1 Introduction

Suppose $S \subset I\!R^n$ is a smooth surface with boundary, having the same topological type as the closed unit disc $\overline{D}$. Is there in some sense an "optimal" parametrisation $Y : \overline{\Omega} \to S$ where we consider both arbitrary (simply connected) $\Omega \subset I\!R^2$ and arbitrary $Y : \overline{\Omega} \to S$? We give a positive answer to various formulations of this question, which we call the *Cartography Problem.*

For any disc-like $\overline{\Omega} \subset I\!R^2$ there is a *conformal* diffeomorphism $Y : \overline{\Omega} \to S$. Moreover, such Y minimises $\int_\Omega |DY|^2$ among all diffeomorphisms $Y : \overline{\Omega} \to S$ for which $Y|_{\partial\Omega}$ is monotone. It is clear, however, from the invariance of $\int_\Omega |DY|^2$ under conformal changes of the domain, that an optimal domain Ω_0 cannot be obtained by minimising $\int_\Omega |DY|^2$ for different Ω.

Instead, motivated by the idea that for an optimal representation (Ω_0, Y_0) the map Y_0 should be as close to linear as possible, we consider minimising the *Hessian Functional*

$$\int_\Omega |D^2Y|^2$$

over suitable (Ω, Y). But note that if $\rho_r : I\!R^2 \to I\!R^2$ is the scaling map $\rho_r(x) = rx$ then

$$\int_{\rho_r(\Omega)} |D^2(Y \circ \rho_r^{-1})|^2 = r^{-2} \int_\Omega |D^2Y|^2 \to 0 \quad \text{as} \quad r \to \infty.$$

For this reason we need to impose some further constraint on Ω.

We are thus led to the following minimisation problems:

M1 *Suppose $S \subset I\!R^n$ is a smooth immersed surface with boundary and S has the same topological type as the closed unit disc $\overline{D}$.*

Minimise

$$\int_\Omega |D^2Y|^2$$

over pairs (Ω, Y), where $\Omega \subset I\!R^2$ is diffeomorphic to D, $Y : \overline{\Omega} \to S$ is a **conformal** *diffeomorphism, and either $|\Omega| = \pi$ or $\mathcal{H}^1(\partial\Omega) = 2\pi$.*

For example, let $S \subset I\!R^3$ be the surface given by

$$S = \{(r\cos\theta, r\sin\theta, \epsilon\theta) : 1/\sqrt{3} \le r \le 1, 0 \le \theta \le 3\pi\},$$

where $\epsilon > 0$. Then for ϵ sufficiently small we expect the problem $M1$ to have the solution (Ω_0, Y_0) approximately described by

$$\Omega_0 \approx \{(r\cos\theta, r\sin\theta) : 1/\sqrt{3} < r < 1, 0 < \theta < 3\pi\} \subset I\!R^2,$$

i.e. Ω_0 is counted with multiplicity 2 in the region corresponding to both $0 < \theta < \pi$ and $2\pi < \theta < 3\pi$, and

$$Y_0(r\cos\theta, r\sin\theta) \approx (r\cos\theta, r\sin\theta, \epsilon\theta),$$

i.e. Y_0 is a "multiple-valued" function.

Rather than vary the domain Ω, and also in order to handle the difficulty of domains counted with "multiplicity" as above, we reformulate the problem as follows:

Let D be the open unit disc. *Fix* a conformal diffeomorphism $X:\overline{D} \to S$. For any disc-like $\Omega \subset I\!R^2$ there exists a conformal diffeomorphism $F:\overline{D} \to \overline{\Omega}$, and moreover $Y(:= X \circ F^{-1}) : \overline{\Omega} \to S$ is a conformal diffeomorphism. Conversely, for any conformal diffeomorphism $Y : \overline{\Omega} \to S$ the map $F(:= Y^{-1} \circ X) : \overline{D} \to \Omega$ is a conformal diffeomorphism and $Y = X \circ F^{-1}$.

For F as above, let $e^{2h(x)}\delta_{ij}$ be the metric induced on D from F and from the standard metric on Ω. Since $e^{2h}\delta_{ij}$ is a flat metric it follows $\triangle h = 0$.

Conversely, suppose $\triangle h = 0$ where $h : D \to I\!R$. Define $\overline{F} : D \to \mathbf{C} \cong I\!R^2$ by $\overline{F}(z) = \int_0^z e^{h+ik}$ where k is the harmonic conjugate of h for which $k(0) = 0$. Then $\overline{F}$ is conformal and $|\overline{F}'(z)| = e^h$, so that $e^{2h}\delta_{ij}$ is the metric induced on D from $\overline{F}$. Moreover, if h is defined from F as above, then F and $\overline{F}$ agree up to composition with an isometry.

With $g = g_{ij} = e^{2h}\delta_{ij}$, let $\nabla^2_g X$ be the covariant Hessian of X with respect to g, and $|\nabla^2_g X|$ be the length of $\nabla^2_g X$ with respect to the metric g on D and the Euclidean metric on $I\!R^n$. Then $|\nabla^2_g X|(w) = |D^2 Y|(F(w))$ by isometric invariance (or by direct calculation).

Motivated by the preceding, we consider the following minimisation problems:

M2 *Suppose $S \subset I\!R^n$ is a smooth immersed surface with boundary and $X:\overline{D} \to S$ is a fixed conformal diffeomorphism.*

Minimise

$$\mathcal{E}(h) := \int_D |\nabla^2_g X|^2 dg$$

over all functions $h : D \to I\!R$, where $g = g_{ij} = e^{2h}\delta_{ij}$, $\triangle h = 0$, and either $\int_D e^{ph} = \pi$ (some fixed $p > 0$), or $\int_{\partial D} e^{ph} = 2\pi$ (some fixed $p > 0$), or $h(0) = 0$.

We denote by $\mathcal{M}$ the class of competing functions h.

The problems $M1$ and $M2$ are essentially equivalent if we take the constraints $\int_D e^{2h} = \pi$ or $\int_{\partial D} e^h = 2\pi$ respectively.

We will initially *not* assume in $M2$ that the map X is conformal. The problem analogous to $M1$ is then one of finding the "optimal" parametrisation of S in the conformal class corresponding to the pair (D, X).

Note that the trivial case for $M1$ is $S \subset I\!R^2$. Suppose $\Omega = \rho S$ where ρ is chosen so $|\Omega| = \pi$ or $\mathcal{H}^1(\partial\Omega) = 2\pi$. Let $Y(x) = \rho^{-1}x$. Then clearly $\int_\Omega |D^2 Y|^2 = 0$ and (Ω, Y) is the unique (up to isometry) minimiser for $M1$.

Similarly, for $M2$ suppose $X : \overline{D} \to S(\subset I\!R^2)$ is a conformal (for simplicity) parametrisation. Then $\mathcal{E}(h) = 0$ iff $e^h = c|DX|$ for some constant c chosen so $h \in \mathcal{M}$. This follows from observing that X is a scaling map with respect to the standard metric on S and the pull-back metric $g = e^{2h}\delta_{ij} = \frac{1}{2}|DX|^2\delta_{ij}$ on D, and so $\nabla_g^2 X \equiv 0$. If $F(z) := \int_0^z e^{h+ik}$ as before, then $F[D]$ is just a scaled version of S and F is a constant multiple of X (up to isometry).

The problem of minimising higher order geometric quantities involving curvature has been considered recently in a number of situations. In [Hu] the existence of a minimiser of functionals involving the second fundamental form was shown in the context of curvature varifolds. See also [AST] and [Ma]. In [Si] the existence of a smooth torus minimising the Willmore functional is established. The functional considered here is related, see in particular Proposition 6.2.

The outline of the paper is as follows.

In Section 2 we consider some model linear problems (linear in the sense of the associated Euler-Lagrange system). While these problems do not have the same geometric content as the main problem, they are none-the-less interesting and the arguments involved are much simpler. In particular, existence, uniqueness, and regularity are relatively straightforward.

We next show that problems $M1$ and $M2$ have a solution. This is perhaps somewhat surprising. The energy functional in $M1$ involves the *second* derivatives of the map Y. However, passing to the equivalent problems involving the metric h, the energy integrand $E(h)$ in $M2$ is quadratic in e^{-h} and its first derivatives. The precise expression, see Proposition 4.2, is

$$E(h) = \sum_{\alpha=1}^{n} \left| e^{-h} D^2 X^\alpha + D(e^{-h}) \odot DX^\alpha \right|^2 . \tag{1}$$

The quantity $D(e^{-h}) \odot DX^\alpha$ is a type of symmetric product of the vectors $D(e^{-h})$ and DX^α and is a 2×2 symmetric matrix (c.f. (5) in Section 3).

Our aim is to prove the existence of a minimiser of $\mathcal{E}(h)$ for $h \in \mathcal{M}$. One could work with the function $\psi = e^{-h}$, but this is not so convenient. In particular, one then has the *open* constraint $\psi > 0$ on D and the constraint $\triangle h = 0$ becomes nonlinear in ψ. These considerations are more important when one considers the question of regularity of solutions.

The first problem then is to find some sort of coercivity type estimate for the e^{-h}. This is not at all clear initially, and depends on an analysis of the precise form of the energy integrand. See Proposition 4.7 (also Lemmas 4.4.*3* and 4.5) for a pointwise estimate and Proposition 5.2 for the $W^{1,2}$ estimate. The latter estimate requires that the surface S *not* be flat, and in fact the estimate degenerates as S becomes flat.

The existence of a minimiser is obtained by considering two cases, see Theorem 5.3. If S is flat one can classify all minimisers. If S is not flat, the problem is to show that a minimising sequence h_j has a limit in $\mathcal{M}$. It follows from the estimates on e^{-h} that the h_j are bounded uniformly from below on compact subsets of D. The integral constraints imposed on members of $\mathcal{M}$ together with Rouché's theorem (or Harnack's inequality) imply that the h_j are bounded uniformly from above on compact subsets. The existence of a minimiser then follows from the particular form of the integral constraints. Analogous results for branched immersions and for univalent maps are also obtained.

In Section 6 we obtain expressions for the energy functional in the case X is conformal. As a consequence the Weierstrass representation of Enneper's surface is shown to be the unique optimal parametrisation. Moreover, stereographic projection of the unit disc onto a spherical cap is also optimal. For the latter one first finds another appropriate form of the energy functional and then, with Mathematica™ as a guide, shows the positivity of various integrals.

Note that a minimising map h is analytic on D (being harmonic) and the map F is hence analytic and locally invertible on D. Thus we automatically have full interior regularity of the minimising metric. In [FH] we consider the question of boundary regularity of minimisers, as well as related Plateau and Dirichlet type problems.

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2 Some Linear Model Problems

Let $D = \{x \in I\!R^2 : |x|^2 < 1\}$ be the unit disk in $I\!R^2$. Let $X : \overline{D} \to I\!R^n$ be a fixed immersion and let S be the corresponding surface.

Let
$$e^{2\overline{h}} = \frac{1}{2}|DX|^2. \tag{2}$$
Thus in case X is conformal, $e^{\overline{h}}$ is the conformal factor of X.

The following result is completely straightforward. It shows that for *any* function $\overline{h}$ in various function spaces there is a unique closest harmonic function h in the relevant norm.

Theorem 2.1 *Suppose $\overline{h} \in L^2(D)$ or $\overline{h} \in W^{1,2}(D)$ respectively. Let*
$$\mathcal{M} = L^2(D) \cap \{h : \triangle h = 0\}$$
or
$$\mathcal{M} = W^{1,2}(D) \cap \{h : \triangle h = 0\}$$
respectively. Let
$$\mathcal{E}(h) = \frac{1}{2}\int_D |h - \overline{h}|^2 \tag{3}$$

or

$$\mathcal{E}(h) = \frac{1}{2}\int_D |h - \overline{h}|^2 + |Dh - D\overline{h}|^2 \tag{4}$$

respectively. Then there exists a unique $h \in \mathcal{M}$ which minimises $\mathcal{E}$.

Proof: Let

$$\gamma = \inf\{\mathcal{E}(h) : h \in \mathcal{M}\}.$$

Let (h_j) be a minising sequence..

Passing to a subsequence, $h_j \rightharpoonup h$ (*say*) weakly in L^2 or $W^{1,2}$ respectively. By regularity theory for harmonic functions, $h_j \to h$ uniformly on compact subsets of D, together with all derivatives. In particular $\triangle h = 0$. Moreover, by Fatou's Lemma, $\mathcal{E}(h) \leq \gamma$, and so $\mathcal{E}(h) = \gamma$ as $h \in \mathcal{M}$. Thus h is the required minimiser.

Uniqueness follows from Taylor's formula in the usual way. Thus in the L^2 case we write

$$\int_D |k - \overline{h}|^2 = \int_D |h - \overline{h}|^2 + 2(h - \overline{h})(k - h) + (k - h)^2.$$

The second integrand on the right integrates to zero by the stationarity of h and so $\mathcal{E}(k) \geq \mathcal{E}(h)$ with equality iff $k = h$. The argument for the $W^{1,2}$ case is similar. ∎

Application Suppose $\overline{h}$ is defined as in (2). If h is the minimiser in either of the previous problems, let k be the harmonic conjugate of h (such that $k(0) = 0$) and define

$$F(z) = \int_0^z e^{h+ik}.$$

Then $F[D]$ (which may be "multiple-valued" as discussed in the Introduction) together with the map $X \circ F^{-1}$ is a natural candidate for the optimal parametrisation of S.

Note, however, that $\mathcal{E}(h)$ here only considers the *intrinsic* curvature of S, unlike the functional considered elsewhere in the paper. One could attempt to account for extrinsic curvature by including, for example, a term in the energy functional of the form $\int_D a^2(x)\, h^2$. But this is not so geometrically natural. The more important point, however, is that the natural geometric quantities are the conformal factors e^h and $e^{\overline{h}}$, rather than h and $\overline{h}$.

Regularity of solutions to the present problems follows from the appropriate Euler-Lagrange system. Note that the Lagrange multiplier f is a member of an appropriate function space.

Theorem 2.2 *Suppose $\overline{h} \in L^2(D)$ and h is the minimiser of (3) in the previous theorem. Then h satisfies the following system of equations:*

$$\begin{array}{llcll} (i) & \triangle f & = & h - \overline{h} & \text{in } D \\ (ii) & \triangle h & = & 0 & \text{in } D \\ (iii) & f & = & 0 & \text{on } \partial D \\ (iv) & \dfrac{\partial f}{\partial \nu} & = & 0 & \text{on } \partial D \end{array}$$

for some $f \in W^{2,2}(D)$, where the equations are interpreted in weak or trace sense. If $\overline{h} \in W^{k,2}(D)$ then $f \in W^{k+2,2}(D)$ and $h \in W^{k,2}(D)$.

PROOF: Define

$$\triangle : L^2(D) \to W_0^{2,2}(D)^*$$

by

$$\langle \triangle k, f \rangle := \int_D k \triangle f$$

for all $f \in W_0^{2,2}(D)$. By standard regularity theory, $k \in \ker \triangle$ iff $\triangle k = 0$ in the classical sense.

Extend $\mathcal{E}$ to all of $L^2(D)$ by the same definition as in (3). The derivative of $\mathcal{E}$ at h is defined by

$$\begin{aligned} \langle d\mathcal{E}(h), k \rangle &:= \left. \frac{d}{dt} \right|_{t=0} \mathcal{E}(h + tk) \\ &= \int_D (h - \overline{h}) k, \end{aligned}$$

and is a bounded linear operator on $L^2(D)$.

If h is the minimiser of $\mathcal{E}(h)$ in $\mathcal{M}$, it follows

$$d\mathcal{E}(h) : \ker \triangle \to \{0\},$$

and so

$$d\mathcal{E}(h) = T \circ \triangle$$

for some bounded linear operator $T : W_0^{2,2}(D)^* \to I\!R$.

By the reflexivity of $W_0^{2,2}(D)$, T corresponds to a function $f \in W_0^{2,2}(D)$ and so for all $k \in L^2(D)$,

$$\begin{aligned} \langle d\mathcal{E}(h), k \rangle &= \langle T, \triangle k \rangle \\ &= \langle \triangle k, f \rangle \\ &= \int_D k \triangle f. \end{aligned}$$

That is,

$$\int_D (h - \overline{h}) k = \int_D k \triangle f$$

for all $k \in L^2(D)$. The Euler-Lagrange system follows.

For the regularity, we note

$$\triangle^2 f = -\triangle \overline{h}$$

in weak form. If $\overline{h} \in W^{k,2}(D)$ it follows, using *(iii)* and *(iv)*, that $f \in W^{k+2,2}(D)$. It then follows from *(i)* that $h \in W^{k,2}(D)$. ∎

We similarly have:

Theorem 2.3 *Suppose $\overline{h} \in W^{1,2}(D)$ and h is the minimiser of (4) in Theorem 2.1. Then h satisfies the following system of equations:*

$$\begin{array}{llcll}
(i) & \triangle f + \triangle h - h & = & \triangle \overline{h} - \overline{h} & \text{in } D \\
(ii) & \triangle h & = & 0 & \text{in } D \\
(iii) & f & = & 0 & \text{on } \partial D \\
(iv) & \dfrac{\partial}{\partial \nu}(f+h) & = & \dfrac{\partial \overline{h}}{\partial \nu} & \text{on } \partial D.
\end{array}$$

for some $f \in W^{1,2}(D)$, where the equations are interpreted in the appropriate weak or trace sense. If $\overline{h} \in W^{k,2}(D)$ then $f \in W^{k,2}(D)$ and $h \in W^{k,2}(D)$.

PROOF: The proof is similar. But now

$$\triangle : W^{1,2}(D) \to W_0^{1,2}(D)^*$$

is defined by

$$\langle \triangle k, f \rangle := -\int_D Dk \cdot Df$$

for all $f \in W_0^{1,2}(D)$.

The derivative of $\mathcal{E}$ at h is given by

$$\langle d\mathcal{E}(h), k \rangle = \int_D (h - \overline{h})k + (Dh - D\overline{h}) \cdot Dk\,.$$

The Lagrange multiplier $f \in W_0^{1,2}(D)$ satisfies

$$\int_D (h - \overline{h})k + (Dh - D\overline{h}) \cdot Dk = -\int_D Dk \cdot Df$$

for all $k \in W^{1,2}(D)$. This is just the weak formulation of *(i)* and *(iv)*.

If $\overline{h} \in W^{3,2}(D)$, it follows from *(i)* and *(iv)* that $f+h \in W^{3,2}(D)$. Writing *(iii)* in the form $h = f + h$, it follws from *(ii)* and *(iii)* that $h \in W^{3,2}(D)$.

If $\overline{h} \in W^{5,2}(D)$, it now follows from *(i)* and *(iv)* that $f + h \in W^{5,2}(D)$. It then follows as before that $h \in W^{5,2}(D)$, etc. ∎

3 Algebraic Preliminaries

In order to study the energy integrand $E(h)$ in (1) we need a little algebra, some of it not completely standard.

The Euclidean inner product and associated norm in $\mathbb{R}^2$ are denoted by

$$a \cdot b \quad \text{and} \quad |a|$$

respectively. The Euclidean inner product and norm on $L(\mathbb{R}^2; \mathbb{R}^2)$ are defined by

$$M \cdot N = \sum_{i,j} M_{ij} N_{ij} \quad \text{and} \quad |M| = (M \cdot M)^{1/2}.$$

If $M = (M^1, \ldots, M^n)$ where each $M^\alpha \in L(I\!R^2; I\!R^2)$, then

$$|M|^2 = \sum_\alpha |M^\alpha|^2.$$

The space of *symmetric* linear maps from $I\!R^2$ to $I\!R^2$ is denoted by

$$\mathcal{S}(I\!R^2; I\!R^2).$$

For $a = (a_1, a_2)$, $b = (b_1, b_2) \in I\!R^2$, we define the *symmetric product* $a \odot b \in \mathcal{S}(I\!R^2; I\!R^2)$ to be the linear map with matrix

$$a \odot b = \begin{bmatrix} a_1b_1 - a_2b_2 & a_1b_2 + a_2b_1 \\ a_1b_2 + a_2b_1 & a_2b_2 - a_1b_1 \end{bmatrix}. \tag{5}$$

(Note that this is different from the usual definition of symmetric product.) It is easily checked that the definition is independent of choice of orthonormal basis. The geometric interpretation is that $a \odot b$ is reflection about the line bisecting a and b, composed with the dilation $|a|\,|b|$.

If $a \in I\!R^2$ and $\overline{c} = (c^1, \ldots, c^n)$ where each $c^\alpha \in I\!R^2$ then we define

$$a \odot \overline{c} = (a \odot c^1, \ldots, a \odot c^n) \in \Big(\mathcal{S}(I\!R^2; I\!R^2)\Big)^n.$$

For any vector $a \in I\!R^2$ let $\tilde{a} = (a_2, -a_1)$ be the vector obtained by rotating a through $-\pi/2$. Then it is easy to check that

$$\begin{aligned} (a \odot b) \cdot (c \odot d) &= 2(a \cdot c)(b \cdot d) + (a \cdot \tilde{c})(\tilde{b} \cdot d) \\ (a \odot b) \cdot (c \odot b) &= 2(a \cdot c)|b|^2 \\ |a \odot b| &= \sqrt{2}|a|\,|b|. \end{aligned} \tag{6}$$

If $M \in \mathcal{S}(I\!R^2; I\!R^2)$ and $b \in I\!R^2$ then we denote by $M^\wedge b$ the unique vector such that

$$M^\wedge b \cdot a = M \cdot (a \odot b)$$

for all $a \in I\!R^2$. It follows

$$\tilde{M}(b) = M^\wedge b,$$

where

$$\tilde{M} = \begin{bmatrix} M_{11} - M_{22} & M_{12} + M_{21} \\ M_{21} + M_{12} & M_{22} - M_{11} \end{bmatrix} \tag{7}$$

and $M = \Big[M_{ij}\Big]$. Moreover,

$$|M^\wedge b|^2 = \Big((M_{11} - M_{22})^2 + (M_{12} + M_{21})^2\Big)|b|^2 = \frac{1}{2}|\tilde{M}|^2|b|^2.$$

Clearly

$$M^\wedge b = 2M(b) - (\mathrm{Tr}\, M)b, \tag{8}$$

where Tr denotes the trace. With respect to an orthonormal basis of eigenvectors of M we have

$$M^\wedge b = (\lambda_1 - \lambda_2) \begin{bmatrix} b_1 \\ -b_2 \end{bmatrix}.$$

If

$$\begin{aligned} \overline{M} &= (M^1, \cdots, M^n), \quad M^\alpha \in \mathcal{S}(I\!R^2; I\!R^2) \\ \overline{b} &= (b^1, \cdots, b^n), \quad b^\alpha \in I\!R^2, \end{aligned}$$

then we define

$$\overline{M}^\wedge \overline{b} = \sum_{\alpha=1}^{n} M^{\alpha\wedge} b^\alpha \ (\in I\!R^2). \tag{9}$$

4 Properties of the Energy Integrand

Let $D = \{x \in I\!R^2 : |x|^2 < 1\}$ be the unit disk in $I\!R^2$. Let $X : \overline{D} \to I\!R^n$ be a fixed immersion and let S be the corresponding surface. In particular $|DX| > 0$ on D. We will always assume that $X \in C^2(\overline{D}; I\!R^n)$. Let $x = (u, v) = (u^1, u^2)$, $X_1 = X_{u^1} = X_u$, $X_2 = X_{u^2} = X_v$, $X_{12} = X_{u_1 u_2}$, etc.

We will fix the following notation throughout the paper.

Definition 4.1 Suppose $h : D \to I\!R$ is *harmonic.*

Let g be the corresponding *flat* metric on D, conformally equivalent to the standard metric, defined by $g_{ij} = e^{2h}\delta_{ij}$.

Let $\mathcal{M}$ denote the class of harmonic h as above, subject to one of the following additional constraints:

$$h(0) = 0 \quad \text{or} \quad \int_D e^{ph} = \pi \quad \text{or} \quad \int_{\partial D} e^{ph} = 2\pi \tag{10}$$

for some fixed $p > 0$.

Let k be the conjugate harmonic function of h such that $k(0) = 0$.

Let

$$f = e^{h+ik}, \quad F(z) = \int_0^z f(\zeta)\, d\zeta.$$

Remarks

1. The choice of constants 2π and π is merely a convenient normalisation and implies that the only constant function in $\mathcal{M}$ is the zero function. In particular, with this normalistion, if $X : \overline{D} \to I\!R^2$ is the identity map then $\mathcal{E}(h) = 0$ iff $h \equiv 0$, i.e. iff $g_{ij} = \delta_{ij}$. This is clear since

$$\mathcal{E}(h) = 2 \int_D |De^{-h}|^2$$

from (16) and (6). In particular, X is the unique (up to isometry) optimal parametrisation of $\overline{D}$.

2. We interpret $\int_{\partial D} e^{ph}$ as follows. Since h is harmonic, it follows e^{ph} is subharmonic and hence
$$r^{-1}\int_{\partial D_r} e^{ph},$$
where $D_r = \{x \in I\!R^2 : |x| < r\}$, is an increasing function of r for $0 < r < 1$. Now *define*
$$\int_{\partial D} e^{ph} = \lim_{r\to 1} r^{-1}\int_{\partial D_r} e^{ph}. \tag{11}$$

3. Recall from the discussion in the Introduction that $g = e^{2h}\delta_{ij}$ is the pullback metric from F. Thus F is one-one means that the metric g defines a domain in $I\!R^2$ which is diffeomorphic to D.

The *Hessian* or *second covariant derivative* of a function $f : D \to I\!R$ with respect to the metric g, in the directions $V_1, V_2 \in T_x(D)$, is defined by
$$\nabla_g^2 f(V_1, V_2) = (V_1 V_2 - \nabla_{V_1} V_2)\, f, \tag{12}$$
where ∇ is the Levi-Civita connection associated to g. The Hessian is symmetric and bilinear, and for fixed f the value at x is independent of the extension of V_1 and V_2 to a neighbourhood of X used to evaluate the right side of (12). In case g is the Euclidean metric, the Hessian is denoted by
$$D^2 f,$$
and is given by the usual matrix of second derivatives. The *length* of the Hessian is defined by
$$|\nabla_g^2 f|^2 = \sum_{i,j} |\nabla_g^2 f(\tau_i, \tau_j)|^2,$$
where (τ_1, τ_2) is an orthonormal frame with respect to g.

The Hessian of X is defined componentwise by
$$\nabla_g^2 X = (\nabla_g^2 X^1, \ldots, \nabla_g^2 X^n), \tag{13}$$
and the *length* of $\nabla_g^2 X$ is given by
$$|\nabla_g^2 X|^2 = |\nabla_g^2 X^1|^2 + \ldots + |\nabla_g^2 X^n|^2.$$
In case g is the standard metric, the Hessian is
$$D^2 X = (D^2 X^1, \ldots, D^2 X^n).$$

The *Energy Functional* is defined by
$$\mathcal{E}(h) := \int_D |\nabla_g^2 X|^2 dg = \int_D e^{2h} |\nabla_g^2 X|^2, \tag{14}$$
where $g = g_{ij} = e^{2h}\delta_{ij}$ and $\triangle h = 0$. Here and elsewhere, integration is with respect to Lebesgue measure unless noted otherwise.

Finally, the *Energy Integrand* is denoted by
$$E(h) := e^{2h} |\nabla_g^2 X|^2. \tag{15}$$

Proposition 4.2 *One has*

$$\nabla_g^2 X = e^{-2h}\left(D^2X - Dh \odot DX\right),$$

and in particular

$$\begin{aligned} E(h) &= e^{-2h}\left|D^2X - Dh \odot DX\right|^2 &(16)\\ &= \left|e^{-h}D^2X + D(e^{-h}) \odot DX\right|^2. &(17)\end{aligned}$$

PROOF: Let Γ_{ij}^k be the Christoffel symbols for g. Recall that

$$\Gamma_{jk}^i = \frac{1}{2}\left(\partial_j g_{pk} + \partial_k g_{pj} - \partial_p g_{jk}\right) g^{pi}.$$

Standard calculations then give

$$\begin{array}{l} \Gamma_{11}^1 = -\Gamma_{22}^1 = \Gamma_{12}^2 = \Gamma_{21}^2 = h_{u^1} = h_1, \\ \Gamma_{22}^2 = -\Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{21}^1 = h_{u^2} = h_2. \end{array} \qquad (18)$$

Let (τ_1, τ_2) be the orthonormal frame for g given by

$$\tau_i = e^{-h}\frac{\partial}{\partial u^i}, \quad i = 1, 2.$$

Since $\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial u^k}$ it follows from (12), (13), (18) and the bilinearity of $\nabla_g^2 X$ that

$$\begin{aligned} \nabla_g^2 X(\tau_1,\tau_1) &= e^{-2h}\left(X_{11} - h_1X_1 + h_2X_2\right),\\ \nabla_g^2 X(\tau_1,\tau_2) &= e^{-2h}\left(X_{12} - h_2X_1 - h_1X_2\right),\\ \nabla_g^2 X(\tau_2,\tau_1) &= e^{-2h}\left(X_{12} - h_2X_1 - h_1X_2\right),\\ \nabla_g^2 X(\tau_2,\tau_2) &= e^{-2h}\left(X_{22} + h_1X_1 - h_2X_2\right). \end{aligned}$$

Hence by (5)

$$\nabla_g^2 X = e^{-2h}(D^2X - Dh \odot DX).$$

The result follows from (15). ∎

Because of the form of the energy integrand (16) we will be interested in quadratic expressions in $w \in \mathbb{R}^2$ of the following type:

Definition 4.3 Let

$$\begin{aligned} \overline{M} &= (M^1, \cdots, M^n), \quad M^\alpha \in \mathcal{S}(\mathbb{R}^2; \mathbb{R}^2),\\ \overline{b} &= (b^1, \cdots, b^n), \quad b^\alpha \in \mathbb{R}^2. \end{aligned}$$

Then for $w \in I\!R^2$,

$$Q(w) := |\overline{M} - w \odot \overline{b}|^2 = \sum_{\alpha=1}^{n} Q^\alpha(w)$$

where

$$Q^\alpha(w) = |M^\alpha - w \odot b^\alpha|^2.$$

The minimum value of $Q(w)$ is denoted by $Q_{\min}$. If $\overline{b} \neq 0$ it then follows from Lemma 4.4.*2(a)* that $Q(w)$ has a unique minimum point, denoted by $w_{\min}$. Similarly from Lemma 4.4.*1(a)*, if $b^\alpha \neq 0$ the minimum value $Q^\alpha_{\min}$ of $Q^\alpha(w)$ is taken at a unique point $w^\alpha_{\min}$.

The following Lemma allows us to express $Q(w)$ as a sum of various positive terms, and also gives some useful lower bounds for $Q(w)$. In particular, note *2(c)* and the last inequality in *3*.

Lemma 4.4

1. *(a)* $Q^\alpha(w) = |M^\alpha|^2 - 2M^{\alpha\wedge} b^\alpha \cdot w + 2|b^\alpha|^2|w|^2$.

(b) *If* $b^a \neq 0$ *then*

$$\begin{aligned} w^\alpha_{\min} &= M^{\alpha\wedge} b^\alpha / 2|b^\alpha|^2 \\ &= M^\alpha(b^\alpha)/|b^\alpha|^2 - (Tr\, M^\alpha) b^\alpha / (2|b^\alpha|^2) \end{aligned}$$

(c) $$Q^\alpha(w) \begin{cases} = \ |M^\alpha|^2 \geq \frac{1}{2}(Tr\, M^\alpha)^2 & |b^\alpha| = 0 \\ = \ \frac{1}{2}(Tr\, M^\alpha)^2 + 2|b^\alpha|^2\, |w - w^\alpha_{\min}|^2 & |b^\alpha| \neq 0 \end{cases}$$

2. *(a)* $Q(w) = |\overline{M}|^2 - 2\overline{M}^\wedge \overline{b} \cdot w + 2|w|^2 |\overline{b}|^2$

(b) *If* $|\overline{b}| \neq 0$ *then* $Q(w)$ *takes its minimum at the unique point*

$$\begin{aligned} w_{\min} &= \textstyle\sum_\alpha |b^\alpha|^2 w^\alpha_{\min} / |\overline{b}|^2 \\ &= \textstyle\sum_\alpha M^{\alpha\wedge} b^\alpha / 2|\overline{b}|^2 \\ &= \textstyle\sum_\alpha M^\alpha(b^\alpha)/|\overline{b}|^2 - \sum_\alpha (Tr\, M^\alpha) b^\alpha / (2|\overline{b}|^2) \end{aligned}$$

(c) *If* $|\overline{b}| \neq 0$ *then*

$$\begin{aligned} Q(w) \ = \ & \textstyle\sum_{\{\alpha : b^\alpha = 0\}} |M^\alpha|^2 \\ & + \textstyle\sum_{\{\alpha : b^\alpha \neq 0\}} \left(\frac{1}{2}(Tr\, M^\alpha)^2 + 2|b^\alpha|^2 |w_{\min} - w^\alpha_{\min}|^2 \right) \\ & + 2|\overline{b}|^2 |w - w_{\min}|^2 \end{aligned}$$

3. $$\begin{aligned} Q(w) &\geq \textstyle\sum_\alpha \frac{1}{2}(Tr\, M^\alpha)^2 + 2|\overline{b}|^2 |w - w_{\min}|^2 \\ &\geq |\overline{b}|^2 \left| w - \textstyle\sum_a M^\alpha(b^\alpha)/|\overline{b}|^2 \right|^2 \\ &\qquad + |\overline{b}|^2 \left| w - \textstyle\sum_a M^\alpha(b^\alpha)/|\overline{b}|^2 + \sum_\alpha (Tr\, M^\alpha) b^\alpha / (|\overline{b}|^2) \right|^2 \\ &\geq |\overline{b}|^2 \left| w - \textstyle\sum_a M^\alpha(b^\alpha)/|\overline{b}|^2 \right|^2 \end{aligned}$$

Proof: We have

$$\begin{aligned} Q^\alpha(w) &= |M^\alpha|^2 - 2M^\alpha \cdot w \odot b^\alpha + |w \odot b^\alpha|^2 \\ &= |M^\alpha|^2 - 2M^{\alpha\wedge} b^\alpha \cdot w + 2|w|^2 |b^\alpha|^2. \end{aligned}$$

Differentiating this and setting the derivative to zero gives the first expression for w^α_{min}; the second expression follows from (8).

The expression for $Q^\alpha(w)$ when $|b^\alpha| = 0$ is trivial.

Since $D^2Q^\alpha(w) = 4|b^\alpha|^2 I$, it follows from Taylor's formula that if $|b^a| \neq 0$ then

$$Q^\alpha(w) = Q^\alpha_{\text{min}} + 2|b^\alpha|^2|w - w^\alpha_{\text{min}}|^2.$$

Moreover,

$$Q^\alpha_{\text{min}} = |M^\alpha - w^\alpha_{\text{min}} \odot b^\alpha|^2.$$

Straightforward calculations from (7) show

$$M^\alpha{}^\wedge b^\alpha \odot b^\alpha = \tilde{M}^\alpha(b^\alpha) \odot b^\alpha = |b^\alpha|^2\tilde{M}^\alpha,$$

and so

$$M^\alpha - w^\alpha_{\text{min}} \odot b^\alpha = M^\alpha - \frac{1}{2}\tilde{M}^\alpha = \begin{bmatrix} \frac{1}{2}(M^\alpha_{11} + M^\alpha_{22}) & 0 \\ 0 & \frac{1}{2}(M^\alpha_{11} + M^\alpha_{22}) \end{bmatrix}.$$

Hence

$$Q^\alpha_{\text{min}} = \frac{1}{2}\left(\operatorname{Tr} M^\alpha\right)^2.$$

This gives the expression for $Q^\alpha(w)$ if $|b^\alpha| \neq 0$.

The expression *2(a)* for $Q(w)$ follows from *1(a)* and (9). The first expression for w_{min} follows from differentiating the sum of the terms in *1(c)* and setting the derivative to zero. The second expression follows from (8).

Since $D^2Q(w) = 4|\bar{b}|^2 I$ from *2(a)*, we have

$$Q(w) = Q_{\text{min}} + 2|\bar{b}|^2|w - w_{\text{min}}|^2.$$

The expression for Q_{min} is obtained by setting $w = w_{\text{min}}$ in *1(c)*, and this then gives *2(c)*.

Finally, the first inequality for $Q(w)$ follows from *2(c)*. It then follows from the third expression for w_{min} and the Cauchy-Schwartz inequality that

$$\begin{aligned}
Q(w) &\geq \frac{1}{2}\sum_\alpha(\operatorname{Tr} M^\alpha)^2 + 2|\bar{b}|^2\left|w - \frac{\sum_\alpha M^\alpha(b^\alpha)}{|\bar{b}|^2} + \frac{\sum_\alpha(\operatorname{Tr} M^\alpha)\, b^\alpha}{2|\bar{b}|^2}\right|^2 \\
&= \frac{1}{2}\sum_\alpha(\operatorname{Tr} M^\alpha)^2 + |\bar{b}|^2\left|w - \frac{\sum_\alpha M^\alpha(b^\alpha)}{|\bar{b}|^2}\right|^2 - \frac{1}{2}|\bar{b}|^2\left|\frac{\sum_\alpha(\operatorname{Tr} M^\alpha)\, b^\alpha}{|\bar{b}|^2}\right|^2 \\
&\quad + |\bar{b}|^2\left|w - \frac{\sum_\alpha M^\alpha(b^\alpha)}{|\bar{b}|^2} + \frac{\sum_\alpha(\operatorname{Tr} M^\alpha)\, b^\alpha}{|\bar{b}|^2}\right|^2 \\
&\geq |\bar{b}|^2\left|w - \frac{\sum_\alpha M^\alpha(b^\alpha)}{|\bar{b}|^2}\right|^2 + |\bar{b}|^2\left|w - \frac{\sum_\alpha M^\alpha(b^\alpha)}{|\bar{b}|^2} + \frac{\sum_\alpha(\operatorname{Tr} M^\alpha)\, b^\alpha}{|\bar{b}|^2}\right|^2 \\
&\geq |\bar{b}|^2\left|w - \frac{\sum_\alpha M^\alpha(b^\alpha)}{|\bar{b}|^2}\right|^2.
\end{aligned}$$

∎

Notation For $x \in D$ and $w \in I\!R^2$ let $Q(x,w)$ be the quadratic expression in Definition 4.3 with $\overline{M}$ there replaced by $D^2X(x)$ and $\overline{b}$ replaced by DX. Thus

$$E(h) = e^{-2h}Q(\cdot\,; Dh).$$

For fixed x let $Q_{\min}(x)$ denote the minimum value of $Q(x;\cdot)$ and (recall $|DX(x)| \neq 0$) let $w_{\min}(x)$ be the unique point at which this minimum is taken (see the previous Lemma).

The following observation is useful.

Lemma 4.5

$$\begin{aligned} D^2X^\alpha(DX^\alpha) &= |DX^\alpha|\, D|DX^\alpha| \quad \text{for } \alpha = 1,\ldots,n \\ \sum_\alpha D^2X^\alpha(DX^\alpha) &= |DX|\, D|DX|. \end{aligned}$$

Proof:

$$\begin{aligned} D^2X^\alpha(DX^\alpha) &= \left(\sum_j D_{1j}X^\alpha D_jX^\alpha,\ \sum_j D_{2j}X^\alpha D_jX^\alpha\right) \\ &= \frac{1}{2}\Big(D_1(|DX^\alpha|^2),\ D_2(|DX^\alpha|^2)\Big) \\ &= \frac{1}{2}D|DX^\alpha|^2 = |DX^\alpha|\, D|DX^\alpha|. \end{aligned}$$

This gives the first result. The second follows from summing the penultimate expression above. ∎

We now have various expressions involving the energy integrand.

Proposition 4.6

$$\begin{aligned} w^\alpha_{\min}(\cdot) &= \frac{1}{2}|DX^\alpha|^{-2}D^2X^\alpha{}^\wedge DX^\alpha = D\log|DX^\alpha| - \frac{1}{2}|DX^\alpha|^{-2}\triangle X^\alpha\, DX^\alpha, \\ w_{\min}(\cdot) &= \frac{1}{2}|DX|^{-2}D^2X^\wedge DX = D\log|DX| - \frac{1}{2}|DX|^{-2}\triangle X\cdot DX, \\ Q_{\min}(\cdot) &= |D^2X|^2 - 2\left|D|DX| - \frac{1}{2}\triangle X\cdot\frac{DX}{|DX|}\right|^2, \\ E(h) &= e^{-2h}Q_{\min} + 2\left|D(|DX|e^{-h}) - \frac{1}{2}e^{-h}\triangle X\cdot\frac{DX}{|DX|}\right|^2, \end{aligned}$$

where $\triangle X\cdot DX := (\sum_\alpha \triangle X^\alpha D_1X^\alpha,\ \sum_\alpha \triangle X^\alpha D_2X^\alpha)$.

In case X is conformal, $\triangle X\cdot DX = (0,0)$ *and the corresponding expressions simplify accordingly.*

PROOF: The given expressions follow from Lemma 4.4, the previous Lemma and (8).

In case X is conformal then $\triangle X \cdot DX = (0,0)$, since the mean curvature vector is the Laplacian on the surface S of the position vector and by conformality this is a multiple of the Laplacian on the disc D. Alternatively, one can differentiate the conformality relations $X_1 \cdot X_2 = 0$ and $X_1 \cdot X_1 = X_2 \cdot X_2$ and then add or subtract the appropriate equations. ∎

The following pointwise coercivity type estimate is important.

Proposition 4.7

$$E(h) \geq \left| D\left(e^{-h}|DX|\right)\right|^2 .$$

PROOF: From Lemmas 4.4.*3* and 4.5 we have

$$\begin{aligned} E(h) &= e^{-2h}\left|D^2X - Dh \odot DX\right|^2 \\ &\geq e^{-2h}\left||DX|\,Dh - \frac{\sum_\alpha D^2X^\alpha(DX^\alpha)}{|DX|}\right|^2 \\ &= \left||DX|\,D(e^{-h}) + e^{-h}D|DX|\right|^2 \\ &= \left|D(e^{-h}|DX|)\right|^2 . \end{aligned}$$

∎

We next see how $Q_{\min}(\cdot)$ measures the flatness of the immersion X. First we have a pointwise result.

Lemma 4.8 *Suppose $x \in D$ and $Q_{\min}(x) = 0$. Then*

1. *if $|DX^\alpha(x)| = 0$ then $D^2X^\alpha(x) = 0$,*
2. *if $|DX^\alpha(x)| \neq 0$ then $\triangle X^\alpha(x) = 0$ and $D\log|DX^\alpha(x)| = D\log|DX(x)|$.*

Moreover, $w_{\min}(x) = D\log|DX(x)|$.

Conversely, if the implications 1 and 2 hold then $Q_{\min}(x) = 0$.

PROOF: Suppose $Q_{\min}(x) = 0$. The first implication is immediate from Lemma 4.4.*2(c)*. If $|DX^\alpha(x)| \neq 0$ then $\triangle X(x) = 0$ from Lemma 4.4.*2(c)*. From Proposition 4.6

$$w^\alpha_{\min}(\cdot) = D\log|DX^\alpha| \quad \text{and} \quad w_{\min}(\cdot) = D\log|DX|.$$

The rest of the first part of the proposition follows from Lemma 4.4.*2(c)* and by noting Proposition 4.6.

The converse is similar. ∎

We now show that $Q_{\min}(\cdot) \equiv 0$ iff X is the composition of a flat conformal map with an affine transformation.

Proposition 4.9 *Suppose $\Omega \subset D$ is simply connected and open.*

If $Q_{\min}(\cdot) \equiv 0$ in Ω then $X[\Omega]$ is contained in a plane. Moreover, $X|_\Omega = L \circ G$ where $G : \Omega \to \mathbf{C}$ is holomorphic and L is affine. Also, $w_{\min}(\cdot) = D \log |DX| = D \log |G'|$.

Conversely, if $X|_\Omega = L \circ G$ where $G : \Omega \to \mathbf{C}$ is holomorphic and L is affine then $Q_{\min}(\cdot) \equiv 0$ in Ω.

Proof: Assume $Q_{\min}(\cdot) \equiv 0$ in Ω. From the previous Lemma, $\triangle X = 0$ in Ω. Without loss of generality we can assume $0 \in \Omega$ and $X(0) = 0$. For $\alpha = 1, \ldots, n$ let $\tilde{X}^\alpha$ be the harmonic conjugate of X^α such that $\tilde{X}^\alpha(0) = 0$. Since X is an harmonic immersion we can assume $|DX^\alpha(x)| > 0$ except at isolated points, as otherwise X^α is constant on Ω in which case we can ignore the coordinate function X^α.

On the set

$$\Omega' = \{x \in \Omega : |DX^\alpha(x)| > 0 \ \forall \alpha\}.$$

we have from Proposition 4.8 that

$$D \log |DX^\alpha| = D \log |DX^1|$$

for $\alpha = 2, \ldots, n$. hence there exist constants c_α such that

$$|DX^\alpha| = e^{c_\alpha} |DX^1|$$

on Ω', and hence on Ω by continuity.

Let $\phi_\alpha = X^\alpha + i\tilde{X}^\alpha$ and note that the ϕ_α are holomorphic. We have

$$|\phi'_\alpha| = |DX^\alpha| = e^{c_\alpha} |DX^1| = e^{c_\alpha} |\phi'_1|.$$

Thus the holomorphic functions ϕ'_α / ϕ'_1 are constants, and so there exist real constants A_α and θ_α and complex constants B_α such that

$$\phi_\alpha = A_\alpha e^{i\theta_\alpha} \phi_1 + B_\alpha.$$

Since $\phi_\alpha(0) = \phi_1(0) = 0$, we have $B_\alpha = 0$. Taking the real part of each side we have

$$\begin{aligned} X &= (X^1, \ldots, X^n) \\ &= (1, A_2 \cos\theta_2, \ldots, A_n \cos\theta_n) X^1 + (0, -A_2 \sin\theta_2, \ldots, -A_n \sin\theta_n) \tilde{X}^1. \end{aligned}$$

That is, X is contained in the plane spanned by $(1, A_2 \cos\theta_2, \ldots, A_n \cos\theta_n)$ and $(0, -A_2 \sin\theta_2, \ldots, -A_n \sin\theta_n)$.

By means of an orthonormal transformation of $I\!R^n$ we may assume $X[D] \subset I\!R^2$. Then we can write

$$\begin{aligned} \begin{bmatrix} X^1 \\ X^2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ A\cos\theta & -A\sin\theta \end{bmatrix} \begin{bmatrix} X^1 \\ \tilde{X}^1 \end{bmatrix} \\ &= T \circ G, \end{aligned}$$

where $G = X^1 + i\tilde{X}^1$. This establishes the required form for $X|_\Omega$. It follows that $|DX| = c|G'|$ for some constant c, and hence from Proposition 4.8 we have $w_{\min}(x) = D \log |DX| = D \log |G'|$.

For the converse assume $X|_\Omega = L \circ G$ where $G : \Omega \to \mathbf{C}$ is holomorphic and L is affine. By means of an orthonormal transformation in $I\!R^n$ we may assume $X[\Omega] \subset I\!R^2$. From the form of $X|_\Omega$ there exist non-zero constants $\overline{c}_1, \overline{c}_2$ such that

$$|DX^\alpha| = \overline{c}_\alpha |DX|$$

and

$$\triangle X^\alpha = 0$$

for $\alpha = 1, 2$. The result now follows from Proposition 4.8 since $|DX^\alpha(x)| \neq 0$. ∎

5 Existence of a Minimiser

First we need an auxiliary lemma.

Lemma 5.1 *Let $\Omega \subset I\!R^n$ be a bounded convex domain. Suppose $u \in W^{1,2}(\Omega)$ and $E \subset \Omega$ a measurable set such that $|E| > 0$. Then*

$$\int_\Omega |u|^2 \leq c \left(|E|^{-2} \int_\Omega |Du|^2 + |E|^{-1} \int_E |u|^2 \right),$$

where c depends only on n and the diameter of Ω.

Proof: From [GT] (Lemma 7.16, pp 162–3)

$$|u(x)| \leq c|E|^{-1} \left(\int_\Omega \frac{|Du(y)|}{|x-y|^{n-1}} dy + \int_E |u| \right).$$

Integrating over Ω with respect to x,

$$\int_\Omega |u| \leq c|E|^{-1} \left(\int_\Omega |Du| + \int_E |u| \right).$$

Replacing u by u^2,

$$\begin{aligned} \int_\Omega |u|^2 &\leq c|E|^{-1} \left(\int_\Omega |u||Du| + \int_E |u|^2 \right) \\ &\leq \epsilon \int_\Omega |u|^2 + c|E|^{-2} \int_\Omega |Du|^2 + c|E|^{-1} \int_E |u|^2, \end{aligned}$$

and the result follows. ∎

The following coercivity estimate is important, and depends on the various hypotheses as noted in the subsequent remarks.

Proposition 5.2 *Suppose $h \in \mathcal{M}$, $X \in C^2(\overline{D}; \mathbb{R}^n)$, $|DX| > 0$ on $\overline{D}$, and $Q_{\min}(\cdot) \not\equiv 0$ on D. Then*

$$\|e^{-h}\|^2_{W^{1,2}(D)} \leq c\,\mathcal{E}(h),$$

where $c = c\,(X)$.

Proof: There is a measurable $E \subset D$ such that $|E| > 0$ and $Q_{\min} \geq \delta > 0$ on E. By the hypotheses on X there are positive constants θ and M such that $\theta \leq |DX| \leq M$ and $|D|DX|| \leq M$ on $\overline{D}$.

In the following, constants c will depend only on δ, θ, M and $|E|$.

From Proposition 4.7 and the previous Lemma, and recalling the Notation following Lemma 4.4, we have

$$\begin{aligned}
\int_D e^{-2h} &\leq c \int_D |DX|^2 e^{-2h} \\
&\leq c\left(\int_D \left|D\left(|DX|e^{-h}\right)\right|^2 + \int_E |DX|^2 e^{-2h}\right) \\
&\leq c\left(\mathcal{E}(h) + \int_E Q_{\min}(\cdot)e^{-2h}\right) \\
&\leq c\left(\mathcal{E}(h) + \int_E Q(\cdot\,; Dh)e^{-2h}\right) \\
&\leq c\,\mathcal{E}(h).
\end{aligned}$$

Since

$$\begin{aligned}
\left|De^{-h}\right| &= \left|D\left(|DX|^{-1}\right)|DX|e^{-h} + |DX|^{-1}D\left(|DX|e^{-h}\right)\right| \\
&\leq c\left(e^{-h} + D\left(|DX|e^{-h}\right)\right),
\end{aligned}$$

it follows from Proposition 4.7 again that

$$\begin{aligned}
\int_D \left|De^{-h}\right|^2 &\leq c\int_D \left(e^{-2h} + \left|D\left(|DX|e^{-h}\right)\right|^2\right) \\
&\leq c\,\mathcal{E}(h).
\end{aligned}$$

This completes the proof. ∎

Remarks 1 The hypothesis $|DX| > 0$ on $\overline{D}$, and not just on D, is necessary.

To see this let

$$X = G\colon D \to \mathbf{C} \cong \mathbb{R}^2$$

be the holomorphic function given by $G(z) = (1-z)^2$. The pull-back metric induced on D via G is

$$g_{ij} := |G'|^2\delta_{ij} = 4|1-z|^2\delta_{ij} = e^{2\log(2|1-z|)}\delta_{ij}.$$

Let $h = \log(2|1-z|)$. Since X is an isometry from D (with the pull-back metric g) to $X[D]$ (with the standard metric), it follows that $\nabla_g^2 X \equiv 0$ on D and so

$$\mathcal{E}(h) = \int_D |\nabla_g^2 X|^2 dg = 0.$$

However, $e^{-h} = \frac{1}{2}|1-z|^{-1} \notin L^2(D)$.

2 The condition $Q_{\min}(x_0) \neq 0$ for some $x_0 \in D$ is also necessary. For example, let $X : D \to I\!R^2$ be the identity map. Then

$$Q(x; w) = |w \odot e_1|^2 + |w \odot e_2|^2 = 4|w|^2,$$

where e_1, e_2 are the usual basis vectors in $I\!R^2$, and so $Q_{\min}(\cdot) \equiv 0$. Moreover, if h is the zero function,

$$\|e^{-h}\|_{L^2(D)} = \pi \quad \text{and} \quad \mathcal{E}(h) = 0.$$

We can now prove the following result, where $\mathcal{M}$ is any of the classes from Definition 4.1.

Theorem 5.3 *Suppose $X : D \to I\!R^n$ is an immersion where $X \in C^2(\overline{D}; I\!R^n)$ and $|DX| > 0$ on $\overline{D}$. Then there exists $h_0 \in \mathcal{M}$ such that*

$$\mathcal{E}(h_0) = \inf_{h \in \mathcal{M}} \mathcal{E}(h).$$

PROOF: Recall

$$\mathcal{E}(h) = \int_D e^{-2h} \left| D^2 X - Dh \odot DX \right|^2 = \int_D e^{-2h} Q(x; Dh),$$

using the Notation following Lemma 4.4.

We consider separately the cases $Q_{\min}(x) = 0$ for all $x \in D$ and $Q_{\min}(x) > 0$ for some $x \in D$.

First suppose $Q_{\min} \equiv 0$ in D.

Then from Proposition 4.9, $X = L \circ G$ where $G : D \to \mathbf{C}$ is holomorphic and L is affine. Moreover, $Q(\cdot\,; Dh_0) \equiv 0$ if $h_0 = \log|G'| + c$ for some constant c. Such h_0 is harmonic (note that $|G'| \neq 0$ since X is an immersion) and we may choose c so that $h_0 \in \mathcal{M}$. Thus h_0 is the required minimiser and in this case $\mathcal{E}(h_0) = 0$.

Next suppose $Q_{\min}(x_0) > 0$ for some $x_0 \in D$.

Let

$$\gamma = \inf_{h \in \mathcal{M}} \mathcal{E}(h).$$

Let $\{h_j\}$ be a minimising sequence. Let k_j be the harmonic conjugate of h_j such that $k_j(0) = 0$. Let

$$\phi_j = e^{-(h_j + ik_j)}.$$

Then ϕ_j is holomorphic, $|\phi_j| = e^{-h_j}$, $|D\phi_j| = |De^{-h_j}|$, and hence from Proposition 5.2,

$$\|\phi_j\|_{W^{1,2}(D)} \le M$$

for some M independent of j.

Passing to a subsequence,

$$\phi_j \to \phi_0 \text{ (say)} \quad \text{weakly in } W^{1,2}(D).$$

Moreover, ϕ_0 is holomorphic and $\phi_j \to \phi_0$ uniformly on compact subsets of D and similarly for their derivatives.

Since $\phi_j \neq 0$ on D it follows from Rouché's theorem that either $\phi_0 \equiv 0$ on D or $\phi_0 \neq 0$ on D. If $\phi_0 \equiv 0$ on D then $h_j \to \infty$ uniformly on compact subsets of D, contradicting $h_j \in \mathcal{M}$ (see Definition 4.1).

Hence $\phi_0 \neq 0$ on D and $h_0 := -\log|\phi_0|$ is finite and harmonic. Moreover, $h_j \to h_0$ uniformly on compact subsets and similarly for their derivatives.

By Fatou's Lemma

$$\begin{aligned}
\mathcal{E}(h_0) &= \int_D e^{-2h_0} \left| D^2X - Dh_0 \odot DX \right|^2 \\
&\le \lim_{j\to\infty} \int_D e^{-2h_j} \left| D^2X - Dh_j \odot DX \right|^2 \\
&= \gamma.
\end{aligned}$$

Note that $\gamma > 0$ as otherwise $\mathcal{E}(h_0) = 0$. But then $E(h_0) = 0$ and so $Q_{\min}(\cdot) \equiv 0$, contradiction.

Thus to show h_0 is the required minimiser we need only check that $h_0 \in \mathcal{M}$.

If $h_j(0) = 0$ for all j then clearly $h_0(0) = 0$.

If $\int_D e^{ph_j} = \pi$ for all j then $\int_D e^{ph_0} = c_0 \le \pi$ by Fatou's Lemma. But if $c_0 < \pi$ then

$$\overline{h}_0 := h_0 + \frac{1}{p} \log \frac{\pi}{c}$$

satisfies

$$\int_D e^{p\overline{h}_0} = \pi,$$

and

$$E(\overline{h}_0) = \left(\frac{c_0}{\pi}\right)^{2/p} \mathcal{E}(h_0) < \gamma,$$

using $\gamma > 0$. This contradicts the definition of γ, and so $\int_D e^{ph} = \pi$.

By a similar argument, if $\int_{\partial D} e^{ph_j} = 2\pi$ for all j then $\int_{\partial D} e^{ph_0} = 2\pi$ (recall the definition of $\int_{\partial D} e^{ph}$ following Definition 4.1). ∎

Remark We have in fact shown that

either (i) $\inf_{h\in\mathcal{M}} \mathcal{E}(h) = 0$, which happens iff $X = L \circ G$ where L is affine and $G : D \to \mathbf{C}$ is holomorphic, and in this case $h = \log|G'| + c$ for some constant c;

or *(ii)* $\inf_{h\in\mathcal{M}} \mathcal{E}(h) > 0$, in which case any minimising sequence has a subsequence which converges uniformly on compact subsets, together with all derivatives, to a minimiser h.

Extensions 1 With the normalisation condition $\int_D e^{ph} = \pi$ or $\int_{\partial D} e^{ph} = 2\pi$ one can weaken the requirement $|DX| > 0$ on $\overline{D}$ to $|DX| > 0$ on D, thus allowing *boundary branch points*.

Proposition 5.2 is no longer valid, but the proof of the previous Theorem in case $Q_{\min}(x_0) > 0$ for some $x_0 \in D$ can be modified as follows.

First define

$$\psi_j = e^{p(h_j+ik_j)/2}$$

where k_j is the harmonic conjugate of h_j. Then the ψ_j are uniformly bounded in $L^2(D)$ (note that $⨍_D e^{ph} \le ⨍_{\partial D} e^{ph}$). Hence, passing to a subsequence,

$$\psi_j \to \psi_0 \text{ (say)} \quad \text{weakly in } L^2(D).$$

Moreover, ψ_0 is holomorphic and $\psi_j \to \psi_0$ uniformly on compact subsets and similarly for their derivatives.

Since $\psi_j \ne 0$ on D it follows from Rouché's Theorem that either $\psi_0 \equiv 0$ on D or $\psi_0 \ne 0$ on D. If $\psi_0 \equiv 0$ then $h_j \to -\infty$ uniformly on compact subsets of D. Since $Q_{\min} \not\equiv 0$, there is a compact $E \subset D$ such that $|E| > 0$ and $Q_{\min} \ge \delta > 0$ on E. Then

$$\mathcal{E}(h_j) \ge \int_E e^{-2h_j} Q_{\min} \to \infty,$$

contradicting the fact that h_j is a minimising sequence. Hence $\psi_0 \ne 0$ on D.

Thus $h_0 := 2(\log|\psi_0|)/p$ is finite and harmonic. The proof that h_0 is a minimiser now proceeds as before.

2 With the same normalisation condition $\int_D e^{ph} = \pi$ or $\int_{\partial D} e^{ph} = 2\pi$ and assuming $Q_{\min} \not\equiv 0$ (in particular if the parametrised surface $X[D]$ is not flat), then one can completely drop the requirement $|DX| > 0$ on D in the previous Theorem. In particular, one can allow *interior branch points*.

In particular, if X is a conformal branched immersion and $F(z) := \int_0^z e^{h_0+ik_0}$ where h_0 is the minimiser and k_0 is the harmonic conjugate, then $(F[D], X \circ F^{-1})$ can be taken as the optimal branched conformal immersion for the surface $X[\overline{D}]$. As in the case of no branch points, $F[D]$ may be a domain with "multiplicity".

3 In case X is one-one it is natural to consider the class $\mathcal{M}^*$ of $h \in \mathcal{M}$ such that if $F = \int e^{h+ik}$ then F is one-one (since in this case the parametrisation $(F[D], X \circ F^{-1})$ is also one-one). One can then prove the existence of a minimiser in $\mathcal{M}^*$ as follows.

First suppose $Q_{\min} \equiv 0$. Then using the notation from the proof of the Theorem, if $F_0 = \int e^{h_0+ik_0}$, one has

$$F_0 = e^c G = e^c\, L \circ X.$$

In particular, F_0 is one-one.

If $Q_{\min}(x_0) > 0$ for some $x_0 \in D$, writing $\phi_0 = e^{-(h_0+ik_0)}$, let

$$F_j(z) = \int_0^z e^{h_j+ik_j} = \int_0^z {\phi_j}^{-1}, \quad F_0(z) = \int_0^z e^{h_0+ik_0} = \int_0^z {\phi_0}^{-1}.$$

Then $F_j \to F_0$ uniformly on compact subsets of D. Hence if each F_j is one-one on D then F_0 is also one-one on D by Rouché's Theorem.

6 The Conformal Case and Examples

Let S be the surface given by the immersion $X : D \to I\!R^n$. The metric on S is given by $\overline{g}_{ij} = X_i \cdot X_j$. Let Γ^k_{ij} be the corresponding Christoffel symbols. Let A be the second fundamental form. Then

$$X_{ij} = \sum_k \Gamma^k_{ij} X_k + A(X_i, X_j). \tag{19}$$

If $n = 3$ then $|A|^2 = \kappa_1^2 + \kappa_2^2$ where κ_1 and κ_2 are the principal curvatures. The mean curvature is $H = (\kappa_1+\kappa_2)/2$ and the Gauss curvature is $K = \kappa_1\kappa_2$.

In the remainder of this section we assume X is conformal.

To simplify notation we set

$$\Lambda = |DX|^2/2,$$

so that $\sqrt{\Lambda}$ is the conformal factor associated with X.

We will derive several useful expressions for the energy integrand $E(h)$. First recall (see Proposition 4.6) that $\triangle X \cdot X = (0,0)$ in this case. Moreover, we have the following useful observation

Lemma 6.1 *If $X : D \to I\!R^n$ is conformal then*

$$|D^2X|^2 = 2|\,D|DX|\,|^2 + \frac{1}{4}|DX|^4|A|^2.$$

If $n = 3$, then

$$|D^2X|^2 = 4\Lambda^2H^2 + \triangle\Lambda.$$

Proof: From the conformality of X, $\overline{g}_{ij} = \Lambda\delta_{ij}$. By standard computations (as for (18)) we have

$$\begin{array}{c} \Gamma^1_{11} = -\Gamma^1_{22} = \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}D_1\Lambda/\Lambda, \\ \Gamma^2_{22} = -\Gamma^2_{11} = \Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2}D_2\Lambda/\Lambda. \end{array}$$

By an orthonormal transformation of $I\!R^n$ we may assume at a fixed $x \in D$ that $X_1 = \sqrt{\Lambda}e_1$, $X_2 = \sqrt{\Lambda}e_2$. From (19) it follows that

$$\begin{aligned} |D^2X|^2 &= \Lambda^{-1}|D\Lambda|^2 + \Lambda^2|A|^2 \\ &= 2|D|DX||^2 + \frac{1}{4}|DX|^4|A|^2. \end{aligned}$$

If $n = 3$, then
$$|D^2X|^2 = \Lambda^{-1}|D\Lambda|^2 + \Lambda^2(4H^2 - 2K).$$
Since (e.g. [DHKW] pp 26,27)
$$K = -\frac{1}{\Lambda}\triangle \log\sqrt{\Lambda} = -\frac{1}{2\Lambda^2}\left(\triangle\Lambda - \frac{1}{\Lambda}|D\Lambda|^2\right),$$
we have
$$|D^2X|^2 = 4\Lambda^2H^2 + \triangle\Lambda.$$
▮

Proposition 6.2 *If $X : D \to I\!R^n$ is conformal then*
$$\begin{aligned} E(h) &= e^{-2h}\left(|D^2X|^2 - 2|\,D|DX|\,|^2\right) + 2\left|D(e^{-h}|DX|)\right|^2 \\ &= \frac{1}{4}e^{-2h}|DX|^4|A|^2 + 2\left|D(e^{-h}|DX|)\right|^2. \end{aligned}$$
If $n = 3$ then
$$E(h) = 4e^{-2h}\Lambda^2H^2 + \triangle(e^{-2h}\Lambda).$$

PROOF: The first two formulae are immediate from Proposition 4.6, the fact $\triangle X \cdot DX = (0, 0)$ and the previous Lemma.

For the third formula we compute using the previous Lemma that
$$\begin{aligned} E(h) &= e^{-2h}\left(|D^2X|^2 - 4|D\sqrt{\Lambda}|^2\right) + 4\left|D(e^{-h}\sqrt{\Lambda})\right|^2 \\ &= e^{-2h}\left(4\Lambda^2H^2 + \triangle\Lambda - 4|D\sqrt{\Lambda}|^2\right) + \\ &\qquad 4e^{-2h}|D\sqrt{\Lambda}|^2 - 8e^{-2h}D\sqrt{\Lambda}\cdot Dh\sqrt{\Lambda} + 4\Lambda e^{-2h}|Dh|^2 \\ &= e^{-2h}\left(4\Lambda^2H^2 + \triangle\Lambda - 4Dh\cdot D\Lambda + 4\Lambda|Dh|^2\right) \\ &= 4e^{-2h}\Lambda^2H^2 + e^{-2h}\triangle\Lambda + 2De^{-2h}\cdot D\Lambda + \Lambda\triangle e^{-2h} \\ &\qquad \left(\text{noting } \triangle e^{-2h} = 4e^{-2h}|Dh|^2 \text{ since } \triangle h = 0\right) \\ &= 4e^{-2h}\Lambda^2H^2 + \triangle(e^{-2h}\Lambda). \end{aligned}$$
This completes the proof. ▮

With the help of these expressions for a conformal immersion we now show that the standard Weierstrass representation of Enneper's surface is optimal.

Proposition 6.3 *For $R > 0$ and $z \in \overline{D}$ let*
$$X_R(z) = \mathcal{R}e\int_0^z \left(\frac{1}{2}(1 - R^2\zeta^2), \frac{i}{2}(1 + R^2\zeta^2), R\zeta\right)d\zeta,$$
be the standard parametrisation of a part of Enneper's surface.

Then $h \equiv 0$ is the unique minimiser of $\mathcal{E}(h)$. Equivalently, (D, X_R) is the unique optimal conformal parametrisation.

PROOF: We have

$$\begin{aligned}\Lambda = \frac{1}{2}|DX_R(z)|^2 &= \frac{1}{4}\left(|1-R^2z^2|^2+|1+R^2z^2|^2+4R^2|z|^2\right)\\ &= \frac{1}{2}\left(1+R^2|z|^2\right)^2 = \frac{1}{2}\left(1+R^2r^2\right)^2,\end{aligned}$$

where $r=|z|$. In particular, Λ is rotationally symmetric.

From Proposition 6.2, if 0 denotes the zero function,

$$\begin{aligned}\mathcal{E}(h)-\mathcal{E}(0) &= \int_D \triangle\left((e^{-2h}-1)\Lambda\right)\\ &= \int_{\partial D}\frac{\partial}{\partial r}\left((e^{-2h}-1)\Lambda\right)\\ &= \frac{\partial\Lambda}{\partial r}(1)\int_{\partial D}(e^{-2h}-1)+\Lambda(1)\int_D \triangle e^{-2h}. \qquad (20)\end{aligned}$$

The normalisation conditions $\int_D e^{ph}=\pi$ and $\int_{\partial D} e^{ph}=2\pi$ imply $h(0)\leq 0$ by the subharmonicity of e^{ph}. Thus all three normalisation conditions imply $e^{-2h(0)}-1\geq 0$ and hence the first integrand in (20) is positive, again by subharmonicity. The second integral is also positive. Finally $\frac{\partial\Lambda}{\partial r}(1) = 2R^2(1+R^2)>0$ and $\Lambda(1)=(1+R^2)^2/2>0$.

It follows that the zero function is the *unique* minimiser, since $\triangle e^{-2h} = 4|Dh|^2e^{-2h}=0$ iff $h\equiv 0$. ∎

Stereographic projection of the unit disc onto a spherical cap in S^2 is given by

$$X = X_R : D\to S_R$$

where

$$X_R(x,y)=\frac{(2Rx,2Ry,R^2(x^2+y^2)-1))}{1+R^2(x^2+y^2)}.$$

The image $S_R = X[D]$ is the lower spherical cap

$$S_R = S^2\cap\left\{(u,v,w): w<\frac{R^2-1}{R^2+1}\right\}.$$

Straightforward computations show that

$$\Lambda = \Lambda_R = \frac{4R^2}{(1+r^2R^2)^2}, \qquad (21)$$

$$\triangle\Lambda = \Lambda''+r^{-1}\Lambda' = \frac{32(-1+2r^2R^2)R^4}{(1+r^2R^2)^4}. \qquad (22)$$

Denote by

$$\mathcal{E}_R(h)$$

the energy functional corresponding to X_R.

We first compute another form for the energy functional in the general case that Λ is rotationally symmetric.

Lemma 6.4 *If $X : D \to I\!R^3$ is conformal and Λ is rotationally symmetric, then*

$$\mathcal{E}(h) = \int_D (4\Lambda^2 H^2 + \triangle\Lambda) e^{-2h} + \int_D (2\Lambda(1) - \Lambda)\triangle e^{-2h}.$$

Proof: From Proposition 6.2,

$$\begin{aligned}
\mathcal{E}(h) &= \int_D 4e^{-2h}\Lambda^2 H^2 + \int_D \triangle(\Lambda e^{-2h}) \\
&= \int_D (4\Lambda^2 H^2 + \triangle\Lambda) e^{-2h} + 2D(\Lambda - \Lambda(1)) \cdot De^{-2h} + \Lambda\triangle e^{-2h} \\
&= \int_D (4\Lambda^2 H^2 + \triangle\Lambda) e^{-2h} - 2(\Lambda - \Lambda(1))\triangle e^{-2h} + \Lambda\triangle e^{-2h} \\
&= \int_D (4\Lambda^2 H^2 + \triangle\Lambda) e^{-2h} + (2\Lambda(1) - \Lambda)\triangle e^{-2h}.
\end{aligned}$$

∎

Theorem 6.5 *The function $h \equiv 0$ is the unique minimiser of $\mathcal{E}_R(h)$. Equivalently, the map $X_R : D \to S_R$ is the unique optimal conformal parametrisation of S_R.*

Proof: From (21), (22) and the previous Lemma, if 0 denotes the zero function, then

$$\begin{aligned}
\mathcal{E}_R(h) - \mathcal{E}_R(0) &= \int_D h_R(r)\left(e^{-2h} - 1\right) + \int_D g_R(r)\triangle e^{-2h} \\
&= \int_0^1 r h_R(r)\left(\int_0^{2\pi} e^{-2h} - 1\, d\theta\right) dr \\
&\quad + \int_0^1 r g_R(r)\left(\int_0^{2\pi} \triangle e^{-2h} d\theta\right) dr, \qquad (23)
\end{aligned}$$

where

$$\begin{aligned}
h(r) = h_R(r) &= \frac{32(1 + 2r^2R^2)R^4}{(1 + r^2R^2)^4}, \\
g(r) = g_R(r) &= \frac{8R^2}{(1 + R^2)^2} - \frac{4R^2}{(1 + r^2R^2)^2}.
\end{aligned}$$

Note for future reference that

$$h_R(r) > 0 \quad \text{if } 0 \le r \le 1, \qquad (24)$$

$$\int_0^1 r g_R(r)\, dr = \frac{2(1 - R^2)R^2}{(1 + R^2)^2}, \qquad (25)$$

and for $r \in [0, 1]$

$$g_R(r) \begin{cases} \le 0 & \text{if } r \le \sqrt{\frac{1}{\sqrt{2}} - \frac{\sqrt{2}-1}{R^2}} \\ \ge 0 & \text{if } r \ge \sqrt{\frac{1}{\sqrt{2}} - \frac{\sqrt{2}-1}{R^2}}. \end{cases} \qquad (26)$$

If $R \leq \sqrt{2-\sqrt{2}}$ then the second alternative in (26) holds for all $r \in [0,1]$.

Let k be the harmonic conjugate of h such that $k(0) = 0$ and define

$$f = e^{-(h+ik)}. \tag{27}$$

Then f is holomorphic, $|f|^2 = e^{-2h}$, and

$$\triangle e^{-2h} = \triangle |f|^2 = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\left(f\overline{f}\right) = 4|f'|^2. \tag{28}$$

Since f is holomorphic we can write

$$f(z) = \sum_{n\geq 0} a_n z^n = \sum_{n\geq 0} a_n r^n(\cos n\theta + i\sin n\theta).$$

Since $|f(0)| = e^{-h(0)}$ and $h(0) \leq 0$ (see the penultimate paragraph of the proof of Proposition 6.3) it follows

$$|f(0)| = |a_0| \geq 1. \tag{29}$$

Moreover

$$\begin{aligned}
\int_0^{2\pi} |f|^2 d\theta &= 2\pi \sum_{n\geq 0} |a_n|^2 r^{2n}, \\
\int_0^{2\pi} |f'|^2 d\theta &= 2\pi \sum_{n\geq 1} n^2 |a_n|^2 r^{2n-2}.
\end{aligned}$$

Hence (c.f. (23))

$$\int_0^1 r h_R(r)\left(\int_0^{2\pi} e^{-2h} - 1 \, d\theta\right) dr \geq 2\pi \sum_{n\geq 1} |a_n|^2 \int_0^1 r^{2n+1} h_R(r)\,,$$

using (29). Also, using (28),

$$\int_0^1 r g_R(r)\left(\int_0^{2\pi} \triangle e^{-2h} d\theta\right) dr = 8\pi \sum_{n\geq 1} |a_n|^2 \int_0^1 n^2 r^{2n-1} g_R(r).$$

Hence

$$\mathcal{E}_R(h) - \mathcal{E}_R(0) \geq 2\pi \sum_{n\geq 1} |a_n|^2 \int_0^1 F_{R,n}(r), \tag{30}$$

where

$$F_{R,n}(r) := r^{2n+1} h_R(r) + 4n^2 r^{2n-1} g_R(r). \tag{31}$$

For $R \leq 1$ it follows from (24), (25), (26) and the observation r^k is a positive increasing function of $r \in [0,1]$ if $k \geq 0$, that

$$\int_0^1 F_{R,n}(r)dr > 0 \quad \text{if } n \geq 1.$$

This proves the Theorem in this case.

For general $R > 0$, direct and tedious computation (see the following Lemma) shows

$$\int_0^1 F_{R,n}(r)dr > 0 \quad \text{if } n = 1, 2, 3 \tag{32}$$

$$\int_0^1 r^7 g_R(r)dr > 0. \tag{33}$$

It follows from this, (24), (31) and once again the fact that r^k is a positive increasing function of r, that

$$\int_0^1 F_{R,n}(r)dr > 0 \quad \text{if } n \geq 1 \text{ and } R \geq 1.$$

This completes the proof of the Theorem. ∎

We remark that if $n = 1, 2, 3$ then in fact $\int_0^1 r^{2n-1} g_R(r)dr < 0$ for sufficiently large R. But this is compensated for by the positivity of the integral $\int_0^1 r^{2n+1} h_R(r)dr$, c.f. (30) and (31).

The following result was first proved by Mathematica™.

Lemma 6.6 *Using the notation of the previous Theorem, one has for $R > 0$ that*

$$\int_0^1 F_{R,n}(r)dr > 0 \quad \textit{if } n = 1, 2, 3 \tag{34}$$

$$\int_0^1 r^7 g_R(r)dr > 0. \tag{35}$$

Proof: With the aid of Mathematica™, one computes

$$\begin{aligned}
\int F_{R,1}(r)dr &= \frac{-16}{3(1+R^2r^2)^3} + \frac{24}{(1+R^2r^2)^2} - \frac{24}{1+R^2r^2} + \frac{16\,(1+R^2r^2)}{(1+R^2)^2}, \\
\int F_{R,2}(r)dr &= \frac{64R^2r^4}{2(1+R^2)^4} + \frac{128R^4r^4}{2(1+R^2)^4} + \frac{64R^6r^4}{2(1+R^2)^4} + \frac{16}{3R^2(1+R^2r^2)^3} \\
&\quad - \frac{32}{R^2(1+R^2r^2)^2} + \frac{48}{R^2\,(1+R^2r^2)}, \\
\int F_{R,3}(r)dr &= \frac{8}{3}R^{-4}\Big(1+R^2\Big)^{-2}\Big(1+R^2r^2\Big)^{-3} \cdot \\
&\quad \Bigg(\Big(-14 - 28R^2 - 14R^4 - 54R^2r^2 - 108R^4r^2 - 54R^6r^2 \\
&\quad - 72R^4r^4 - 144R^6r^4 - 72R^8r^4 - 27R^6r^6 - 90R^8r^6 \\
&\quad - 45R^{10}r^6 + 39R^8r^8 - 30R^{10}r^8 - 15R^{12}r^8 \\
&\quad + 54R^{10}r^{10} + 18R^{12}r^{12}\Big) \\
&\quad + 12\log(1+R^2r^2)\Big(1 + 2R^2 + R^4 + 3R^2r^2 + 6R^4r^2 + \\
&\quad + 3R^6r^2 + 3R^4r^4 + 6R^6r^4 + 3R^8r^4 + R^6r^6 + 2R^8r^6 \\
&\quad + R^{10}r^6\Big)\Bigg).
\end{aligned}$$

Differentiating back with Mathematica™, one checks these results.

Hence

$$
\begin{aligned}
\int_0^1 F_{R,1}(r)dr &= \frac{8R^2\left(3+3R^2+2R^4\right)}{3(1+R^2)^3},\\
\int_0^1 F_{R,2}(r)dr &= \frac{16R^2\left(3+2R^2\right)}{3(1+R^2)^3},\\
\int_0^1 F_{R,3}(r)dr &= \frac{8}{3}R^{-4}\left(1+R^2\right)^{-3}\Big(-12R^2-30R^4-13R^6+3R^8\\
&\qquad +12\log(1+R^2)\left(1+3R^2+3R^4+R^6\right)\Big).
\end{aligned}
$$

The first two of these expressions are clearly positive for all R. For the third denote the numerator by $p(R^2)$. Then

$$
\begin{aligned}
p(x) &= -12x-30x^2-13x^3+3x^4+12\left(1+3x+3x^2+x^3\right)\log(1+x)\\
p'(x) &= -36x-27x^2+12x^3+36(1+2x+x^2)\log(1+x)\\
p''(x) &= -18x+36x^2+72(1+x)\log(1+x)\\
p'''(x) &= 54+72x+72\log(1+x).
\end{aligned}
$$

Thus $p(0)=p'(0)=p''(0)=0$ and $p'''(x)>0$ if $x>0$. It follows $p(R^2)>0$ if $R>0$. This completes the proof of (34).

We next compute with Mathematica™, differentiating back as a check, that

$$
\begin{aligned}
\int r^7 g_R(r)dr &= R^{-6}\left(1+R^2\right)^{-2}\left(1+R^2r^2\right)^{-1}\cdot\\
&\quad \Big(\Big(-2-4R^2-2R^4+4R^2r^2+8R^4r^2+4R^6r^2\\
&\qquad +3R^4r^4+6R^6r^4+3R^8r^4-R^6r^6-2R^8r^6\\
&\qquad -R^{10}r^6+R^8r^8+R^{10}r^{10}\Big)\\
&\quad -6\log(1+R^2r^2)\left(1+2R^2+R^4+R^2r^2+2R^4r^2+R^6r^2\right)\Big).
\end{aligned}
$$

Hence

$$
\int_0^1 r^7 g_R(r)dr = \frac{6R^2+9R^4+2R^6-6\log(1+R^2)\left(1+2R^2+R^4\right)}{R^6(1+R^2)^2}.
$$

Denoting the numerator by $q(R^2)$ one obtains

$$
\begin{aligned}
q(x) &= 6x+9x^2+2x^3-6(1+2x+x^2)\log(1+x),\\
q'(x) &= 12x+6x^2-12\log(1+x)-12x\log(1+x),\\
q''(x) &= 12x-12\log(1+x).
\end{aligned}
$$

Thus $q(0)=q'(0)=0$ and $q''(x)>0$ if $x>0$. It follows $q(R^2)>0$ if $R>0$.

This completes the proof of the Lemma. ∎

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