcal{H}^{N-1}$$

where π_y is the approximate tangent space to S at y.

Allard characterized singular points of area stationary sets S as those points x such that, for any ball $B_\varrho(x)$, either the tilt–excess is sufficiently large or the density

$$\frac{\mathcal{H}^{N-1}(S \cap B_\varrho(x))}{\omega_{N-1}\varrho^{N-1}}$$

is sufficiently larger than 1. In the special case of Almgren minimisers we can neglect the density condition, as the following corollary shows.

Corollary 4.2 *There exists an absolute constant $\delta_0 > 0$ such that*

$$\Sigma(S) = \{x \in S : \ \mathcal{T}(S, x, \varrho) \geq \delta_0 \ \ \forall \varrho > 0\}$$

for any Almgren area minimiser S.

Proof. The inclusion $\supset$ holds for any choice of $\delta_0 > 0$; we will prove that for δ_0 small enough the opposite one holds by a simple contradiction argument. Assume that (up to homotheties and translations) Almgren area minimisers S_h and numbers $\delta_h > 0$ exist such that $0 \in \Sigma(S_h)$, $\mathcal{T}(S_h, 0, 1) < \delta_h$ and $\delta_h \downarrow 0$. By Theorem 4.1 we can assume that the varifolds V_h associated to S_h converge to the varifold associated to some Almgren area minimiser S. The continuity of $\mathcal{T}$ under varifold convergence implies $\mathcal{T}(S, 0, 1) = 0$, hence $S \cap B_1$ is a $(N-1)$-disk. In particular

$$\lim_{h\to\infty} \mathcal{T}(S_h, 0, 1/2) = 0 \qquad \text{and} \qquad \lim_{h\to\infty} \mathcal{H}^{N-1}(S_h \cap B_{1/2}) = \frac{\omega_{N-1}}{2^{N-1}}$$

and therefore, by Allard's regularity criterion, $0 \notin \Sigma(S_h)$ for h large enough. □

Theorem 4.3 *For any Almgren area minimising set S we have $\mathcal{H}$-dim$(\Sigma(S)) \leq N - 2$.*

Proof. We apply the abstract version of Federer's dimension reduction argument in Theorem A.4 of [21] with the set of characteristic functions

$$\mathcal{F} := \{\chi_C : \ C \ \text{ is an Almgren area minimising set}\}$$

endowed with the convergence

$$\chi_{C_h} \to \chi_C \qquad \Longleftrightarrow \qquad \lim_{h\to\infty} \int_{C_h} g\, d\mathcal{H}^{N-1} = \int_C g\, d\mathcal{H}^{N-1} \quad \forall g \in C_c(\mathbb{R}^N)$$

and with the "singularity map" $\mathrm{sing}(\chi_C) = \Sigma(C)$.

It is easy to check that the assumptions A.1 (scaling invariance of $\mathcal{F}$), A.3(2) (scaling invariance of ϕ) and A.3(1) ($\mathrm{sing}(\phi) = \emptyset$ if ϕ is the characteristic function of an hyperplane) of the theorem are satisfied. The validity of assumption A.2 (existence of homogeneous degree zero tangent functions) is the content of Theorem 4.1(iii). Assumption A.3(3) (upper semicontinuity of $\phi \mapsto \mathrm{sing}(\phi)$) is a direct consequence of the varifold convergence and of the representation of $\Sigma(C)$ given in Corollary 4.2. □

5 Limit behaviour of sequences of quasi-minimisers

Let u be a function in $SBV_{\mathrm{loc}}(\Omega)$. In the following we shall set

$$F(u,\Omega) = \int_\Omega |\nabla u|^2\, dx + \mathcal{H}^{N-1}(S_u)\,.$$

We say that u is a *quasi-minimiser* of the functional F in Ω if there exists a constant $\omega \geq 0$ such that

$$F(u, B_\varrho(x)) \leq F(v, B_\varrho(x)) + \omega\varrho^N \tag{5.1}$$

whenever $B_\varrho(x) \subset\subset \Omega$ and v is any function in $SBV_{\mathrm{loc}}(\Omega)$ such that $\mathrm{supp}(u-v) \subset\subset B_\varrho(x)$. If $\omega = 0$ then u will be called a *local minimiser* of F in Ω.

We recall that if $u \in SBV(\Omega)$ is a minimiser of the Mumford–Shah functional

$$\int_\Omega |\nabla u|^2\, dx + \alpha \int_\Omega |u-g|^q\, dx + \mathcal{H}^{N-1}(S_u \cap \Omega)\,, \tag{5.2}$$

where $g \in L^\infty(\Omega)$, $\alpha > 0$, $q \geq 1$, then it is easy to check (see [6, Section 7.2]) that u is a quasi-minimiser satisfying (5.1) with $\omega = 2^q\alpha\omega_N\|g\|_\infty^q$.

In this section we study the limit behaviour of a sequence (u_h) of quasi-minimisers of the functional F whose volume energies $\int_\Omega |\nabla u|^2 dx$ vanish as $h \to \infty$ and we prove that, up to a subsequence, the corresponding jump sets S_{u_h} converge weak* locally to an Almgren area minimiser. This result is then applied to the case when the sequence is obtained by blowing up a quasi-minimiser at a singular point of the jump set $\overline{S}_u$. This fact can be used to estimate the dimension of a subset of the singular set of $\overline{S}_u$ where the rescaled volume energy vanishes asymptotically. A consequence of this estimate (see Corollary 5.7) is that if u is a local minimiser of F such that ∇u is in L^p for some $p > 2$ then the dimension of the singular set $\Sigma(u)$ is less than or equal to $\max\{N-2, N-p/2\}$.

Remark 5.1 (Scaling of quasi-minimisers) If $u \in SBV(B_\varrho(x))$ and we set

$$u_\varrho(y) = \varrho^{-1/2}u(x_0 + \varrho y) \qquad \forall y \in B_1\,,$$

then $u_\varrho \in SBV(B_1)$, $S_{u_\varrho} = (S_u - x_0)/\varrho$ and moreover the Dirichlet integral and the area term in the functional F both rescale by ϱ^{1-N}, hence

$$F(u_\varrho, B_1) = \varrho^{1-N}F(u, B_\varrho(x_0))\,.$$

From this inequality it follows also that if $u \in \mathcal{M}_\omega(\Omega)$ is a quasi-minimiser, then $u_\varrho \in \mathcal{M}_{\varrho\omega}(\Omega_\varrho))$ with $\Omega_\varrho = (\Omega - x_0)/\varrho$.

The following Euler–Lagrange inequality can be easily checked by comparing the energy of a quasi-minimiser u in $B_\varrho(x_0)$ with the energy of $u(\Phi_\varepsilon^{-1}(y))$, where $\Phi_\varepsilon(x) = x + \varepsilon\eta(x)$ and η is a Lipschitz map with compact support in $B_\varrho(x_0)$ (see [6, Section 7.4]).

Proposition 5.2 *If $u \in \mathcal{M}_\omega(\Omega)$ is a quasi-minimiser, $B_\varrho(x_0) \subset\subset \Omega$, $F(u, B_\varrho(x_0)) \le M$ and $\eta \in \mathrm{Lip}(B_\varrho(x_0), \mathbb{R}^N)$ has compact support in $B_\varrho(x_0)$, there exist $\varepsilon(\eta) > 0$ and $c(\eta, M)$ such that if $0 < |\varepsilon| < \varepsilon(\eta)$*

$$\varepsilon \int_{B_\varrho(x_0)} \left[|\nabla u|^2 \mathrm{div}\eta - 2\langle \nabla u, \nabla u \cdot \nabla\eta\rangle\right] dx + \varepsilon \int_{S_u} \mathrm{div}^{S_u}\eta \, d\mathcal{H}^{N-1} \ge -c(\eta, M)\varepsilon^2 - \omega\varrho^N . \tag{5.3}$$

Let us now consider the limit behaviour of a sequence of quasi-minimisers whose volume energies are infinitesimal. To simplify the presentation of proofs we have split our result in two parts. First, in the next proposition, we prove that limit of the jump sets is area stationary and then in Theorem 5.4 we show that this set is an area minimiser in the Almgren sense.

Proposition 5.3 *Let $u_h \in \mathcal{M}_{\omega_h}(\Omega)$ be a sequence of quasi-minimisers such that*

$$\nabla u_h \to 0 \text{ in } L^2_{\mathrm{loc}}(\Omega, \mathbb{R}^N), \qquad \omega_h \to 0 ,$$

$$\mathcal{H}^{N-1} \llcorner S_{u_h} \to \mu \text{ weakly}^* \text{ locally in } \Omega .$$

Then there exists a countably $\mathcal{H}^{N-1}$-rectifiable set $C \subset \Omega$ such that $\mu = \mathcal{H}^{N-1} \llcorner C$. Moreover C is area stationary, i.e.

$$\int_C \mathrm{div}^C \eta \, d\mathcal{H}^{N-1} = 0 \qquad \forall \eta \in C^1_0(\mathbb{R}^N, \mathbb{R}^N) . \tag{5.4}$$

Proof. The proof can be achieved arguing as in Theorem 8.8 of [6], where the stronger assumption that the quadratic oscillation of tangent planes was infinitesimal was made (with the stronger conclusion that C is a locally finite union of m-lanes). However the arguments used in the proof of that theorem still work in this more general situation. □

Theorem 5.4 *Let $u_h \in \mathcal{M}_{\omega_h}(\mathbb{R}^N)$ be a sequence of quasi-minimisers of F satisfying in $\mathbb{R}^N$ the assumptions of Proposition 5.3. Then the set C in the conclusion of the proposition is an Almgren area minimiser.*

Proof. Let us fix a Lipschitz map $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ such that $\{x \in \mathbb{R}^N : \varphi(x) \ne x\} \subset\subset B_R$. To prove that

$$\mathcal{H}^{N-1}(C \cap B_R) \le \mathcal{H}^{N-1}\left(\varphi(C \cap B_R)\right)$$

we may always assume, with no loss of generality, that $\varphi(B_R) \subset B_R$ and that $\mathcal{H}^{N-1}(C \cap \partial B_R) = 0$. Let us fix $\varepsilon \in (0, 1/2)$ and let us follow the argument of the proof of part (ii) of Theorem 4.1. In this way we can find a finite number of pairwise disjoint cylinders $C_i \subset\subset B_R$, $i = 1, \dots, m$, and of open sets $U_i \subset\subset C_i$ such that

$$\lim_{h\to\infty} \mathcal{H}^{N-1}((S_{u_h} \cap B_R) \setminus \cup_{i=1}^m C_i) < \varepsilon , \tag{5.5}$$

and such that for all i (see 4.9)

$$\mathcal{H}^{N-1}(C \cap C_i \setminus U_i) < c(N)\varepsilon \mathcal{H}^{N-1}(C \cap C_i) . \tag{5.6}$$

Moreover we can construct a Lipschitz map $\psi : \mathbb{R}^N \to \mathbb{R}^N$ such that

$$(5.7) \quad \mathrm{Lip}(\psi) < c(N), \qquad \psi(x) = x \quad \forall x \in \mathbb{R}^N \setminus \cup_{i=1}^m C_i, \qquad \psi(C_i) = C_i \quad \forall i = 1, \ldots, m$$

and such that for all i (see 4.8)

$$(5.8) \qquad \psi(S_{u_h} \cap C_i) \subset \psi(S_{u_h} \cap C_i \setminus U_i) \cup (C \cap C_i)\,.$$

Recalling the Deformation Lemma 2.4, we have also that if $v \in SBV(C_i)$, then $\psi_\# v \in SBV(C_i)$, that $\psi_\# v$ has the same trace of v on ∂C_i and that

$$(5.9) \qquad \int_{C_i} |\nabla(\psi_\# v)|^2 \, dy \leq c(N) \int_{C_i \setminus U_i} |\nabla v|^2 \, dx\,.$$

Finally, we have also, with the same notation used in the proof of Theorem 4.1,

$$(5.10) \qquad S_{\psi_\# v} \subset \Gamma_{\tilde{g}_i} \cup \psi(S_v)\,,$$

where $\Gamma_{\tilde{g}_i} \subset C_i$ is a Lipschitz graph with the property that

$$(5.11) \qquad \mathcal{H}^{N-1}(\Gamma_{\tilde{g}_i} \triangle (C \cap C_i)) \leq c(N) \varepsilon \mathcal{H}^{N-1}(C \cap C_i)\,.$$

Let us now set

$$v_h = \psi_\# u_h$$

and let us apply Theorem 3.1 to $S = C \cap B_R$ and to the map φ, thus getting two diffeomorphisms Φ, Ψ and a Lipschitz map γ as in the statement of that theorem. In particular we have

$$(5.12) \qquad \mathcal{H}^{N-1}((\Phi \circ \gamma \circ \Psi^{-1})(C \cap B_R)) < \mathcal{H}^{N-1}(\varphi(C \cap B_R)) + \varepsilon\,.$$

Then we set

$$w_h = \Phi_\#(\gamma_\#((\Psi^{-1})_\# v_h))\,.$$

Since Φ and Ψ are diffeomorphisms and γ is a piecewise affine function such that $\det \nabla \gamma > 0$ in each region where $\nabla \gamma$ is constant, from Theorem 2.3 it follows that $w_h \in SBV_{\mathrm{loc}}(\mathbb{R}^N)$ and that $\mathcal{H}^{N-1}\left(S_{w_h} \setminus (\Phi \circ \gamma \circ \Psi^{-1})(S_{v_h})\right) = 0$ for all h. Moreover w_h coincides with v_h (and thus with u_h) outside B_R and from (2.4), (5.9) we have

$$\int_{B_R} |\nabla w_h|^2 \, dx \leq c \int_{B_R} |\nabla v_h|^2 \, dx \leq c' \int_{B_R} |\nabla u_h|^2 \, dx\,,$$

where c' is a constant depending only on N and on γ. Therefore from the quasi-minimality of u_h, comparing $F(u_h, B_R)$ with $F(w_h, B_R)$, we have

$$(5.13) \qquad \begin{aligned} \lim_{h\to\infty} \mathcal{H}^{N-1}(S_{u_h} \cap B_R) &\leq \liminf_{h\to\infty} \mathcal{H}^{N-1}\left((\Phi \circ \gamma \circ \Psi^{-1})(S_{v_h} \cap B_R)\right) \\ &\leq (1+\varepsilon)^{N-1} \liminf_{h\to\infty} \mathcal{H}^{N-1}\left((\gamma \circ \Psi^{-1})(S_{v_h} \cap B_R)\right)\,. \end{aligned}$$

Recalling (5.7), we have that

$$S_{v_h} \cap B_R = \left(S_{u_h} \cap B_R \setminus \bigcup_{i=1}^m C_i \right) \cup (S_{v_h} \cap \bigcup_{i=1}^m C_i)$$

and from (5.10) and (5.8) we have the following inclusion

$$
\begin{aligned}
S_{v_h} \cap C_i = (S_{v_h} \cap C_i \setminus \Gamma_{\tilde g_i}) \cup (S_{v_h} \cap \Gamma_{\tilde g_i}) &\subset \psi(S_{u_h} \cap C_i) \cup (\Gamma_{\tilde g_i} \setminus (C \cap C_i)) \cup (C \cap C_i) \\
&\subset \psi(S_{u_h} \cap C_i \setminus U_i) \cup (\Gamma_{\tilde g_i} \setminus (C \cap C_i)) \cup (C \cap C_i) .
\end{aligned}
$$

Therefore, from (5.5), (5.6) and (5.11),we get

$$
\begin{aligned}
\liminf_{h\to\infty} \mathcal{H}^{N-1}\big((\gamma\circ\Psi^{-1})(S_{v_h} \cap B_R)\big) &\le [\mathrm{Lip}(\gamma\circ\Psi^{-1})]^{N-1} \lim_{h\to\infty} \mathcal{H}^{N-1}(S_{u_h} \cap B_R \setminus \cup_{i=1}^m C_i) \\
&\quad + [\mathrm{Lip}(\gamma\circ\Psi^{-1}\circ\psi)]^{N-1} \sum_{i=1}^m \limsup_{h\to\infty} \mathcal{H}^{N-1}(S_{u_h} \cap C_i \setminus U_i) \\
&\quad + [\mathrm{Lip}(\gamma\circ\Psi^{-1})]^{N-1} \sum_{i=1}^m \mathcal{H}^{N-1}(\Gamma_{\tilde g_i} \setminus (C\cap C_i)) + \mathcal{H}^{N-1}((\gamma\circ\Psi^{-1})(C \cap \cup_{i-1}^m C_i)) \\
&\le C_1\varepsilon\left[1 + \mathcal{H}^{N-1}(C\cap B_R)\right] + \mathcal{H}^{N-1}((\gamma\circ\Psi^{-1})(C\cap B_R)) ,
\end{aligned}
$$

where C_1 depends only on N and $\mathrm{Lip}(\gamma)$, hence only on N, $\mathrm{Lip}(\varphi)$ and R. Finally from the inequality above, (5.13) and (5.12) we obtain

$$
\begin{aligned}
\mathcal{H}^{N-1}(C\cap B_R) &= \lim_{h\to\infty} \mathcal{H}^{N-1}(S_{u_h} \cap B_R) \\
&\le c\varepsilon\left[1 + \mathcal{H}^{N-1}(C\cap B_R)\right] + (1+\varepsilon)^{2(N-1)} \mathcal{H}^{N-1}((\Phi\circ\gamma\circ\Psi^{-1})(C\cap B_R)) \\
&\le C_2\varepsilon\left[1 + \mathcal{H}^{N-1}(C\cap B_R)\right] + \mathcal{H}^{N-1}(\varphi(C\cap B_R)) ,
\end{aligned}
$$

with C_2 again depending only on N, $\mathrm{Lip}(\varphi)$ and R. The result then follows letting $\varepsilon \downarrow 0$. □

Theorem 5.5 *Let $u \in \mathcal{M}_\omega(\Omega)$ and let $x \in \overline{S}_u$ be a point such that*

$$
\lim_{\varrho\downarrow 0} \varrho^{1-N} \int_{B_\varrho(x)} |\nabla u|^2\, dy = 0 . \tag{5.14}
$$

Then for any sequence $\varrho_h \to 0$ there exist a subsequence ϱ_{h_j} and a closed set C such that

$$
\mathcal{H}^{N-1} \llcorner \frac{S_u - x}{\varrho_{h_j}} \to \mathcal{H}^{N-1} \llcorner C \qquad \text{weak}^* \text{ locally in } \mathbb{R}^N .
$$

Moreover C is an Almgren area minimiser.

Proof. We recall the *energy upper bound* (see [6, Section 7.2]) which states that if $u \in \mathcal{M}_\omega(B_\varrho(x))$ then

$$
\int_{B_\varrho(x)} |\nabla u|^2\, dy + \mathcal{H}^{N-1}(S_u \cap B_\varrho(x)) \le N\omega_N \varrho^{N-1} + \omega\varrho^N . \tag{5.15}
$$

Given the sequence ϱ_h, let us set $u_h(y) = \varrho_h^{-1/2} u(x + \varrho_h y)$ for $y \in (\Omega - x)/\varrho_h$. From Remark 5.1 it follows that $u_h \in \mathcal{M}_{\varrho_h\omega}((\Omega - x)/\varrho_h)$, while the assumption (5.14) implies that $|\nabla u_h| \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$. Moreover the energy upper bound (5.15) implies that the measures $\mathcal{H}^{N-1} \llcorner (S_u - x)/\varrho_h = \mathcal{H}^{N-1} \llcorner S_{u_h}$ are locally equibounded in $\mathbb{R}^N$. Therefore (up to a not relabelled subsequence) we may assume that the measures $\mathcal{H}^{N-1} \llcorner S_{u_h}$ converge weak* locally in $\mathbb{R}^N$ to a Radon measure μ. Then Proposition 5.3 and Theorem 5.4 imply that $\mu = \mathcal{H}^{N-1} \llcorner C$ for some Almgren area minimising set C. □

We recall the following regularity result proved in [4], [5] (see also [6, Ch. 8]). For any $u \in \mathcal{M}_\omega(\Omega)$ there exists an $\mathcal{H}^{N-1}$-negligible set $\Sigma(u) \subset \overline{S}_u \cap \Omega$, relatively closed in Ω, such that $\overline{S}_u \cap \Omega \setminus \Sigma(u)$ is an $(N-1)$-manifold of class $C^{1,1/4}$. Moreover there exist ε_0, R_0 depending only on ω and N such that

$$\Sigma(u) = \{x \in \overline{S}_u \cap \Omega : \mathcal{D}(x,\varrho) + \mathcal{A}(x,\varrho) \geq \varepsilon_0 \text{ for all } \varrho < R_0\}, \tag{5.16}$$

where the quantities $\mathcal{D}$ and $\mathcal{A}$ are defined in (1.1).

Theorem 5.6 *Let $u \in \mathcal{M}_\omega(\Omega)$ and let*

$$\Sigma' = \left\{ x \in \Sigma(u) : \lim_{\varrho \downarrow 0} \varrho^{1-N} \int_{B_\varrho(x)} |\nabla u|^2 \, dy = 0 \right\}.$$

Then $\mathcal{H}$-dim$(\Sigma') \leq N-2$.

Proof. Let $s \in (N-2, N-1)$. We claim that $\mathcal{H}^s(\Sigma') = 0$. To prove this claim we argue by contradiction, assuming that $\mathcal{H}^s(\Sigma') > 0$. If this is true then we have also $\mathcal{H}^s_\infty(\Sigma') > 0$ and (see [21, Theorem 3.6]) for $\mathcal{H}^s$-a.e. $x \in \Sigma'$

$$\limsup_{\varrho \downarrow 0} \varrho^{-s} \mathcal{H}^s_\infty(\Sigma' \cap B_\varrho) \geq \frac{\omega_s}{2^s}. \tag{5.17}$$

Let us fix a point $x \in \Sigma'$ such that (5.17) holds and let assume for simplicity that $x = 0$. Let us also denote by ϱ_h an infinitesimal sequence such that

$$\mathcal{H}^s_\infty(\Sigma' \cap \overline{B}_{\varrho_h}) \geq \frac{\omega_s}{2^{s+1}} \varrho_h^s. \tag{5.18}$$

Then from Theorem 5.5 it follows that, up to a subsequence, $\mathcal{H}^{N-1} \llcorner S_u/\varrho_h \to \mathcal{H}^{N-1} \llcorner C$ weak* locally in $\mathbb{R}^N$, where C is an Almgren area minimising set. Let us set $\Sigma'_h = \Sigma'/\varrho_h$. Given any open set A containing $\Sigma(C) \cap \overline{B}_1$, let us show the existence of h_0 such that

$$\Sigma'_h \cap \overline{B}_1 \subset A \qquad \forall h \geq h_0. \tag{5.19}$$

In fact, otherwise we could find a sequence of points $x_{h_j} \in \Sigma'_{h_j} \cap \overline{B}_1 \setminus A$ converging to a point $x_0 \notin \Sigma(C)$. Since the approximate tangent plane $\pi^C_{x_0}$ to C at x_0 exists, there exists ϱ such that

$$\varrho^{-1-N} \int_{C \cap B_\varrho(x_0)} \text{dist}^2(y, \pi^C_{x_0}) \, d\mathcal{H}^{N-1} < \varepsilon_0,$$

where ε_0 is as in (5.16). Hence we have that

$$\lim_{j \to \infty} \varrho^{-1-N} \int_{S_u/\varrho_{h_j} \cap B_\varrho(x_{h_j})} \text{dist}^2(y, \pi^C_{x_0}) \, d\mathcal{H}^{N-1} < \varepsilon_0.$$

Therefore, by (5.16), for j large enough $x_{h_j} \notin \Sigma'_{h_j}$. This contradiction shows (5.19) and then from (5.18) it follows that

$$\mathcal{H}^s\left(\Sigma(C) \cap \overline{B}_1\right) \geq \mathcal{H}^s_\infty\left(\Sigma(C) \cap \overline{B}_1\right) \geq \limsup_{h \to \infty} \mathcal{H}^s_\infty\left(\Sigma'_h \cap \overline{B}_1\right) \geq \frac{\omega_s}{2^{s+1}}.$$

Then, the contradiction follows by Theorem 4.3. $\square$

Assuming higher integrability of the gradient we can obtain an estimate on the Hausdorff dimension of the full singular set $\Sigma(u)$.

Corollary 5.7 *Let $u \in \mathcal{M}_\omega(\Omega)$. If $\nabla u \in L^p_{\rm loc}(\Omega; \mathbb{R}^N)$ for some $p > 2$ then*

$$\mathcal{H}\text{-dim}(\Sigma(u)) \le \max\{N-2, N-p/2\}\,.$$

Proof. Let us fix $s \in (N - p/2, N-1)$. We set

$$\Lambda_s = \{x \in \Omega : \limsup_{\varrho \downarrow 0} \varrho^{-s} \int_{B_\varrho(x)} |\nabla u|^p\,dy > 0\}$$

and recall that $\mathcal{H}^s(\Lambda_s) = 0$. Then the result will follow from (5.16) and from Theorem 5.6 if we show that $\Sigma(u) \setminus \Lambda_s \subset \Sigma'$. In fact, notice that if $x \notin \Lambda_s$ then we have

$$\varrho^{1-N} \int_{B_\varrho(x)} |\nabla u|^2\,dy \le \omega_N^{1-2/p} \varrho^\delta \left(\varrho^{-s} \int_{B_\varrho(x)} |\nabla u|^p\,dy\right)^{2/p},$$

where $\delta = 1 + 2(s-N)/p > 0$, and the right hand side of this inequality is infinitesimal as $\varrho \downarrow 0$. □

We conclude this section showing that if u is a quasi-minimiser of F, then at any singular point of $\overline{S}_u$ where the rescaled Dirichlet integral $\mathcal{D}(x,\varrho)$ goes to zero there exists a blow-up limit C of $\overline{S}_u$ which is a cone. In two dimensions this property, together with the fact that C is an Almgren area minimiser, implies that C is a *propeller*, i.e. the set consisting of three half-lines meeting at a point with equal angles.

Proposition 5.8 *Let u be a quasi-minimiser of the functional F and let $x \in \Sigma_u$ be a point satisfying (5.14). Then there exists a sequence $\varrho_h \downarrow 0$ such that $\mathcal{H}^{N-1} \llcorner (S_u - x)/\varrho_h \to \mathcal{H}^{N-1} \llcorner C$, where C is an Almgren area minimising cone. Moreover:*
(a) if $N = 2$, C is a propeller;
(b) if $N = 3$, C is either the three sheeted cone consisting of three half planes meeting along a line at equal angles or is the cone over the 1-skeleton of a tethraedron with vertex at the center of the tethraedron.

Proof. Let us fix $x \in \Sigma_u$, such that (5.14) holds and let r_i be an infinitesimal sequence such that $\mathcal{H}^{N-1} \llcorner (S_u - x)/r_i \to \mathcal{H}^{N-1} \llcorner \tilde{C}$, where by Theorem 5.5 $\tilde{C}$ is an Almgren area minimiser. Moreover from the proof of Theorem 5.6 it is clear that 0 is a singular point of $\tilde{C}$. From [22, Corollary II.2] we know that there exists an increasing sequence n_h, with $n_h \in \mathbb{N}$, such that the sets $n_h \tilde{C}$ converge to an Almgren area minimiser tangent cone C as $h \to \infty$. Since for all h $\mathcal{H}^{N-1} \llcorner n_h (S_u - x)/r_i \to \mathcal{H}^{N-1} \llcorner n_h \tilde{C}$ as $i \to \infty$, we get easily that there exists an infinitesimal sequence $\varrho_h = n_h / r_{i_h}$ such that $\mathcal{H}^{N-1} \llcorner (S_u - x)/\varrho_h \to \mathcal{H}^{N-1} \llcorner C$.
The last part of the assertion then follows again from [22, Proposition II.3] □

Remark 5.9 If u is a local minimiser of F satisfying the assumptions of Corollary 5.7, the conclusion of Proposition 5.8 can be strengthened. In fact it is possible to show that if $x \in \Sigma_u$ is a point satisfying (5.14) and ϱ_h is *any* infinitesimal sequence such that $\mathcal{H}^{N-1} \llcorner (S_u - x)/\varrho_h \to \mathcal{H}^{N-1} \llcorner C$, then C is a *cone* and hence, by Proposition 5.4, an Almgren area minimising cone. The proof can be obtained by deriving a suitable monotonicity formula for S_u and then passing to the limit in that formula.

6 Final remarks

In this section we prove that if $N = 2$ and $u \in SBV_{\rm loc}(\Omega)$ is a local minimiser of the functional F then ∇u is p-summable for any $p < 4$ in a neighbourhood of any *crack tip* point or any *triple junction.*

Lemma 6.1 *Let $A \subset \mathbb{R}^2$ be a connected open set and let $\Gamma \subset \mathbb{R}^2$ be a $C^{1,1}$ graph such that $A \setminus \Gamma = A_1 \cup A_2$, where A_1 and A_2 are connected open sets. Let $u \in W^{1,2}(A\setminus\Gamma)$ be a weak solution of the equations*

$$(6.1) \qquad \int_{A_i} \langle \nabla u, \nabla\eta\rangle = 0 \qquad \forall \eta \in C^1_0(A),\ i = 1,2\ .$$

For all $p > 2$ there exists $c_p > 0$ depending only on p, Γ such that if $P_0 \in \Gamma$ and $B_{2\varrho}(P_0) \subset A$, then

$$(6.2) \qquad ⨍_{B_\varrho(P_0)} |\nabla u|^p \le c_p \left(⨍_{B_{2\varrho}(P_0)} |\nabla u|^2 \right)^{p/2} \ .$$

Proof. By rotating and translating we may always assume that $P_0 = (0,0)$ and that $\Gamma = \{(x,y) : a \le x \le b,\ y = \varphi(x)\}$ for some $\varphi \in C^{1,1}([a,b])$. We set $L = \sqrt{1+\|\varphi'\|^2_\infty}$ and $R_t = (-t,t) \times (-4Lt, 4Lt)$ for $t > 0$. Let us fix ϱ so that $R_{2\varrho} \subset A$. Let us set also $\Phi(x,y) = (x, y-\varphi(x))$, $U_2 = \Phi(R_{2\varrho})$, $U_1 = \Phi(R_\varrho)$, while T denotes the x-axis and, if $U \subset \mathbb{R}^2$ is any open set, $U^\pm$ is the set of points of U respectively above or below T. Moreover it is easy to check that there exists a strictly positive constant c depending only on L such that

$$(6.3) \qquad \mathrm{dist}(\partial U_2, U_1) \ge c\varrho \ .$$

The function $v(r,s) = u\left(\Phi^{-1}(r,s)\right)$ is a weak solution of the equation

$$(6.4) \qquad \int_{U_2^+} a_{ij} \nabla_i v \nabla_j \eta \, dr ds = 0$$

for all $\eta \in C^1(\overline{U}_2^+)$ vanishing in a neighbourhood of $\partial U_2^+ \setminus T$, where $a_{11} = 1$, $a_{12} = a_{21} = -\varphi'(r)$, $a_{22} = 1 + \varphi'^2(r)$. Let us extend v and the coefficients a_{ij} to U_2 setting for all $(r,s) \in U_2^-$

$$v(r,s) = v(r,-s), \quad a_{11} = 1, \quad a_{12} = a_{21} = \varphi'(r), \quad a_{22} = 1+\varphi'^2(r) \ .$$

In this way we get immediately that for all $\eta \in C^1_0(U_2)$

$$\int_{U_2} a_{ij}\nabla_i v \nabla_j \eta \, dr ds = 0 \ .$$

By a standard difference quotient argument we then have that $\nabla_s v \in W^{1,2}_{\rm loc}(U_2)$ and that for all $\eta \in C^1_0(U_2)$

$$(6.5) \qquad \int_{U_2} |\nabla(\nabla_s v)|^2 \eta^2 \, dr ds \le c \int_{U_2} |\nabla v|^2 |\nabla \eta|^2 \, dr ds \ ,$$

where c depends on $L, \mathrm{Lip}(\varphi')$. By (6.3) we can find a function $\eta \in C^1_0(U_2)$, such that $0 \le \eta \le 1$, $\eta \equiv 1$ on U_1 and $|\nabla\eta| \le c/\varrho$. Inserting this function η in (6.5) we have by the Sobolev–Poincaré inequality that for all $p > 2$

$$(6.6)\qquad \left(\varrho^{-2}\int_{U_1}|\nabla_s v|^p\right)^{1/p} \le c(p)\varrho\left(\varrho^{-2}\int_{U_2}|\nabla(\eta\nabla_s v)|^2\right)^{1/2}$$
$$\le c\varrho\left(\varrho^{-2}\int_{U_2}|\nabla v|^2|\nabla\eta|^2\right)^{1/2} \le c\left(\varrho^{-2}\int_{U_2}|\nabla v|^2\right)^{1/2} \quad .$$

Integrating by parts the equation satisfied by v we have that for all $\eta \in C_0^1(U_2)$

$$\int_{U_2}\nabla_r\eta(a_{11}\nabla_r v + a_{12}\nabla_s v) = -\int_{U_2}\nabla_s\eta(a_{12}\nabla_r v + a_{22}\nabla_s v) = \int_{U_2}\eta\frac{\partial}{\partial s}(a_{12}\nabla_r v + a_{22}\nabla_s v),$$

hence $|\nabla_r(a_{11}\nabla_r v + a_{12}\nabla_s v)| \le c|\nabla_s(\nabla v)|$. In particular we have that $|\nabla^2_{rr}v| \le c[|\nabla v| + |\nabla(\nabla_s v)|]$ and thus that $|\nabla(\nabla_r v)| \le c[|\nabla v| + |\nabla(\nabla_s v)|]$. Therefore, arguing as before we have also that for all $p > 2$

$$\left(\varrho^{-2}\int_{U_1}|\nabla_r v|^p\right)^{1/p} \le c\left(\varrho^{-2}\int_{U_2}|\nabla v|^2\right)^{1/2}$$

and this inequality together with (6.6) immediately implies the assertion. □

Remark 6.2 Notice that in the above lemma, since u is harmonic in $A \setminus \Gamma$, the inequality (6.2) clearly holds with another constant c_p, depending only on p, if the ball $B_{2\varrho}(P_0)$ is contained in $A \setminus \Gamma$.

We are now in position to prove the desired property of SBV minimisers of the functional F.

Proposition 6.3 *Let $A \subset \mathbb{R}^2$ be open and $\Gamma = \cup_{i=1}^M \Gamma_i \subset\subset A$, where each Γ_i is a $C^{1,1}$ graph. Assume that if $i \neq j$ then either $\Gamma_i \cap \Gamma_j = \emptyset$ or they intersect with a strictly positive angle at a finite number of points. Let $u \in SBV_{\rm loc}(A)$ be a local minimiser of F. Then $\nabla u \in L^p_{\rm loc}(A, \mathbb{R}^2)$ for all $p < 4$.*

Proof. We limit ourselves to prove that if $P_0 \in \Gamma_i$ for all $i = 1, \dots, M$ then $\nabla u \in L^p$ for all $p < 4$ in a neighbourhood of P_0, since the other possible cases can be dealt with in a similar (and simpler) way. To this aim notice that we may assume with no loss of generality that $P_0 = (0, 0)$, that P_0 is an endpoint of all the curves Γ_i and that there exist a ball B_R such that in $B_R \setminus \{P_0\}$ the curves do not intersect and do not have other endpoints. Moreover, since the curves intersect each other at P_0 with positive angles it is easy to check that there exists a constant $\nu_0 \in (0, 1/2)$ such that if $3\varrho < R$ then

$$\mathrm{dist}(\Gamma_i \cap B_{3\varrho} \setminus \overline{B}_{\varrho/2}, \Gamma_j \cap B_{3\varrho} \setminus \overline{B}_{\varrho/2}) > \nu_0\varrho \qquad \forall i \neq j \ .$$

Let us fix $\varrho < R/3$ and denote by $\mathcal{F}$ the covering of $B_{2\varrho} \setminus \overline{B}_\varrho$ containing either closed balls of the type $\overline{B}_{\nu_0\varrho/4}(P)$, with $P \in \Gamma_i \cap B_{2\varrho} \setminus \overline{B}_\varrho$ for some i, or balls of the type $\overline{B}_{\nu_0\varrho/8}(P)$, with $P \in B_{2\varrho} \setminus \overline{B}_\varrho$ and $\mathrm{dist}(P, \Gamma_i) \ge \nu_0\varrho/4$ for all i. Notice that in the first case (when $P \in \Gamma_i$ for some i) the open ball $B_{\nu_0\varrho/2}(P)$ does not intersect neither the endpoints of Γ_i nor the other curves and in the second case trivially the open ball $B_{\nu_0\varrho/4}$ does not intersect any of the curves Γ_i. By the Besicovitch covering theorem we can extract a finite number ξ (with ξ an absolute constant) of disjoint subfamilies $\mathcal{F}_h$ of $\mathcal{F}$ so that the family $\mathcal{G} = \cup_{h=1}^{\xi}\mathcal{F}_h$ is still a covering of $B_{2\varrho} \setminus \overline{B}_\varrho$. Since the balls in each family $\mathcal{F}_h$ are pairwise disjoint and have radius comparable with ϱ, we have

$\#(\mathcal{G}) \leq \gamma$, where γ depends only on ξ and ν_0. From Lemma 6.1, Remark 6.2 and the energy upper bound (5.15) for any $p > 2$ we have

$$\int_{B_{2\varrho}\setminus\overline{B}_\varrho} |\nabla u|^p \leq \sum_{B_{r_i}(P_i)\in\mathcal{G}} \int_{B_{r_i}(P_i)} |\nabla u|^p \leq c_p \nu_0^2 \varrho^2 \sum_{B_{r_i}(P_i)\in\mathcal{G}} \left(\fint_{B_{r_i}(P_i)} |\nabla u|^2 \right)^{p/2} \leq c\gamma\varrho^{2-p/2} \ .$$

Therefore from this inequality, if $p < 4$ we may conclude that

$$\int_{B_{R/2}} |\nabla u|^p = \sum_{i=1}^{\infty} \int_{B_{R/2^i}\setminus B_{R/2^{i+1}}} |\nabla u|^p \leq c \sum_{i=1}^{\infty} \left(\frac{R}{2^i}\right)^{2-p/2} < \infty,$$

which proves the assertion. $\square$

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