Non-Collapsing & the Lawson Conjecture

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Outline of the talk:

- Review of hypersurface geometry
- Minimal surfaces and the Lawson conjecture
- Mean curvature flows
- Mean curvature flows are non-collapsing
- Proof of the Lawson conjecture.
Hypersurfaces

Recall that an immersed (embedded) hypersurface of a manifold $N$ is an immersion (embedding) $X : M^n \to N^{n+1}$.

- If $(N, \langle \cdot, \cdot \rangle, D)$ is Riemannian, then $M$ is equipped with the metric

  $$g(u, v) := \langle X_* u, X_* v \rangle .$$

- And the Levi-Civita covariant derivative

  $$\nabla_u v := (D_u X_*)^T .$$

- The normal part of $D$ is the second fundamental form of $X$:

  $$h(u, v) := \langle D_u X_* v, \nu \rangle = - \langle X_* v, D_u \nu \rangle .$$

- The Weingarten map of $X$ is the endomorphism $\mathcal{W} : TM \to TM$ corresponding to the bilinear form $h$. It is given by

  $$\mathcal{W}(u) = D_u \nu .$$

- Its eigenvalues, $\kappa_1 \leq \cdots \leq \kappa_n$, are the principal curvatures of $X$ and their sum, $H$, is the mean curvature of $X$. 
Minimal Surfaces

A hypersurface is a minimal (hyper)surface if its mean curvature vanishes.

- For example, the catenoid, $\text{Cat} \subset \mathbb{R}^3$, parametrised locally by
  \[ X(u, v) = (\cosh u \cos v, \cosh u \sin v, v), \]
  is a minimal hypersurface of $\mathbb{R}^3$.

- We have
  \[
  X_u := X_* \partial_u = (\sinh u \cos v, \sinh u \sin v, 1)
  \]
  \[
  X_v := X_* \partial_v = (-\cosh u \sin v, \cosh u \cos v, 0),
  \]

- so that
  \[
  g_{uu} := g(\partial_u, \partial_u) = \langle X_u, X_u \rangle = \sinh^2 u + 1 = \cosh^2 u
  \]
  \[
  g_{vv} := g(\partial_v, \partial_v) = \langle X_v, X_v \rangle = \cosh^2 u
  \]
  \[
  g_{vu} = g_{uv} := g(\partial_u, \partial_v) = \langle X_u, X_v \rangle = 0
  \]
Minimal Surfaces

A unit normal to Cat is given by

$$\nu = \frac{X_v \times X_u}{|X_v \times X_u|} = \frac{1}{\cosh u} (\cos \nu, \sin \nu, -\sinh u).$$

The second (Euclidean) derivatives of $X$ are

$$X_{uu} := D_u X_u = (\cosh u \cos \nu, \cosh u \sin \nu, 0)$$
$$X_{vv} := D_v X_v = - (\cosh u \cos \nu, \cosh u \sin \nu, 0)$$
$$X_{uv} := D_u X_v = (- \sinh u \sin \nu, \sinh u \cos \nu, 0).$$

We may now calculate the components of the second fundamental form:

$$h_{uu} := h(\partial_u, \partial_u) = \langle X_{uu}, \nu \rangle = 1$$
$$h_{vv} := h(\partial_v, \partial_v) = \langle X_{vv}, \nu \rangle = -1$$
$$h_{uv} := h(\partial_u, \partial_v) = \langle X_{uv}, \nu \rangle = 0.$$

It follows that

$$H = \text{tr}(\mathbf{W}) = g^{ij} h_{ij} = \frac{1}{\cosh^2 u} - \frac{1}{\cosh^2 u} = 0.$$
The Clifford Torus

Another example of a minimal surface is the Clifford torus,

\[
\text{Cliff} := \left\{ z \in S^3 \subset \mathbb{R}^4 : z_1^2 + z_2^2 = z_3^2 + z_4^2 = \frac{1}{2} \right\}.
\]

- Cliff is parametrised locally by
  \[
  X(\theta, \phi) = \frac{1}{\sqrt{2}}(\cos \theta, \sin \theta, \cos \phi, \sin \phi).
  \]

- We have
  \[
  X_\theta = \frac{1}{\sqrt{2}}(-\sin \theta, \cos \theta, 0, 0); \quad X_\phi = \frac{1}{\sqrt{2}}(0, 0, -\sin \phi, \cos \phi).
  \]

- Therefore,
  \[
  g_{\theta\theta} := g(\partial_\theta, \partial_\theta) := \langle X_\theta, X_\theta \rangle = \frac{1}{2}; \quad g_{\phi\phi} = \frac{1}{2}; \quad g_{\theta\phi} = 0.
  \]

- Moreover,
  \[
  X_{\theta\theta} := D_\theta X_\theta = -\frac{1}{\sqrt{2}}(\cos \theta, \sin \theta, 0, 0);
  \]
  \[
  X_{\phi\phi} := D_\phi X_\phi = -\frac{1}{\sqrt{2}}(0, 0, \cos \phi, \sin \phi);
  \]
  \[
  X_{\theta\phi} = X_{\phi\theta} = 0.
  \]
The Clifford Torus

- We see that $X_{ij}$ is normal to Cliff for each $i, j$.
- Therefore $\nabla_i \partial_j = 0$ for each $i, j \Rightarrow$ Cliff is flat.
- Now, by twice applying the ‘Gauss equation’, we have:

$$X_{ij} = \overline{D}_i X^*_j \partial_j - \overline{h}_{ij} \overline{\nu}(X) = \nabla_i \partial_j - h_{ij} \nu - \overline{h}_{ij} \overline{\nu}$$

$$= - h_{ij} \nu - \overline{g}_{ij} X,$$

where $\overline{D}, \overline{h}, \overline{\nu}$, and $\overline{g}$ are the connection, second fundamental form, normal and metric on $S^3$.
- Therefore,

$$- h_{0\theta} \nu = X_{\theta\theta} + \langle X_{\theta}, X_{\theta} \rangle X;$$

$$- h_{\phi\phi} \nu = X_{\phi\phi} + \langle X_{\phi}, X_{\phi} \rangle X;$$

$$- h_{0\phi} \nu = X_{\theta\phi} + \langle X_{\theta}, X_{\phi} \rangle X = 0.$$

- We now have

$$- H\nu = - g^{ij} h_{ij} \nu = - 2h_{0\theta} \nu - 2h_{\phi\phi} \nu = 2(X_{\theta\theta} + X_{\phi\phi} + X) = 0.$$

- Thus Cliff is a minimal submanifold of $S^3$.
- The Clifford torus is also an embedded Lagrangian submanifold of $\mathbb{C}^2$ with its standard symplectic structure.
The Lawson Conjecture

In 1970, H. Blaine Lawson posed the following conjecture:

**Conjecture (Lawson 1970)**

*If $X : \Sigma^2 \rightarrow S^3$ is an embedded minimal torus, then it is (up rotating or reflecting $S^3$) the Clifford torus.*

- Note that this is false without the assumption of embeddedness: There are many immersed minimal tori (constructed using the machinery of integrable systems).
- Lawson also constructed embedded minimal surfaces for any higher genus, including non-unique examples for any non-prime genus.
The Lawson Conjecture

The conjecture was recently proved in the affirmative by Simon Brendle.

- In fact, Brendle proved the following:

**Theorem (Brendle 2012)**

If $X : \Sigma^2 \rightarrow S^3$ is a minimal torus, then it is flat.

- That $X$ is the Clifford torus then follows from a classical rigidity theorem of Lawson:

**Theorem (Lawson 1969)**

*Every complete minimal hypersurface of $S^{n+1}$ with parallel Ricci curvature (that is, $\nabla \text{Ric} = 0$) is a product of spheres of the type*

$$S^p \left( \sqrt{\frac{p}{n}} \right) \times S^q \left( \sqrt{\frac{q}{n}} \right),$$

*where $p + q = n$.*

- Thus, if $n = 2$, the only Ricci parallel minimal surfaces are the great spheres $S^2(1)$ and the Clifford tori $S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2})$. 
Before proving the Lawson conjecture, I will introduce an equation related to the minimal surface equation called the *mean curvature flow*, since the main technique used in Brendle’s proof was originally applied to mean curvature flows.
Mean Curvature Flow

Closey related to the minimal surface equation is the mean curvature flow.

- The mean curvature flow moves initial hypersurfaces in the direction of their mean curvature vector $\vec{H} = -H\nu$. More precisely, a smooth family of immersions $X : M^n \times [0, T) \to N^{n+1}$ is the flow by mean curvature of $X_0 : M^n \to N^{n+1}$ if

$$\frac{\partial X}{\partial t} = -H\nu$$

$$X(x, 0) = X_0(x).$$

- **Motivation:** Since $\vec{H} = \Delta X := \text{div}(X_*)$, MCF is a kind of heat equation, so it should ‘smooth out’ complicated immersions. In fact, MCF is the gradient flow of the area functional.

- Parabolic $\Rightarrow$ maximum principle $\Rightarrow$ comparison principle, preservation of embeddedness, preservation of various convexity conditions, etc;

- If $M$ is compact, $N = \mathbb{R}^n$, singularities always occur (when smoothness is violated), and always because $H \to \infty$ (thus $C^2$ smoothness breaks down);

- **(Huisken, 1984):** If $N = \mathbb{R}^{n+1}$, $n \geq 2$, and $X_0$ is locally uniformly convex ($\kappa_i \geq k > 0$), then $X(\cdot, t) \to p \in \mathbb{R}^{n+1}$ and

$$\tilde{X}(\cdot, t) := \frac{X(\cdot, t) - p}{\sqrt{2n(T - t)}} \xrightarrow{C^\infty} \tilde{X}_T, \quad \tilde{X}_T(M) = S^n.$$

- **Corollary:** Any locally uniformly convex Euclidean hypersurface is diffeomorphic to $S^n$. 
Mean Curvature Flow

What about non-convex initial hypersurfaces?

In general, the picture is quite complicated, but in the case that $H > 0$ more is understood:

- No longer get uniform collapse to a round point as $H \to \infty$.
- Now ‘neck-pinches’ (collapsing cylinders) and ‘degenerate neck-pinches’ (collapsing ‘cigars’), and worse (e.g. ‘bubble sheets’, ‘cylinder sheets’ and ‘cigar sheets’), are possible.
- Note that topological information is lost in the latter rescalings.
- However, once we know what the tumor looks like, it can be removed by surgery and the flow continued.

- (Huisken-Sinestrari 2009) If $X_0(M^n) \subset \mathbb{R}^{n+1}$ is uniformly 2-convex ($\kappa_1 + \kappa_2 \geq k > 0$), then $M$ is diffeomorphic to a sphere or a connected sum of tori $S^{n-1} \times S^1$.

- Moral: Understanding what singularities look like yields useful topological information.
- These ideas originate with Hamilton’s Ricci flow program for proving the Poincaré conjecture.
Non-Collapsing

If $X_0 : M \to \mathbb{R}^{n+1}$ is an embedding, additional useful information on singularities may be obtained by taking into account the geometry of the region $\Omega_t$ enclosed by the hypersurface $M_t := X_t(M) := X(M, t)$.

- A compact embedding $X : M \to \mathbb{R}^{n+1}$ separates into a precompact ‘interior’ region $\Omega$ and a non-compact ‘exterior’ region $\Omega^c$.

- Observe that a ball $B \subset \mathbb{R}^{n+1}$ of boundary curvature $k = 1/r$ which touches $X(M)$ at $X(x)$ is contained in $\Omega$ iff

$$\left\| \left( X(x) - k^{-1} \nu(x) \right) - X(y) \right\|^2 \geq k^{-2} \quad \text{for all } y \neq x$$

$$\iff \left\| X(x) - X(y) \right\|^2 - 2k^{-1} \langle X(x) - X(y), \nu(x) \rangle \geq 0 \quad \text{for all } y \neq x$$

$$\iff \frac{2 \langle X(y) - X(x), \nu(x) \rangle}{\left\| X(x) - X(y) \right\|^2} \leq k \quad \text{for all } y \neq x.$$

- Let’s define

$$k(x, y) := \frac{2 \langle X(y) - X(x), \nu(x) \rangle}{\left\| X(x) - X(y) \right\|^2}.$$

- Then

$$\bar{k}(x) := \sup_{y \neq x} k(x, y)$$

is the boundary curvature of the largest ball contained in $\Omega$ which touches $X(M)$ at $X(x)$. 
Non-Collapsing

- We call $\bar{k}$ the **interior ball curvature** of $X$.
- Similarly, the **exterior ball curvature** of $X$,

$$
\underline{k}(x) := \inf_{y \neq k} k(x, y),
$$

is the boundary curvature of the smallest ball, half-space, or ball compliment that contains $\Omega$ and touches $X(M)$ at $X(x)$.
- We’ll say that $X$ is ‘interior’ non-collapsed if there exists $k > 0$ such that

$$
\bar{k}(x) \leq kH(x)
$$

for all $x \in M$,
- and ‘exterior’ non-collapsed if there exists $k > -\infty$ such that

$$
\underline{k}(x) \geq kH(x)
$$

for all $x \in M$.
- Since $\limsup_{y \to x} k(x, y) = \kappa_n(x)$ and $\liminf_{y \to x} k(x, y) = \kappa_1(x)$, every compact, mean convex hypersurface is interior and exterior non-collapsed.
Non-Collapsing

If $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ is time dependent, then we can define

$$
\overline{k}(x, t) := \sup_{y \neq x} k(x, y, t) := \sup_{y \neq x} \frac{2 \langle X(y, t) - X(x, t), \nu(x, t) \rangle}{\|X(x, t) - X(y, t)\|^2}
$$

and

$$
\underline{k}(x, t) := \inf_{y \neq x} k(x, y, t)
$$

(Sheng-Wang, Andrews) If $X_0$ is a compact, mean convex embedding which is interior (exterior) non-collapsed with constant $k$:

$$
\overline{k}_0 \leq kH_0 \quad (k_0 \geq kH_0)
$$

then the corresponding solution $X_t$ of MCF is interior (exterior) non-collapsed with the same constant $k$ at all times:

$$
\overline{k}(\cdot, t) \leq kH(\cdot, t) \quad (\underline{k}(\cdot, t) \geq kH(\cdot, t)).
$$

Corollary: Since $\Gamma$, the ‘Grim Reaper’ translating curve, is collapsed, mean convex embedded solutions cannot asymptote to the ‘cigar sheet’ $\Gamma \times \mathbb{R}^{n-1}$. 
Non-Collapsing (Proof)

Proposition

The interior (exterior) ball curvature is a viscosity sub-(super-)solution of the equation

\[ \partial_t u = \Delta u + |\nabla|^2 u. \]  \( (0.1) \)

- This is significant because \( H \) is precisely a solution of (0.1).
- Thus, for mean convex solutions, we have (in the viscosity sense)

\[
(\partial_t - \Delta) \left( \frac{k}{H} \right) = \frac{1}{H} (\partial_t - \Delta) k - \frac{k}{H^2} (\partial_t - \Delta) H - \frac{1}{H} \left\langle \nabla \left( \frac{k}{H} \right), \nabla H \right\rangle \n\]

\[
\leq - \frac{1}{H} \left\langle \nabla \left( \frac{k}{H} \right), \nabla H \right\rangle. \n\]

- Therefore, at any spatial maximum of \( k/H \), we have \( \partial_t (k/H) \leq 0 \).
- It follows that \( k/H \) cannot increase above its initial maximum, and we have proved that mean convex mean curvature flow is (exterior) non-collapsed.
Since $\bar{k}$ is not smooth, the notion of viscosity solution is needed.

**Definition**

$f$ is a viscosity subsolution of

$$\partial_t u = \Delta u + |\mathcal{W}|^2 u$$

if given any $(x_0, t_0) \in M \times [0, T)$, every smooth function $\phi$ satisfying $\phi(x_0, t_0) = f(x_0, t_0)$ and $\phi(x, t) \geq f(x, t)$ (for $x$ near $x_0$ and $t \leq t_0$ near $t_0$) satisfies

$$\partial_t \phi \leq \Delta \phi + |\mathcal{W}|^2 \phi$$

at $(x_0, t_0)$.

- The maximum principle holds for viscosity sub/super-solutions.
Non-Collapsing in the Sphere

Since geodesic balls in $S^3$ are the intersection of $S^3$ with balls in $\mathbb{R}^4$, the technique also yields useful results when the ambient space is the sphere.

**Proposition**

Let $X : M^n \times [0, T) \to S^{n+1}$ solve the mean curvature flow in the sphere $S^{n+1}$. Then the interior (exterior) ball curvature $\overline{k}$ ($k$) is a viscosity sub-(super-)solution of

$$(\partial_t - \Delta)u = |\nabla u|^2 u + 2H - nu.$$  (0.2)

- Equation (0.2) is again the equation satisfied by the mean curvature of the solution.
- A non-collapsing statement follows.
Non-Collapsing in the Sphere

Since geodesic balls in $S^3$ are the intersection of $S^3$ with balls in $\mathbb{R}^4$, the technique also yields useful results when the ambient space is the sphere.

Proposition

Let $X : M^n \times [0, T) \to S^{n+1}$ solve the mean curvature flow in the sphere $S^{n+1}$. Then the interior (exterior) ball curvature $k$ ($\bar{k}$) is a viscosity sub-(super-)solution of

$$
(\partial_t - \Delta)u = |\nabla u|^2 u + 2H - nu - \frac{|\nabla u|^2}{u}.
$$

(0.2)

- In fact, it is possible to squeeze an extra gradient term out of the computation.
- It will be useful later.
Proof of the Lawson Conjecture

The key to the proof of Brendle’s theorem is the following observation:

**Proposition**

*If $X : \Sigma \to S^3$ is an embedded minimal torus, then at every point of $\Sigma$ the interior ball curvature is $\kappa_2$ and the exterior ball curvature is $\kappa_1 = -\kappa_2$;*

- We can use this to deduce that $\nabla h = 0$.
- It then follows from the Gauss equation that $\nabla \text{Ric} = 0$.
- The Lawson conjecture then follows from Lawson’s rigidity theorem.
Proof of the Lawson Conjecture

Any minimal surface is a stationary solution of the mean curvature flow. Therefore $\bar{k}$ satisfies (in the viscosity sense)

$$(\partial_t - \Delta)\bar{k} \leq |\mathcal{W}|^2\bar{k} + 2H - 2\bar{k} - \frac{|\nabla\bar{k}|^2}{\bar{k}}$$

$$\iff 0 \leq \Delta \bar{k} + (|\mathcal{W}|^2 - 2)\bar{k} - \frac{|\nabla\bar{k}|^2}{\bar{k}}. \quad (0.3)$$

- However, since $H = 0$, we cannot compare $\bar{k}$ with $H$.
- Brendle instead compares $\bar{k}$ with $\kappa_2$.
- This is a good idea, because a minimal torus in $S^{n+1}$ cannot have umbilic points (proved using the holomorphic Hopf differential), so that $\kappa_2$ is everywhere positive, and smooth: Since $\kappa_1 = -\kappa_2$, we have

$$\kappa_2 = \frac{1}{\sqrt{2}}|\mathcal{W}|.$$ 

- Moreover, the Clifford torus satisfies $\bar{k} = \kappa_2$ (and $\underline{k} = \kappa_1$).
- Even better: (0.3) is satisfied (with equality) by $\kappa_2$. 
Proof of the Lawson Conjecture

Claim:

\[ \Delta \kappa_2 + \left( |W|^2 - 2 \right) \kappa_2 - \frac{\left| \nabla \kappa_2 \right|^2}{\kappa_2} = 0 \]

Proof:

- A simple computation gives

\[ \Delta (|W|^2) = 2|W| \langle \Delta W, W \rangle + |\nabla W|^2. \]

- On the other hand, by commuting derivatives \( \nabla_i \nabla_j W_{kl} - \nabla_k \nabla_l W_{ij} \) and taking the trace over \( k, l \), we can obtain the Simons identity:

\[ \Delta W_{ij} + (|W|^2 - 2) W_{ij} = 0, \]

- from which it follows that

\[ \Delta (|W|^2) = 2 \langle \Delta W, W \rangle + 2|\nabla W|^2 = 2|\nabla W|^2 - 2(|W|^2 - 2)|W|^2. \]

- Equating the two expressions for \( \Delta |W|^2 \) and applying \( \kappa_2 = |W|/\sqrt{2} \) yields the claim.
Proof of the Lawson Conjecture

Putting this together, we find (in the viscosity sense),

\[ 0 \geq \Delta \left( \frac{\kappa}{\kappa_2} \right) + \left( \frac{\kappa}{\kappa_2} \right) \| \nabla \left( \frac{\kappa}{\kappa_2} \right) \|^2. \]

- By the strong maximum principle, \( \frac{\kappa}{\kappa_2} \) is constant, so that \( \kappa = C \kappa_2 \).
- But \( \kappa = \kappa_2 \) at a point where \( \kappa_2 \) attains a maximum.
- We conclude \( \kappa = \kappa_2 \).
- A similar argument using the exterior ball curvature yields \( \kappa = \kappa_1 = -\kappa_2 \).
- We have proved:

**Lemma**

Let \( X : \Sigma \to S^3 \) be an embedded minimal torus. Then at each point \( X(x) \), the largest interior ball touching \( X(\Sigma) \) at \( X(x) \) has curvature \( \kappa_2 \) and the largest exterior ball touching \( X(\Sigma) \) at \( X(x) \) has curvature \( -\kappa_1 \).

- We will use this to show that \( \nabla h = 0 \).
We have proved that
\[
\kappa_2(x) = \overline{k}(x) \geq \frac{2 \langle X(x) - X(y), \nu(x) \rangle}{|X(x) - X(y)|^2}
\]
for all \( y \in M \setminus \{x\} \).

It follows that
\[
0 \leq 2Z(x, y) := \kappa_2(x, t)|X(x) - X(y)|^2 + \langle X(y) - X(x), \nu(x) \rangle
= 2\kappa_2(x)(1 - \langle X(x), X(y) \rangle) + 2 \langle X(y), \nu(x) \rangle
\]

Fix \( x \in \Sigma \) and let \( e_1 \) and \( e_2 \) be the principal directions at \( x \). That is, \( g(e_i, e_j) = \delta_{ij} \), and \( h(e_i, e_j) = \kappa_i \delta_{ij} \).

Now let \( \gamma \) be a geodesic in \( \Sigma \) with \( \gamma(0) = x \) and \( \gamma'(0) = e_2 \), and consider the function
\[
f(s) := Z(x, \gamma(s)) := \kappa_2(x)(1 - \langle X(x), X(\gamma(s)) \rangle) + \langle X(\gamma(s)), \nu(x) \rangle
\]
Proof of the Lawson Conjecture

We have (N.b: \( D = \overline{D} - \overline{h} = \nabla - h - \overline{h}\overline{\nu} \), where \( \overline{h} = \overline{g} \) and \( \overline{\nu} = \text{Id} \))

\[
\begin{align*}
\frac{d f}{dx} & = - \langle \kappa^2(x) X(x) - \nu(x), X \ast \gamma' \rangle; \\
\frac{d^2 f}{dx^2} & = \langle \kappa^2(x) X(x) - \nu(x), \gamma(s) + h(\gamma', \gamma') \nu(\gamma) \rangle; \\
\frac{d^3 f}{dx^3} & = \langle \kappa^2(x) X(x) - \nu(x), X \ast \gamma' + \nabla_{\gamma'} h(\gamma', \gamma') \nu(\gamma) \rangle \\
& \quad + \langle \kappa^2(x) X(x) - \nu(x), h(\gamma', \gamma') W(\gamma') \rangle.
\end{align*}
\]

- Now observe that \( f(0) = f'(0) = f''(0) = 0 \). Since \( f \geq 0 \) it follows that \( f'''(0) = 0 \).
- But this implies \( \nabla_1 h_{11} \) vanishes at \( x \). Since \( x \) was arbitrary, we must have \( \nabla_1 h_{11} = 0 \) everywhere.
Proof of the Lawson Conjecture

A similar argument using the corresponding result for the exterior region yields $\nabla^2 h_{22} = 0$ everywhere.

- Now extend $e_i$ to a parallel frame near $x$. Then $0 = e_i h(e_1, e_2) = \nabla_i h_{12}$ at $x$.
- It follows that $\nabla h = 0$.
- The Gauss equation

$$R_{ijkl} = \overline{R}_{ijkl} + h_{ij} h_{jl} - h_{ij} h_{kl}$$

now implies that $\nabla R = \nabla \overline{R} = 0$.

- In particular, $\nabla \text{Ric} = 0$.
- The Lawson conjecture now follows from Lawson’s rigidity theorem.