Curvature Flows and Applications

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If you’re wondering where I’ve been...

Getting a promotion (East China Normal University)
If you’re wondering where I’ve been...

Getting fit (Riding from Cambridge to Munich)
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The Heat Equation

The source of inspiration diffused throughout everything that follows is the classical heat equation.

- Let $u(x, t)$ be the temperature of a point $x$ on a nice ($n$-dimensional, because we’re mathematicians) system $\Omega \subset \mathbb{R}^n$ at time $t \in \mathbb{R}$.

- Then, under certain postulates, $u$ evolves according to the heat equation:

  $$\frac{\partial u}{\partial t} = \Delta u := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_i}.$$ 

- Specifying appropriate initial data $u_{t=0}$ and boundary data $u_{\partial \Omega}$ allows us to determine $u$ everywhere for positive times.

- For simplicity, we will assume spatially periodic boundary conditions. Effectively, this corresponds to looking at the equation on a torus, so we can ignore issues to do with the boundary.
The Heat Equation

Jean Baptiste Joseph Fourier (1768–1830), whose 1807 treatise contained the first detailed theory of heat.
Properties of Solutions of the Heat Equation

The heat equation has some wonderfully useful properties which carry over to much more general parabolic (or ‘heat-type’) equations:

1. **The Maximum Principle**: At a point where \( u \) attains a maximum in space (that is, in \( x \)), the second derivatives in each direction are non-positive. By the heat equation, the time derivative is non-positive. It follows that the maximum temperature does not increase as time passes.

2. **Smoothing**: Derivatives of \( u \) are bounded for \( t > 0 \) in terms of bounds on \( u \). It follows that, even if you start with a heat distribution which is discontinuous, it immediately becomes smooth (that’s right, \( C^\infty \! \)!).

3. **Eventual simplicity**: Solutions converge smoothly (in \( C^\infty \)) to constants as \( t \to \infty \).

A further useful property which holds for many but not all heat-type equations is the gradient property:

4. **Gradient Flow**: The heat equation is the flow of steepest decrease of the Dirichlet Energy:

\[
E(u) := \frac{1}{2} \int_{\Omega} ||\nabla u||^2 \, dx \quad (1.1)
\]
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The Curve Shortening Flow

Now, instead of a function $u$, let us consider a regular planar curve $\gamma : I \rightarrow \mathbb{R}^2$.

- Recall that the curvature $k$ of $\gamma$ at a point $x \in I$ is one on the radius of the circle that best fits $\gamma$ at $\gamma(x)$:

- Its sign is fixed by a choice of orientation (cont. unit normal field $\nu$) for the curve.

- If we traverse the curve with unit speed, then the curvature is the signed magnitude of the acceleration:

$$\gamma''(x) = -k(x)\nu(x).$$

- Therefore the natural ‘heat equation for curves’ is the curve shortening flow:

$$\frac{\partial \gamma}{\partial t}(x, t) = -k(x, t)\nu(x, t).$$

(CSF)
The Curve Shortening Flow
The Curve Shortening Flow

The nice diffusion properties of the heat equation are shared by (CSF):

1. **The Maximum Principle:**
   (i) If two curves are initially disjoint, they remain so;
   (ii) Simple (Non-self-intersecting) curves do not develop self-intersections;
   (iii) Initially convex (that is, $k > 0$) curves remain convex.

2. **Smoothing:**
   (i) Derivatives of $k$ are bounded for $t > 0$ whenever $k$ is also bounded;
   (ii) Initial curves with bounded $k$ become immediately smooth.

3. **Eventual simplicity** (AKA Grayson’s Theorem): Closed simple curves converge to points $p \in \mathbb{R}^2$. After rescaling to keep total length constant, they converge smoothly to circles.

4. **Gradient Flow:** The curve shortening flow is the flow of steepest decrease of curve-length.
Grayson’s Theorem

http://www.math.wisc.edu/~angenent/curveshortening/
Mean Curvature Flow

The next equation I want to introduce is the mean curvature flow, which is a natural generalisation of the curve shortening flow to higher dimensions.

- The mean curvature, $H$, of a hypersurface $M^n \subset \mathbb{R}^{n+1}$ at a point $X \in M^n$ is $n$ times the average of the geodesic curvatures in all directions tangent to $M$ at the point $X$.
- Equivalently, $H$ is the sum the $n$ principal curvatures $\kappa_i$.
- Analytically, these are the eigenvalues of the second fundamental form $h$, which is given in coordinates by

$$h_{ij} = \left\langle \frac{\partial^2 X}{\partial x^i \partial x^j}, \nu \right\rangle.$$

- Geometrically, $\kappa_i$ are the geodesic curvatures in the principal directions (the eigenvectors of $h$).
- It is useful to consider the case of a surface $M^2 \subset \mathbb{R}^3$:

Then $\kappa_{\text{max}}$ is the geodesic curvature along the direction in which the surface curves most ‘inwards’ (in this case $e_2$), and $\kappa_{\text{min}}$ is the geodesic curvature along the direction in which the surface curves most ‘outwards’ (in this case $e_1$).
A natural generalisation of the curve shortening flow is therefore given by the *mean curvature flow*:

\[
\frac{\partial X}{\partial t}(x, t) = -H(x, t)\nu(x, t),
\]

(MCF)

where \( X(\cdot, t) : M^n \to \mathbb{R}^{n+1} \) is an evolving immersion of \( M^n \).

▶ In fact, since the second fundamental form is constructed from the second derivatives of \( X \), we have

\[-H\nu = \Delta X := \text{div} (DX).\]

▶ So (MCF) is the natural ‘heat equation for a hypersurface’.
The Mean Curvature Flow

Our nice diffusion properties also hold for the mean curvature flow:

1. **The Maximum Principle:**
   1.1 If two hypersurfaces are initially disjoint, they remain so;
   1.2 Embedded hypersurfaces remain embedded (no self-intersections);
   1.3 Initially mean convex (that is, $H > 0$) hypersurfaces remain mean convex;
   1.4 Initially convex (that is, all geodesic curvatures are positive. Equivalently, the hypersurface bounds a convex region) hypersurfaces remain convex.

2. **Smoothing:**
   2.1 Derivatives of curvature are bounded for $t > 0$ whenever curvature is bounded;
   2.2 Initial hypersurfaces with bounded curvature become immediately smooth.

3. **Eventual simplicity** (AKA Huisken’s Theorem): Convex, embedded, closed hypersurfaces converge to points $p \in \mathbb{R}^{n+1}$. After rescaling to keep the area constant, they converge smoothly to spheres.

4. **Gradient Flow:** The mean curvature flow is the flow of steepest decrease of surface area.
Extensions of the Mean Curvature Flow

We note that the mean curvature flow makes sense in more general settings.

- Every $C^2$ immersion $X$ of a manifold $M^n$ into a Riemannian manifold $(N^{n+k}, \langle \cdot, \cdot \rangle)$ equips the submanifold $X(M)$ with a mean curvature vector $\vec{H} = -H^\alpha \nu_\alpha$ (here $\alpha$ is an index which is summed over the $k$ normal directions).
- Therefore the mean curvature flow makes sense as an evolution equation for immersed submanifolds of Riemannian manifolds.
- However, the ambient geometry and the extra dimensions lead to analytical difficulties.
- Moreover, some of the nice diffusion properties fail; for example, the ambient curvature can stop the flow from preserving positivity of various curvatures of $X$, and the extra room to move means that self-intersections can develop.
There is a rather general procedure for producing heat-like curvature flows.

- We wish to evolve hypersurfaces $M^n$ of $\mathbb{R}^{n+1}$ (or $(N^{n+1}, \langle \cdot, \cdot \rangle)$).
- Then any (smooth) symmetric function $f$ of $n$ variables which is monotone increasing in each variable defines a suitable speed function:

$$F(x, t) := f(\kappa_1(x, t), \ldots, \kappa_n(x, t)),$$

where $\kappa_i$ are the principal curvatures of $X$.

- This yields a general class of curvature flows:

$$\frac{\partial X}{\partial t}(x, t) = -F(x, t)\nu(x, t).$$
Let us note some examples.

Examples:

- Mean curvature flow: \( F = H = \sum_{i=1}^{n} \kappa_i \).
- Harmonic mean curvature flow: \( F = \left( \sum_{i=1}^{n} \frac{1}{\kappa_i} \right)^{-1} \) (convex hypersurfaces).
- Gauß curvature flow: \( F = K = \kappa_1 \ldots \kappa_n \) (convex hypersurfaces).
- Inverse mean curvature flow: \( F = -H^{-1} \) (mean convex hypersurfaces).
- These are just a few natural examples that will come up in what follows. There are many, many more...
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We will now describe some applications of curvature flows. We begin with the most direct:

- Recall that the curve shortening flow shrinks smooth, closed, simple curves to round points.
- This implies that any smooth, closed, simple curve in the plane is diffeomorphic to a circle and bounds a smooth disk.
- This is not true in higher dimensions, but Huisken’s Theorem implies that any smooth, convex, closed, embedded hypersurface of $\mathbb{R}^{n+1}$ is diffeomorphic to a sphere and bounds a smooth $(n+1)$-disk.
- Using surgery to continue the flow through singularities, Huisken and Sinestrari were able to conclude that any smooth, closed, immersed two-convex hypersurface $M^n \subset \mathbb{R}^{n+1}$, $n \geq 3$, is diffeomorphic to either a sphere $S^n$ bounding a smooth $(n+1)$-disk, or a connected sum of tori $S^{n-1} \times S^1$ bounding a smooth $(n+1)$-handlebody.
- **Two-Convex**: The sum of the smallest two principal curvatures is positive.
Isoperimetry

The isoperimetric inequality states that the disk has the smallest area amongst domains of fixed perimeter.

- More precisely, given any (not too crazy) domain $\Omega \subset \mathbb{R}^2$, we have

$$L^2(\Omega) - 4\pi A(\Omega) \geq 0$$

(IPI)

with equality iff $\Omega \subset \mathbb{R}^2$ is a disk.

- If the curve $\omega$ bounding $\Omega$ is $C^2$, then (IPI) is a simple application of CSF:
  1. It is easy to derive evolution equations for the length and area of $\omega$:

$$\frac{\partial}{\partial t} L(\omega) = - \int k^2 \, ds \quad \text{and} \quad \frac{\partial}{\partial t} A(\omega) = - \int k \, ds = -2\pi.$$

  2. Thus

$$\frac{\partial}{\partial t} (L^2 - 4\pi A) = -2L \int k^2 \, ds + 8\pi^2.$$

  3. But, by the Cauchy-Schwarz inequality,

$$L^{-1} \int k^2 \, ds \leq \left( L^{-1} \int k \, ds \right) \cdot \left( L^{-1} \int k \, ds \right),$$

with equality if and only if $k$ is constant (iff $\omega$ is a circle).

  4. Thus

$$\frac{\partial}{\partial t} (L^2 - 4\pi A) \leq -2 \left( \int k \, ds \right)^2 + 8\pi^2 = -2 \cdot (2\pi)^2 + 8\pi^2 = 0.$$

  5. Thus the isoperimetric defect is strictly decreasing unless $\omega$ is a circle.

  6. The claim now follows from Grayson’s Theorem.
The curve shortening flow becomes more interesting when we add topology:

▶ If the initial curve is looped around a ‘topological hole’ in the ambient manifold, such as the hole in a doughnut, then it can no longer be smoothly contracted to a point.

▶ Instead, the energy of the curve goes to zero as $t \to \infty$; therefore, if a limiting curve exists, it is a closed geodesic. We immediately obtain:

▶ **Theorem:** Let $(M^2, g)$ be a compact Riemannian 2-manifold. Then $(M, g)$ has a closed geodesic in every non-trivial homotopy class.
The Sphere Theorem

It is known that if a manifold is simply connected and has constant positive sectional curvatures, then it is a sphere with the standard Riemannian metric.

- In the 1940’s, Heinrich Hopf asked whether we can also wiggle the geometry a little, instead only requiring that the sectional curvatures be close to some constant. Say $1 - \varepsilon < K \leq 1$.

- Work of Rauch, Berger and Klingenberg confirmed the conjecture, with the optimal value of $\varepsilon$:

- **Theorem**: A simply connected Riemannian manifold with sectional curvatures in the range $(1/4, 1]$ is homeomorphic to a sphere. If the value $1/4$ is allowed, there are counterexamples.

- It remained an open conjecture for over half a century that the conclusion of homeomorphism should be improvable to diffeomorphism.

- The problem was solved in the affirmative by Brendle and Schoen in 2010 using the Ricci Flow.
Proving the (Homeomorphic) Sphere Theorem Using Harmonic Mean Curvature Flow

The sphere theorem has also been proved by Andrews using curvature flows:

- Using the pinching assumption, it is not difficult to construct a large disk $D_p(r)$ in $M$ whose boundary is smooth and convex in the ‘outwards’ direction.

- We would like to flow this boundary in the outwards direction to a point via a suitable curvature flow.
- This would demonstrate that the manifold is formed from gluing two disks together along their boundaries, and hence is a sphere.
- The mean curvature flow doesn’t work, but there’s at least one flow speed that does the job; namely, the harmonic mean curvature: $F = \left( \sum_i \frac{1}{\kappa_i} \right)^{-1}$.
- Note that the conclusion in this case is stronger than homeomorphism: The manifold is diffeomorphic to a twisted sphere (two disks glued by a diffeomorphism along their boundary).
- But this is still slightly weaker than diffeomorphism.
The Riemannian Penrose Inequality

The arena of general relativity is a four dimensional curved spacetime (Lorentzian manifold). Here comes a lightning introduction:

- The initial data formulation decomposes spacetime as a three dimensional Riemannian manifold (space) \((\Sigma^3, g)\) whose metric \(g\) evolves in time.
- The existence of a closed minimal surface \(\tau^2\) (that is, \(H = 0\)) in that 3-space is an indication of a black hole forming in the spacetime.
- This is because the mean curvature of the surface is a measure of the focusing of light rays emanating from points of the surface. We call \(\tau\) a horizon.
- There is no general local notion of mass in general relativity; however, the Schwarzschild solution does have a natural mass parameter.
- An isolated gravitating system appears to a distant observer as a Schwarzschild solution. The mass of that Schwarzschild solution is called the ADM mass \(m\) of the system.
- The Riemannian Penrose inequality estimates the ADM mass of a 3-space in terms of the total area of its black hole horizons:

\[
m \geq \sqrt{A/16\pi}.
\]

with equality only in the Schwarzschild spacetime.
- The Riemannian Penrose inequality is a special case of the unsettled Penrose Conjecture.
Proof of the Riemannian Penrose Inequality using Inverse Mean Curvature Flow

Following an idea of Geroch, Huisken and Ilmanen proved the Riemannian Penrose Inequality (for connected horizons) using the inverse mean curvature flow: $F = -1/H$.

- Geroch considered a functional on closed surfaces $\tau^2 \subset \Sigma^3$ (now called the Geroch mass) which interpolates between the left hand side of the Penrose inequality for large surfaces and the right hand side for horizons.

- He also observed that the Geroch mass is monotone non-decreasing under the inverse mean curvature flow (which expands, rather than contracts, a surface).

- Therefore, the idea is to flow the horizon out to infinity by inverse mean curvature, so that the final Geroch mass (the ADM mass) is not less than the initial Geroch mass (the root of the normalised area of the horizon).

- Unfortunately, the hypersurface can become singular before reaching infinity. Huisken and Ilmanen were able to overcome this problem by formulating and analysing a suitable weak notion of the flow.

- The case of multiple horizons was treated by a conformal flow invented by H. Bray.
Lens Making and the Fate of the Rolling Stones

For centuries, highly spherical objects such as lenses and ball bearings have been constructed by a process involving the rubbing two or more of bodies together, which has a tendency to homogenise the curvature, and hence make the bodies umbilical.

- For example, very spherical lenses may be constructed by rubbing a convex surface against a concave surface. The two surfaces will wear until they fit together along a spherical surface.

- One method of producing highly spherical ball bearings was to ‘stir’ a barrel filled with roughly round balls. The modern method involves rolling the bearings between two plates with random indentations.

- It has been suggested by Firey (who considered the wearing of stones on a beach) that the surfaces of the bodies involved in such processes should evolve by the Gauß curvature flow.

- Firey conjectured that the Gauß curvature flow should wear the stones down to small round pebbles, as is observed.

- The conjecture was finally confirmed by Andrews:

  **Theorem:** Let $M_0$ be the boundary of a convex body in $\mathbb{R}^3$. There exista a unique viscosity solution of the Gauß curvature flow starting from $M_0$. The solution eventually becomes smooth, thereafter becoming asymptotically round as it shrinks to a point in finite time.
Bouncing of Charged Droplets

Two oppositely charged droplets of water in an oil will tend to drift together under the influence of their charges.

- As they make contact, one might expect them to coalesce and form one large droplet, and this indeed happens when the charge difference is sufficiently small.
- However it has been observed that for large enough charge differentials, the droplets bounce off each other as they make contact.

- Helmendsorfer and Topping have accounted for this behaviour by assuming that the surface of the coalesced droplet evolves by trying to minimise its surface area (and hence by mean curvature flow).
Image Processing

Curve shortening flows have also proved useful in the field of image processing.

- A problem in image processing involves roughness of image data.
- Natural images are in general very noisy. This can lead to computing problems when algorithms are based on differential features.
- The problem can be overcome by smoothing the contours in the image by running the CSF for a short time to eliminate the small oscillations without significantly changing the global shape of the contours.

An important application is in image recognition software.
Further Applications

There are many more interesting applications of curvature flows including, but not limited to:

▶ Many more topological applications. In particular, ‘local-global’ results. Research in this area is very active.
▶ The link with isoperimetry is also much deeper.
▶ Construction of minimal surfaces and lagrangian submanifolds.
▶ Modelling various physical processes involving surface tension
FIN.