Quantum Foundations Talking Points 31/5/19

§§3.2–3.5 of Notes

Pairs of Qubits

- General joint state:
  $\alpha_{00}|0\rangle \otimes |0\rangle + \alpha_{01}|0\rangle \otimes |1\rangle + \alpha_{10}|1\rangle \otimes |0\rangle + \alpha_{11}|1\rangle \otimes |1\rangle\,, \qquad \sum_{j,k}|\alpha_{jk}|^2 = 1$
  $= \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle\,.$
- physical realisations (polarised light, energy levels, spin)
- may be physically separated

Bell States

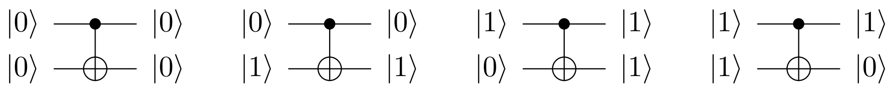
- $|\Phi^+\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)$, $|\Phi^-\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle - |11\rangle\big)$, $|\Psi^+\rangle = \frac{1}{\sqrt{2}}\big(|01\rangle + |10\rangle\big)$, $|\Psi^-\rangle = \frac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big)$.
- Not product states, are "maximally" entangled
- orthonormal basis

Invariance Properties

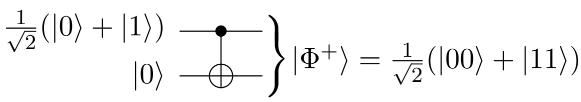
- Invariance property for $|v\rangle = U|0\rangle$, $|v^\perp\rangle = U|1\rangle$, <u>real</u> unitary $U = \begin{bmatrix}\alpha & -\beta\\ \beta & \alpha\end{bmatrix}$.
  Then $|\Phi^+\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big) = \frac{1}{\sqrt{2}}\big(\,|vv\rangle + |v^\perp v^\perp\rangle\big)$
- $|\Psi^-\rangle$ has full invariance
- Measure Alice's qubit w.r.t. any $|v\rangle$, Bob's qubit gives same result with $|v\rangle$. Synchronised coins.
- Can be classically explained, but not so other correlations.

CNOT

- unitary operating on <u>pair</u> of qubits, both need to be present
- $|00\rangle \mapsto |00\rangle \qquad |01\rangle \mapsto |01\rangle \qquad |10\rangle \mapsto |11\rangle \qquad |11\rangle \mapsto |10\rangle$

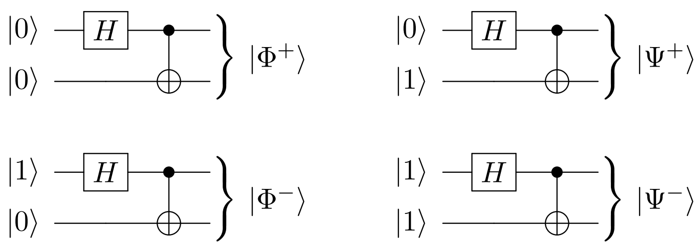


- control and target
- produces entanglement



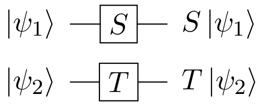
- universal property
- matrix $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$.

Preparing Bell States



Tensor Products of Operators

- Example: $H \otimes I$
- In general: $S \otimes T$,



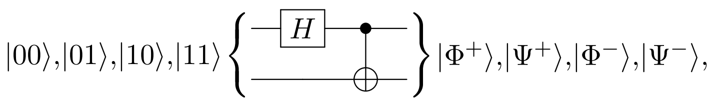
$$(S \otimes T)(|\psi_1\rangle \otimes |\psi_2\rangle) = S|\psi_1\rangle \otimes T|\psi_2\rangle$$

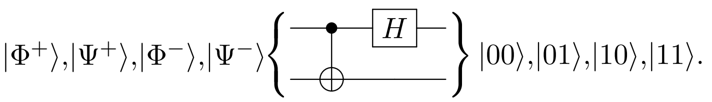
- $S \otimes T = \begin{bmatrix} S_{11}T_{11} & S_{11}T_{12} & S_{12}T_{11} & S_{12}T_{12} \\ S_{11}T_{21} & S_{11}T_{22} & S_{12}T_{21} & S_{12}T_{22} \\ S_{21}T_{11} & S_{21}T_{12} & S_{22}T_{11} & S_{22}T_{12} \\ S_{21}T_{21} & S_{21}T_{22} & S_{22}T_{21} & S_{22}T_{22} \end{bmatrix} = \begin{bmatrix} S_{11}T & S_{12}T \\ S_{21}T & S_{22}T \end{bmatrix}.$
- CNOT is not a tensor product (tensor products don't entangle, or from matrix representation)
- Tensoring preserves unitary and hermitian properties

Measuring Entangled States — Partial Measurement Rule

- Single qubit measured w.r.t. $\{|v\rangle, |v^\perp\rangle\}$:
  $\alpha|v\rangle + \beta|v^\perp\rangle \to \alpha|v\rangle + \cancel{\beta|v^\perp\rangle} \to \frac{\alpha}{|\alpha|}|v\rangle \sim |v\rangle$
  Similarly for outcome $|v^\perp\rangle$.
- *Measure first qubit*
  $$a|vw\rangle + b|vw^\perp\rangle + c|v^\perp w\rangle + d|v^\perp w^\perp\rangle = |v\rangle \otimes \big(a|w\rangle + b|w^\perp\rangle\big) + |v^\perp\rangle \otimes \big(c|w\rangle + d|w^\perp\rangle\big)$$
  $$\to |v\rangle \otimes \big(a|w\rangle + b|w^\perp\rangle\big) + \cancel{|v^\perp\rangle \otimes \big(c|w\rangle + d|w^\perp\rangle\big)}$$
  $$\sim |v\rangle \otimes \left(\frac{a}{\sqrt{|a|^2+|b|^2}}|w\rangle + \frac{b}{\sqrt{|a|^2+|b|^2}}|w^\perp\rangle\right) \quad \text{with probability } |a|^2+|b|^2,$$
  Similarly for outcome $|v^\perp\rangle$.
- *Measure second qubit*
  $|v\rangle \otimes |w\rangle$ with probability $\big(|a|^2+|b|^2\big)\frac{|a|^2}{|a|^2+|b|^2} = |a|^2$
  $|v\rangle \otimes |w^\perp\rangle$ with probability $\big(|a|^2+|b|^2\big)\frac{|b|^2}{|a|^2+|b|^2} = |b|^2$
  Similarly $|v^\perp\rangle \otimes |w\rangle$, $|v^\perp\rangle \otimes |w^\perp\rangle$.
- In computational basis, measurement operator is $\begin{bmatrix} \lambda_{00} & 0 & 0 & 0 \\ 0 & \lambda_{01} & 0 & 0 \\ 0 & 0 & \lambda_{10} & 0 \\ 0 & 0 & 0 & \lambda_{11} \end{bmatrix}$ provided λ_{ij} all distinct.
- Bell $|\Phi^+\rangle$ in computational basis, outcome $|00\rangle$ or $|11\rangle$ each with probability 1/2.

Measuring w.r.t. Bell Basis

- 

hence


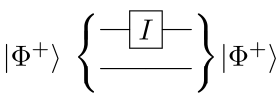
Explain.
- Hence physically realise measurement in the Bell basis

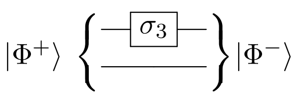
Superdense Coding Comments

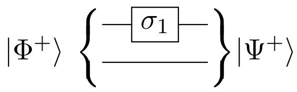
- "normally" 1 qubit codes up 1 bit
- A & B share Bell pair, A transforms her qubit , sends to Bob, he extracts **2** bits of info.

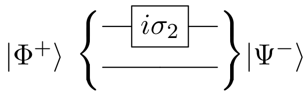
Superdense Coding Method

- A & B share a pair of qubits in Bell state $|\Phi^+\rangle$
- To send 00, 01, 10 or 11 to B, A applies $I \otimes I$, $\sigma_3 \otimes I$, $\sigma_1 \otimes I$, or $i\sigma_2 \otimes I$ respectively to $|\Phi^+\rangle$.









- A sends her transformed qubit to B
- B measures the pair in the Bell basis

Teleportation Comments

- Alice has qubit $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$, wants to send state to Bob
- No electronic connection, just mail postal service
- Share a $|\Phi^+\rangle$ state
- A can send 2 classical bits to Bob, he can then reconstruct $|\psi\rangle$
- *Impressive*!!

Teleportation Method

- $|\psi_0\rangle = |\psi\rangle \otimes |\Phi^+\rangle = \frac{1}{\sqrt{2}}\big(\alpha\,|000\rangle + \alpha\,|011\rangle + \beta\,|100\rangle + \beta\,|111\rangle\,\big)$
  $= \frac{1}{2}\Big(\,|\Phi^+\rangle\,\big(\alpha\,|0\rangle + \beta\,|1\rangle\,\big) + |\Phi^-\rangle\,\big(\alpha\,|0\rangle - \beta\,|1\rangle\,\big) + |\Psi^+\rangle\,\big(\beta\,|0\rangle + \alpha\,|1\rangle\,\big) + |\Psi^-\rangle\,\big(-\beta\,|0\rangle + \alpha\,|1\rangle\,\big)\Big)\ .$
- Alice measures in Bell basis, gets one of 4 results, mails which result to Bob
- Bob applies gates $I, \sigma_3, \sigma_1, i\sigma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ resp. Always gets $\alpha\,|0\rangle + \beta\,|1\rangle$. *Impressive*!!
- *Note*: Bob not entangled until Alice measures in Bell basis.