The Elements of Finite Elements

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October 95

1 Finite elements and finite differences

The problem is to find approximations to solutions of P.D.E.'s defined over some domain $\Omega \subset \mathbb{R}^n$ and to examine the error in various norms.

There are two main approaches; *finite differences* and *finite elements*.

Finite difference methods are characterised by replacing derivatives by difference quotients.

Finite element methods are characterised:

- by replacing a variational problem in $H^1(\Omega)$ by the analogous problem in some *finite dimensional* "discrete" function space X approximating $H^1(\Omega)$ (often a subspace) — the differential operator remains unchanged;
- X has the property that it arises from a triangulation of Ω, and has a basis consisting of functions defined locally and having small support.

Finite elements are much more versatile if $n \geq 2$, at least for variational problems, are well adapted to domains with complicated boundary, and have a well-developed mathematical theory.

2 The smooth problem

Consider the model elliptic problem (Poissons's equation):

Find $u \in H^1(\Omega)$ such that

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \ (\subset \mathbb{R}^n), && (1)\\ u &= g \quad \text{on } \partial\Omega, && (2) \end{aligned}$$

where f and g are smooth, and $\partial\Omega$ is smooth. In weak form this is

$$\begin{aligned} \int_\Omega Du\, D\psi &= \int_\Omega f\psi \quad \forall \psi \in H^1_0(\Omega), && (3)\\ u &= g \quad \text{on } \partial\Omega. && (4) \end{aligned}$$

3 Triangulations

In order to give a discrete analogue of (3) and (4) we first triangulate Ω. See Figure 1.

For various $h > 0$, let $\mathcal{T}_h$ be a triangulation of Ω in a sense that can easily be made precise. Each triangle has side $\leq h$ with interior angles are bounded away from zero independently of h.

$$\Omega_h := \bigcup_{T \in \mathcal{T}_h} T.$$

If Ω is *convex* then $\Omega_h \subset \Omega$. If Ω has *piecewise linear boundary* then $\Omega_h = \Omega$, provided each corner of $\partial\Omega$ is a node of $\mathcal{T}_h$.

The nodes of $\mathcal{T}_h$ are denoted by

$$v_1, \ldots, v_M, v_{M+1}, \ldots, v_N,$$

where $v_1, \ldots, v_M$ are *interior nodes* and $v_{M+1}, \ldots, v_N$ are *boundary nodes*.

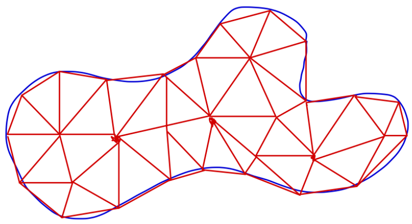

Figure 1: Ω in blue and Ω_h in red.

4 Finite element spaces

1. Define the *finite element space* or *discrete space* by

$$X_h := \{v \in C^0(\Omega_h) \mid v \in P_1(T)\ \forall T \in \mathcal{T}_h\},$$

where $P_1(T)$ denotes the set of polynomials of degree one over T. Thus $v \in X_h$ iff v is continuous and piecewise linear over Ω_h. Clearly,

$$X_h \subset H^1(\Omega_h).$$

2.

$$X_{h0} := \{v \in X_h \mid v = 0 \text{ on } \partial\Omega_h\}.$$

Clearly $X_{h0} \subset H_0^1(\Omega_h)$, and $X_{h0} \subset H_0^1(\Omega)$ if Ω is convex and functions in X_{h0} are extended to be zero in $\Omega \setminus \Omega_h$.

3. A natural basis for X_h consists of the *hat functions* $\phi_i \in X_h$, for $i = 1, \ldots, N$, defined by

$$\phi_i(v_j) = \delta_{ij}.$$

4. A basis of X_{h0} consists of those ϕ_i for $1 \le i \le M$, i.e. such that v_i are interior nodes.

5 The discrete problem

The simplest and most commonly treated case is when $\partial\Omega$ is piecewise linear, so $\Omega = \Omega_h$; and $g = 0$ on $\partial\Omega_h$. Keep this case in mind in the following.

The analogue of (3) and (4) is:

Find $u_h \in H^1(\Omega_h)$ such that

$$\int_{\Omega_h} Du_h\, D\psi_h = \int_{\Omega_h} f\psi_h \quad \forall \psi_h \in X_{h0}, \tag{5}$$

$$u_h = I_h g \quad \text{on } \partial\Omega_h, \tag{6}$$

where $I_h g$ is the piecewise linear interpolant of g on $\partial\Omega_h$, characterised by

$$I_h g(v_i) = g(v_i), \quad \text{for } i = M+1, \ldots, N.$$

To compute u_h, let

$$u_h = \sum_{i=1}^{N} \alpha_i \phi_i.$$

Thus $\alpha_i = u_h(v_i)$. The problem is to find the α_i.

It is sufficient to take as test functions ψ_h in (5) the basis functions $\phi_1, \dots, \phi_M$, and so

$$\begin{aligned} \sum_i \alpha_i \underbrace{\int_{\Omega_h} D\phi_i \, D\phi_j}_{a_{ij}} &= \underbrace{\int_{\Omega_h} f\phi_j}_{b_j} \quad j = 1, \dots, M, \\ \alpha_j &= g(v_j) \quad j = M+1, \dots, N. \end{aligned}$$

This is a system of N equations for N unknowns α_i. It is easily checked to be positive definite and so has a unique solution.

Remarks

1. The matrix $[a_{ij}]$ is called the *stiffness matrix* and the vector (b_i) is called the *load vector*.
2. $\int_{\Omega_h} f\phi_i$ is usually not known exactly, but can be computed to any desired degree of accuracy.
3. The matrix $[a_{ij}]$ is *sparse*, i.e. for any i only 5 or 6 of the j are such that $a_{ij} \neq 0$. This is extremely important computationally, as a sparse $N \times N$ linear system can often be solved (iteratively) in the order of $O(N)$ operations, rather than $O(N^3)$ operations.

6 Statement of error estimates

In Section 9 we will prove the standard *error estimates*:

Proposition 6.1. Let u and u_h solve the smooth and discrete problems (3)-(4) and (5)-(6) respectively. Then

$$|u - u_h|_{H^1(\Omega_h)} \le ch|u|_{H^2(\Omega_h)}, \tag{7}$$

$$\|u - u_h\|_{L^2(\Omega_h)} \le ch^2|u|_{H^2(\Omega_h)}. \tag{8}$$

Here and elsewhere, $|\cdot|$ denotes a semi-norm, and $\|\cdot\|$ denotes a norm. It is often convenient to work with semi-norms since they scale well.

Remarks

1. These results are optimal with respect to powers of h, in the sense that up to this order of error there is typically *no* member of X_h closer to u; see Remarks 3 and 4 in Section 7.
2. If Ω has piecewise linear boundary, then we can replace Ω_h by Ω in (7) and (8).
3. If Ω is convex, then $\Omega_h \subset \Omega$. Moreover, for any function u,

   $$\|u\|_{H^1(\Omega\setminus\Omega_h)} \le ch\|u\|_{H^2(\Omega)}.$$

   as follows from trace theory and integration over the boundary strip. Together with (7) and (8) this can be interpreted as implying that u_h provides an $O(h)$ approximation to u in the H^1 norm.

   Concerning an $O(h^2)$ approximation in the L^2 norm; suppose u_h is extended from each boundary triangle T to the corresponding "curved" triangle $\tilde{T}$ which has part of its boundary on $\partial\Omega$, so that the extended function $\tilde{u}_h$ is linear on $\tilde{T}$.

   Then trace theory and integration over the boundary strip, together with the fact $\|u-u_h\|_{L^2(\partial D_h)} \le ch|u|_{H^1(\partial D_h)}$ (c.f. Section 7), implies

   $$\|u - u_h\|_{L^2(\Omega\setminus\Omega_h)} \le ch^2\|u\|_{H^1(\Omega)}.$$

   This, together with (8), can be interpreted as saying that u_h provides an $O(h^2)$ approximation to u in the L^2 norm.

4. In case Ω is not convex, or piecewise linear, there are analogous results concerning $O(h)$ and $O(h^2)$ approximation in the H^1 norm and L^2 norm respectively, but the statements and proofs are a little more complicated.

5. Computationally, one usually refines an initial fairly course grid by repeatedly halving the grid size. The correct power of h in (7) and (8) typically shows up after a few iterations by computing the difference between $u_h - u_{h/2}$ in the appropriate semi-norm or norm. Moreover, one can then estimate c.

7 Interpolation and approximation theory

Before proving the error estimates in the previous section, we need to consider to what extent an arbitrary "smooth" function can be approximated by discrete functions.

If $u \in H^2(\Omega)$ and $n = 1, 2, 3$ then u is continuous and so one can define the piecewise linear interpolant $I_h u$ by

$$I_h u \in X_h, \quad I_h u(v_i) = u(v_i) \text{ for } i = 1, \ldots, N.$$

Then we have the following:

Proposition 7.1. For $n = 1, 2, 3$

$$|u - I_h u|_{H^1(\Omega_h)} \leq ch|u|_{H^2(\Omega_h)}, \tag{9}$$

$$\|u - I_h u\|_{L^2(\Omega_h)} \leq ch^2|u|_{H^2(\Omega_h)}. \tag{10}$$

Remarks

1. An important case is when $\partial\Omega$ is piecewise linear, and so $\Omega = \Omega_h$.

2. The powers of h can be remembered by noting that if Ω_h is replaced by T ($\in \mathcal{T}_h$) in (9) or (10), then both sides of the inequality scale the same in h.

3. In particular, the H^1 distance from X_h to $u \in H^2(\Omega_h)$ is $\leq ch|u|_{H^2(\Omega_h)}$ if $n = 1, 2, 3$. If $n > 3$ then this is still true, but it is necessay to use a different definition of $I_h u$. For example, one can define $I_h u(v_i)$ at nodes v_i to equal the integral average of u over those triangles having v_i as a vertex. This idea is due to Clement, Scott and others; the previous proof has only to be modified a little.

4. The Proposition is optimal in the sense that, *up to powers of* h, one cannot find a member of X_h closer to u than $I_h u$, in either the H^1 semi-norm of L^2 norm. Moreover, one does not obtain better estimates even if u is C^∞.

5. These results generalise, with a similar proof, to $W^{m,p}$ norms and to higher order interpolants. See also Section 10.

Proof. The proof is set out in a manner which is easily generalised to other finite element spaces.

1. *The unscaled estimate*: Let $\hat{T}$ be any "reference" triangle; i.e. a triangle with sides ≤ 1 and interior angles bounded away from zero. (Members of the triangulation $\mathcal{T}$ will later be obtained by scaling an appropriate $\hat{T}$.) Suppose Iv is the linear interpolant of v on $\hat{T}$, i.e. the unique member of $P_1(\hat{T})$ agreeing with v at the vertices of $\hat{T}$. Then

$$\begin{aligned} \|u - Iu\|_{H^1(\hat{T})} &= \|u - \psi - I(u - \psi)\|_{H^1(\hat{T})} \quad \text{for any } \psi \in P_1(\hat{T}), \\ &\quad \text{since } I\psi = \psi \\ &\leq \|u - \psi\|_{H^1(\hat{T})} + \|I(u - \psi)\|_{H^1(\hat{T})} \\ &\leq c\|u - \psi\|_{H^2(\hat{T})} \quad \text{by (a) below} \\ &\leq c|u|_{H^2(\hat{T})} \quad \text{for suitable } \psi \text{, by (b) below.} \end{aligned}$$

(a) *Boundedness of I*: If $v \in H^2(\hat{T})$ then

$$\begin{aligned} \|Iv\|_{H^1(\hat{T})} &\leq c\|Iv\|_{C^0(\hat{T})} \quad \text{since } Iv \in P_1(\hat{T}), \\ & \quad\; \text{a finite dimensional space} \\ &\leq c\|v\|_{C^0(\hat{T})} \quad \text{by definition of } I \\ &\leq c\|v\|_{H^2(\hat{T})} \quad \text{by Sobolev imbedding.} \end{aligned}$$

Moreover, elementary arguments show that c depends only on the upper bound (here 1) on the length of the sides, and on the interior angle bounds for $\hat{T}$; c.f. the first paragraph in the proof of Proposition 8.1.

(b) *A Poincaré inequality*: By a compactness argument $\exists \psi \in P_1(\hat{T})$ such that

$$\|u - \psi\|_{H^2(\hat{T})} \leq c|u|_{H^2(\hat{T})}. \tag{11}$$

As before, c is independent of $\hat{T}$.

2. *The scaled estimate*: It follows from the unscaled estimate that

$$|u - Iu|_{H^1(\hat{T})} \leq c|u|_{H^2(\hat{T})}, \quad \|u - Iu\|_{L^2(\hat{T})} \leq c|u|_{H^2(\hat{T})}.$$

By scaling, squaring, and adding, and taking square roots

$$|u - I_h u|_{H^1(\Omega_h)} \leq ch|u|_{H^2(\Omega_h)}, \quad \|u - I_h u\|_{L^2(\Omega_h)} \leq ch^2|u|_{H^2(\Omega_h)}.$$

Note that $I_h u \in H^1(\Omega_h)$ and hence $u - I_h u \in H^1(\Omega_h)$. In particular, the previous additions are justified.

∎

8 Inverse estimates

Since X_h is finite dimensional, all norms are comparable. In particular, higher norms can be estimated in terms of lower norms, although the constant will depend on h and blow-up as $h \to \infty$, and hence as X_h "approaches" $H^1(\Omega)$. (The rate of blow-up is important; and could perhaps be more utilised in analysing smooth problems.) For example:

Proposition 8.1. If $\psi_h \in X_h$ then

$$|\psi_h|_{H^2} \leq ch^{-1}|\psi_h|_{H^1}, \tag{12}$$
$$|\psi_h|_{H^1} \leq ch^{-1}\|\psi_h\|_{L^2}. \tag{13}$$

Proof. With $\hat{T}$ as before,

$$|\psi|_{H^2(\hat{T})} \leq c|\psi|_{H^1(\hat{T})}$$

for $\psi \in P_1(\hat{T})$. This essentially comes from the finite dimensionality of $P_1(\hat{T})$, which implies all norms are equivalent. (More precisely, take a *fixed* $\hat{T}$. Then the result is true; noting that we can use the H^1 *semi*-norm on the right since it is zero only if ψ is a constant; in which case the H^2 semi-norm is also zero. The fact that we can use a fixed c if the sides of $\hat{T}$ are bounded above and the interior angles are bounded below, follows from elementary considerations by using a controlled one-one map from some fixed $\hat{T}$.)

From scaling,

$$|\psi_h|_{H^2(T)} \leq ch^{-1}|\psi|_{H^1(T)}$$

for any $\psi_h \in P_1(T)$ and $T \in \mathcal{T}_h$. Squaring and summing over $\mathcal{T}_h$ gives (12).

The proof of (13) is similar. ∎

Remarks

1. The powers of h can, as usual, be read off by a scaling argument.
2. The results easily generalise to other norms.

9 Proof of error estimates

To simplify the arguments:

Assume Ω is convex or $\partial\Omega$ is piecewise linear.

Proof. **1** *Proof of* (7): Subtracting (3) and (5)

$$\int_{\Omega_h} (Du - Du_h) D\psi_h = 0 \quad \text{if } \psi_h \in X_{h0},$$

since we may extend ψ_h to be zero in $\Omega \setminus \Omega_h$. Hence

$$\begin{aligned} 0 &= \int_{\Omega_h} (Du - Du_h)(Du_h - DI_h u) \quad \text{as } u_h = I_h u \text{ on } \partial\Omega_h \\ &= \int_{\Omega_h} (Du - Du_h)(Du_h - Du + Du - DI_h u). \end{aligned}$$

Hence

$$\|Du - Du_h\|_{L^2(\Omega_h)} \le \|Du - DI_h u\|_{L^2(\Omega_h)},$$

and so

$$|u - u_h|_{H^1(\Omega_h)} \le ch|u|_{H^2(\Omega_h)}$$

from (9). This proves (7), at least for the most important cases $n = 1, 2, 3$. (If $n > 3$, slightly more involved arguments are needed, since $I_h u$ is not well-defined.)

2 *Proof of* (8) *with h^2 replaced by h*: Note that

$$\begin{aligned} \|u - u_h\|_{L^2(\partial\Omega_h)} &= \|u - I_h u\|_{L^2(\partial\Omega_h)} \\ &\le ch|u|_{H^1(\partial\Omega_h)} \\ &\le ch\|u\|_{H^2(\Omega_h)}. \end{aligned} \tag{14}$$

by interpolation theory similar to that in Section 7 and by trace theory. $\Big($In the case of homogeneous boundary data and piecewise linear boundary, we trivially have $u = u_h = 0$ on $\partial\Omega_h = \partial\Omega$, and so (14) is trivial.$\Big)$ From this, (7) and a Sobolev type estimate,

$$\|u - u_h\|_{L^2(\Omega_h)} \le ch\|u\|_{H^2(\Omega_h)}.$$

3 *Proof of* (8): We use the Aubin-Nitsche "trick" of solving for the error to obtain the optimal power of h in (8).

$\Big($For simplicity we assume piecewise linear boundary $\partial\Omega$ and that $g = 0$, see (2). But a similar proof works for general Ω and g. The differences are that one should work over Ω_h rather than Ω; define $\overline{u}$ by $\Delta\overline{u} = f$ in Ω_h and $\overline{u} = I_h g$ on $\partial\Omega_h$; define w as below except that u is replaced by $\overline{u}$ and Ω by Ω_h; estimate $\|\overline{u} - u_h\|^2_{L^2(\Omega_h)}$ in the way $\|u - u_h\|^2_{L^2(\Omega)}$ is estimated below; and then (straightforwardly) estimate $\|u - \overline{u}\|^2_{L^2(\Omega_h)}$.$\Big)$

Assume $g = 0$ and $\partial\Omega$ is piecewise linear.

Define w by

$$-\Delta w = u - u_h \quad \text{in } \Omega \tag{15}$$

$$w = 0 \quad \text{on } \partial\Omega. \tag{16}$$

In weak form, (15) becomes

$$\int_\Omega Dw\, D\psi = \int_\Omega (u - u_h)\psi \quad \forall \psi \in H^1_0(\Omega). \tag{17}$$

Then

$$\begin{aligned}
\|u-u_h\|^2_{L^2(\Omega)} &= \int_\Omega |u-u_h|^2 \\
&= \int_\Omega (Du-Du_h)Dw \quad \text{from (17) as } u-u_h\in H^1_0(\Omega) \\
&= \int_\Omega (Du-Du_h)(Dw-DI_hw) \quad \text{subtracting (5) and (3),} \\
&\qquad \text{since } I_hw\in H^1_0(\Omega) \\
&\le |u-u_h|_{H^1(\Omega)}\,|w-I_hw|_{H^1(\Omega)} \\
&\le ch|u|_{H^2(\Omega)}\,h|w|_{H^2(\Omega)} \quad \text{by (7) and (9)} \\
&\le ch^2|u|_{H^2(\Omega)}\,\|u-u_h\|_{L^2(\Omega)} \quad \text{by elliptic regularity theory.}
\end{aligned}$$

This gives (8). ∎

10 Generalisations

1 One can consider many other finite element spaces, for example globally C^0 and piecewise quadratic. For higher order problems, e.g. involving the bi-Laplacian, one requires at least global $C^{1,1}$ elements.

2 For $0<\alpha<1$ and $\Omega\subset I\!\!R^n$, let

$$|u|_{H^\alpha(\Omega)} := \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}.$$

For $s=k+\alpha$, $0<\alpha<1$, $k\in\mathcal{N}$, let

$$|u|_{H^s(\Omega)} := |D^k u|_{H^\alpha(\Omega)}.$$

For $s>0$,

$$\|u\|^2_{H^s(\Omega)} := \|u\|^2_{L^2(\Omega)} + |u|^2_{H^s(\Omega)}.$$

Similar definitions apply to $\partial\Omega$, with n replaced by $n-1$.

These spaces are important:

1. If $g\in L^2(\partial\Omega)$ then there exists a unique harmonic extension u to Ω with trace g, and moreover for $s\ge 0$
$$\|u\|_{H^{s+\frac12}(\Omega)} \le c\|g\|_{H^s(\partial\Omega)}.$$
2. Conversely, for *any* $u\in H^{1/2}(\Omega)$, u has a well-defined trace g on $\partial\Omega$ and for $s\ge 0$
$$\|g\|_{H^s(\partial\Omega)} \le c\|u\|_{H^{s+\frac12}(\Omega)}.$$

3 The interpolant I_hu defined in Remark 2 of Section 7 satisfies

$$|u-I_hu|_{H^r} \le ch^{s-r}|u|_{H^s}$$

for $0\le r\le 1$, $r\le s$, $0\le s\le 2$, where $|\cdot|_{H^0} := \|\cdot\|_{L^2}$.

The h^{s-r} comes from scaling considerations as in Step 2 in the proof in Section 7.

The $r\le 1$ comes from the additive requirement in Step 2, which requires $X_h\subset H^r$, and is true iff $r\le 1$.

The $s\le 2$ comes from the Poincaré inequality in Step 1(b), since $\psi\in P_1(T)$.

If I_h is the *standard* interpolation operator, then $0<s\le 2$ is replaced by $1/2<s\le 2$ for $n=1$, by $1<s\le 2$ for $n=2$, and by $3/2<s\le 2$ for $n=3$. The lower bound on s corresponds to the requirement I_hu be well-defined if $u\in H^s$, i.e. the requirement $H^s\subset C^0$.

Similar considerations indicate the interpolation results one can expect in other situations.