Recall that the smoothing property of the MCF shows that solutions \( X : M \times [0, T) \to \mathbb{R}^{n+1} \) exist either indefinitely, or until \( \sup_{x \in M} H(x, \cdot) \to \infty \). If the initial data is compact, then, by the avoidance principle, comparison with a large shrinking sphere demonstrates that the solution becomes extinct after a finite time \( T \), and therefore \( \max_{x \in M} H(x, \cdot) \to \infty \) as \( t \to T \). We will now study the behaviour of solutions near curvature blow-ups. In the convex, compact case, Huisken showed that the Weingarten map of a solution approaches that of the sphere in regions of high curvature. With the help of a gradient estimate for \( H \), it can be shown that this behaviour persists in a neighbourhood of the singular point, which ultimately leads to the convergence statement of Huisken’s theorem. We will see that, at least for mean convex solutions, regions of high curvature are modelled approximately by ‘soliton solutions’ of the MCF.

1. Solitons

A soliton solution of an evolution equation is a solution which moves under a one parameter family of symmetries of the equation. For the MCF, these include translations and homothetic dilations. We’ll refer to the former as translating solutions and the latter as dilating solutions.

Translating solutions are those satisfying
\[
X(x, t) = X(x, 0) + tT
\]
for some translation generator \( T \in \mathbb{R}^{n+1} \), and dilating solutions are those satisfying
\[
X(x, t) = \sqrt{2\sigma t} X(x, 0).
\]
for \( \sigma \in \{-1, 1\} \). Observe that translating solutions are eternal; that is, they exist for all time, whereas expanding solutions (\( \sigma = 1 \)) are immortal, existing for an infinite time in the future, and shrinking solutions (\( \sigma = -1 \)) are ancient, having existed for an infinite time in the past. In order that these ‘solutions’ actually satisfy the MCF, we require that the normal part of their velocity be equal to their mean curvature vector. For translating solutions, we have
\[
\partial_t X = T.
\]
So we require
\[
\langle T, \nu \rangle = -H. \tag{1.1}
\]
For dilating solutions, we have
\[
\partial_t \mathcal{X} = \frac{1}{2t} \mathcal{X}.
\]
So we require
\[
\frac{1}{2t} \langle \mathcal{X}, \nu \rangle = -H. \tag{1.2}
\]
If this holds, then the image satisfies the MCF since we can chose a tangential reparametrisation that satisfies MCF: Consider \( \tilde{X}(x, t) = \mathcal{X}(\phi(x, t), t) \) for some smooth family \( \phi(\cdot, t) \) of diffeomorphisms. Then
\[
\partial_t \tilde{X}(x, t) = \frac{\partial \mathcal{X}}{\partial x^i} \frac{d\phi^i}{dt} + \partial_t \mathcal{X}.
\]
If we choose $\phi$ to solve
\[
\begin{aligned}
\frac{d\phi}{dt} &= -V_i \\
\phi(., 0) &= \text{identity},
\end{aligned}
\]
where $V = T + H\nu$ for a translating solution, or $V = k\mathcal{X} + H\nu$ for a dilating solution, then $\mathcal{X}$ satisfies the mean curvature flow (and has the same image as $\mathcal{X}$).

### 1.1. Examples
See [Eck] for more detailed examples.

1. Spheres $S^n$ and cylinders $S^{n-k} \times \mathbb{R}^k$ ($k < n$) are examples of shrinking solutions.
2. There is a shrinking torus called the *Angenant Torus* [Ang92]. (Not all tori shrink homothetically).
3. Given a cone defined by the graph of a homogeneous function, the unique solution to the MCF (having at most linear growth) starting from this cone is an expansion solution (see [Eck]). Stability of solutions emanating from cones was studied by Clutterbuck and Schnürer [CS11].
4. The *Grim Reaper* curve $\Gamma(\theta, t) = \log \cos \theta + t$ is a translating solution of the curve shortening flow. $\Gamma \times \mathbb{R}^{n-1}$ is a translating solution of MCF in higher dimensions.
5. There are translating ‘cigar’ solutions in higher dimensions. These are rotationally symmetric translating solutions. Unlike the Grim Reaper, they are not bound within some cylinder. For stability of translating solutions, see [CSS07].
6. Minimal hypersurfaces, being stationary, are trivial examples of translating solutions.
7. Note that, by the avoidance principle, there can be no compact translating or expansion solutions.

### 1.2. Classifying Mean Convex Dilating Solutions
Setting $t = -1/2$, equation (1.2) becomes
\[
\langle \mathcal{X}, \nu \rangle = H
\]
This is an elliptic equation for the ‘initial’ hypersurface $\mathcal{X}_{-\frac{1}{2}}$.

**Theorem 1.1** (Huiskens et al. [Mul56 AL86 Hu90 Hu93]). The only mean convex solutions of (1.3) are the cylinders $S^k \times \mathbb{R}^{n-k}$, $0 \leq k \leq n$ and $\gamma_{p,q} \times \mathbb{R}^{n-1}$, where $\{\gamma_{p,q}\}_{p,q \in \mathbb{N}}$ are a family of (non-embedded) shrinking solutions of the curve shortening flow discovered by Abresch and Langer [AL86].

**Proof.** Differentiate (1.3) to obtain
\[
\Delta H = \langle \nabla H, \mathcal{X} \rangle + H - H|\mathcal{X}|^2
\]
The strong maximum principle implies $H > 0$.

Now compute
\[
\Delta \left( \frac{|\mathcal{X}|^2}{H^2} \right) = \frac{2}{H^4} |\nabla H - H \nabla |\mathcal{X}|^2 + \left\langle \nabla - \frac{2}{H} \nabla H, \nabla \left( \frac{|\mathcal{X}|^2}{H^2} \right) \right\rangle
\]
Integrate by parts against $\rho|\mathcal{X}|^2$, where $\rho(y, t) = \exp(-||y||^2/2)$, to obtain
\[
0 = \int_M \rho \left| \nabla \left( \frac{|\mathcal{X}|^2}{H^2} \right) \right|^2 d\mu + 2 \int_M \rho \frac{|\mathcal{X}|^2}{H^2} |\nabla H - H \nabla |\mathcal{X}|^2 d\mu.
\]
Therefore,
\[
\frac{|\mathcal{X}|^2}{H^2} = \text{Const.} \quad \text{and} \quad |\nabla H - H \nabla |\mathcal{X}|^2 = 0.
\]
The second identity implies
\[ |\nabla H|^2 \left( |h|^2 - \sum_{k=1}^{n} h_{1k}^2 \right) \equiv 0 \]
in an orthonormal frame with \( e_1 = \nabla H/|\nabla H| \). It follows that at every point of \( M \) least one of the following holds:
\[ |\nabla H| = 0 \text{ or } |h|^2 = H^2 = h_{11}^2, \] (1.5)
If \( \nabla H \equiv 0 \), then (1.4) implies \( \nabla h \equiv 0 \), which, via the Gauss-Weingarten equation, implies \( \nabla \text{Ric} \equiv 0 \). It then follows from Lawson’s rigidity theorem [La69] that \( M = S^{n-k} \times \mathbb{R}^k \).

If instead there is some \( x_0 \) such that \( |\nabla H| \neq 0 \), then there is a neighbourhood \( V \ni x_0 \) on which \( |\nabla H| \neq 0 \) and \( |h|^2 = h_{11}^2 \). We are led to consider the two subbundles
\[ E_1 = \{ X \in TV : W(X) = HX \} \]
and \[ E_2 = \{ X \in TV : W(X) = 0 \} \]
of \( TV \). Using \( |h\nabla H - H\nabla h|^2 \equiv 0 \), we find that \( \nabla X Y \in \Gamma(E_1) \), and hence \( [X, Y] = \nabla X Y - \nabla Y X \in \Gamma(E_1) \), whenever \( X, Y \in \Gamma(E_1) \). A similar argument implies the same conclusion for \( E_2 \). An application of the Frobenius theorem now shows that \( M \) is isometric to \( L_1 \times L_2 \), where \( L_1 \) is a (1-dimensional) leaf of \( E_1 \) and \( L_2 \cong \mathbb{R}^{n-1} \) is a leaf of \( E_2 \). Moreover, \( L_1 \) lies in some 2-dimensional plane, so that \( H = \langle l, \nu \rangle \) for each \( l \in L_1 \). The only such curves are the family \( \{ \gamma_{p,q} \} \) classified by Abresch and Langer [AL86]. For the full details see [Hu93]. □

1.3. Convex, Eternal Solutions are Translating Solutions. Hamilton observed a relationship between solitons and ‘differential Harnack inequalities’ for heat-type equations. Applying this intuition to the MCF, he was able to prove the following inequality:

**Theorem 1.2 (Differential Harnack Inequality, Hamilton [Ham95a]).** Let \( \mathcal{X} \) be a convex solution of MCF. Then, for any tangent vector field \( V \),
\[ \frac{\partial H}{\partial t} + 2 \langle \nabla H, V \rangle + h(V, V) + \frac{H}{2t} \geq 0. \]

**Proof.** The hard part of the proof is to find the right quantity to estimate. Hamilton was motivated by the belief that many quantities should be sharp on soliton solutions, and he observed that the Harnack quantity
\[ Z(V, V) := \frac{\partial H}{\partial t} + 2 \langle \nabla H, V \rangle + h(V, V) + \frac{H}{2t} \] (1.6)
vanishes on a translating soliton when we choose \( V = T + H\nu \) to be the tangential part of the translation vector. One then simply grinds out an evolution equation for \( Z(V, V) \), to which, miraculously, the maximum principle can be applied. See [Ham95a].

We note that Andrews [An94b] obtained a similar Harnack estimate for a wider class of flows around the same time as Hamilton’s result. His proof is far tidier, demonstrating that, in some sense, the ‘Gauss map parametrisation’ is the natural setting for proving differential Harnack inequalities. □

The differential Harnack inequality provides useful information when applied to convex, eternal solutions of the MCF:

**Corollary 1.3 (Hamilton [Ham95a]).** Let \( \mathcal{X} \) be a convex, eternal solution of MCF for which the supreme value of \( H \) is attained at some finite space-time point. Then \( \mathcal{X} \) is a translating solution.

**Proof.** Since \( \mathcal{X} \) is eternal, we obtain from Hamilton’s Harnack estimate the stronger inequality
\[ \frac{\partial H}{\partial t} + 2 \langle \nabla H, V \rangle + h(V, V) \geq 0 \]
for all $V$. In particular, this holds for $V := -W^{-1}(\nabla H)$; that is,
\[
0 \leq \frac{\partial H}{\partial t} - 2 \langle \nabla H, W^{-1}(\nabla H) \rangle + \langle \nabla H, W^{-1}(\nabla H) \rangle = \frac{\partial H}{\partial t} - \langle \nabla H, W^{-1}(\nabla H) \rangle.
\]
This vanishes at the spacetime point at which $H$ is maximised. But then the strong maximum principle implies that $Z(V,V)$ must be identically zero (recall that (1.6) is proved using the maximum principle). The evolution equation for $Z(V,V)$ then implies the apparently stronger identity
\[
\partial_t W - \nabla V W = 0.
\]
Substituting this into the evolution equation
\[
\partial_t W = \nabla^2 H + HW^2
\]
for the Weingarten map yields the identity
\[
0 = \nabla V - HW.
\]
Using this identity and the definition of $V$, we obtain $DT = 0$, where $T = -H \nu + V$; that is, $T$ is a constant vector in $\mathbb{R}^{n+1}$. The proof is completed by tangentially reparametrising $\mathcal{X}$ such that $\partial_t \mathcal{X} = T$. □

2. Application to Singularities

Recall that the curvature of the shrinking sphere satisfies
\[
|h(x,t)|^2 = \frac{1}{2(T-t)},
\]
where $T = 1/2n$ is the extinction time. More generally, we have the evolution equation
\[
(\partial_t - \Delta)|h|^2 = -2|\nabla h|^2 + 4|h|^4.
\]
An application of the maximum principle yields the estimate
\[
\max_{x \in M} |h(x,t)|^2 \geq \frac{1}{2(T-t)}.
\]
Therefore, it is natural to consider the following dichotomy:

1. There exists $C > 0$ such that $\max_{x \in M} |h(x,t)|^2 \leq C/(T-t)$;
2. There exists no such $C$.

We say that the flow undergoes a type-I singularity in the first case, and a type-II singularity in the second.

2.1. Parabolic Rescalings and Limit Flows. Observe that the MCF is invariant under parabolic rescalings
\[
\mathcal{X}(x,t) \to \alpha \mathcal{X} \left( x, \frac{t}{\alpha^2} \right).
\]
To analyse the shape of singularities, we consider a sequence of rescalings about the singular region. This procedure was first described by Hamilton in the context of the Ricci flow (see [Ham95]) and was subsequently adapted to the mean curvature flow by Huisken and Sinestrari [HS99] (see also [RS]).

First, for each $k \in \mathbb{N}$, we choose a sequence $(t_k)$ of times $t_k \in [0, T - 1/k]$ and a sequence $(x_k)$ of points $x_k \in M$ such that:
\[
|W(x_k, t_k)|^2 = \max_{(x,t) \in M \times [0,T-\frac{1}{k}]} |W(x,t)|^2
\]
if the singularity is of type-I, and
\[
|W(x_k, t_k)|^2 \left( T - \frac{1}{k} - t_k \right) = \max_{(x,t) \in M \times [0,T-1/k]} |W(x,t)|^2 \left( T - \frac{1}{k} - t \right)
\]
Proof. If the singularity is of type-I, then there is some constant $\sigma_k$ such that
\[ \sigma_k := |W(x_k, t_k)|^2 \left( T - \frac{1}{k} - t_k \right) \leq |W(x_k, t_k)|^2 (T - t_k) < C, \]
where $0 < \Sigma < \infty$ if the singularity is of type-I, and $\Sigma = \infty$ if it is of type-II.

We refer to the sequence $(\mathcal{X}_k^i)$ as a blow-up sequence.

We first observe

Lemma 2.1 (Huisken-Sinestrari [HS99a]). As $k \to \infty$, we have
\[ t_k \to T, \quad L_k \to \infty, \quad \alpha_k \to -\infty, \quad \text{and} \quad \sigma_k \to \Sigma, \]
where $0 < \Sigma < \infty$ if the singularity is of type-I, and $\Sigma = \infty$ if it is of type-II.

Proof. If the singularity is of type-I, then there is some constant $C > 0$ such that
\[ \sigma_k := |W(x_k, t_k)|^2 \left( T - \frac{1}{k} - t_k \right) \leq |W(x_k, t_k)|^2 (T - t_k) < C, \]
Consider
\[ \sigma_{k+1} - \sigma_k = L_{k+1}(T - t_{k+1}) - L_k(T - t_k) \geq L_k(t_{k+1} - t_k). \]

Now, for type-I singularities, $t_{k+1} \geq t_k$, since $\max_{M \times [0,T-\frac{1}{k+1}]} |W|$ cannot occur at an earlier time than $\max_{M \times [0,T-\frac{1}{k}]} |W|$. It follows that $\sigma_k$ is increasing, and must therefore approach some finite limit $\Sigma \geq \sigma_0 > 0$.

If instead the singularity is of type-II, then, for all $R > 0$, there exist $t_R \in (0, T)$ and $x_R \in M$ such that
\[ |W(x_R, t_R)|^2 (T - t_R) > 2R. \]

On the other hand, there is some sufficiently large $k_R \in \mathbb{N}$ such that
\[ t_R < T - \frac{1}{k}, \quad |W(x_R, t_R)|^2 (T - \frac{1}{k} - t_R) > R \]
for all $k > k_R$. Therefore, by definition,
\[ \sigma_k = \max_{(x,t) \in M \times [0,T-1/k]} |W(x,t)|^2 \left( T - \frac{1}{k} - t \right) \geq |W(x_R, t_R)|^2 \left( T - \frac{1}{k} - t_R \right) > R \]
for all $k > k_R$. Since $R$ was arbitrary, we find $\sigma_k \to \infty$ as $k \to \infty$.

Since $(T - \frac{1}{k} - t_k)$ is bounded, it follows from the definition of $\sigma_k$ that $L_k \to \infty$ as $k \to \infty$. Therefore, since $|W|$ remains bounded whilst $t < T$, we must have $t_k \to T$. It follows that $\alpha_k \to -\infty$. \hfill \qed

Let’s now compute
\[ \frac{\partial \mathcal{X}_k^i}{\partial t}(x,t) = -L_k^{-\frac{1}{2}} H \left( x, \frac{t}{L_k} + t_k \right) \nu \left( x, \frac{t}{L_k} + t_k \right); \]
\[ \frac{\partial \mathcal{X}_k^i}{\partial x^i}(x,t) = \sqrt{L_k} \frac{\partial \mathcal{X}_k^i}{\partial x^i} \left( x, \frac{t}{L_k} + t_k \right) \Rightarrow (g_k)^{ij}(x,t) = L_k g^{ij} \left( x, \frac{t}{L_k} + t_k \right); \]
\[ \Rightarrow (g_k)^{ij}(x,t) = \frac{1}{L_k} g^{ij} \left( x, \frac{t}{L_k} + t_k \right); \]
and
\[ \nu_k(x, t) = \nu \left( x, \frac{t}{L_k} + t_k \right) \]
\[ \Rightarrow \quad kD_i \nu_k(x, t) = kD_i \nu \left( x, \frac{t}{L_k} + t_k \right) \]
\[ \Rightarrow \quad W_k(x, t) = L_k^{-\frac{1}{2}} W \left( x, \frac{t}{L_k} + t_k \right) \]
\[ \Rightarrow \quad H_k(x, t) = L_k^{-\frac{1}{2}} H \left( x, \frac{t}{L_k} + t_k \right), \]
where we used the script \( k \) to distinguish quantities pertaining to the rescaling \( \mathcal{X}_k \) (in particular, \( kD \) is the pullback of the Euclidean connection along \( \mathcal{X}_k \)).

We also note the following properties of the rescalings:

**Lemma 2.2.** The rescaled family \( \{ \mathcal{X}_k \} \) satisfies the following statements:

1. For each \( k \), \( \mathcal{X}_k(x_k, 0) = 0 \) and \( |W_k(x_k, 0)| = 1 \).
2. For any \( \epsilon > 0 \) there exists \( k_0 \in \mathbb{N} \) such that:
\[ \max_{M \times \{ \alpha_k, \sigma_k \}} |W_k|^2 \leq 1 + \epsilon \quad \forall k \geq k_0. \] (2.1)

**Proof.** Part (1) is immediate from the definitions and our calculation of \( W_k \). To prove part (2), first note that
\[ |W_k(x, t)|^2 = L_k^{-1} |W(x, L_k^{-1} t + t_k)|^2. \]
By the definition of \( L_k \) and the choice of \( (x_k, t_k) \), we also have
\[ |W(x, L_k^{-1} t + t_k)|^2 \left( T - \frac{1}{k} - (L_k^{-1} t + t_k) \right) \leq L_k \left( T - \frac{1}{k} - t_k \right). \]
Therefore,
\[ |W_k(x, t)|^2 \leq \frac{T - \frac{1}{k} - t_k}{T - \frac{1}{k} - t_k - L_k^{-1} t} = \frac{\sigma_k}{\sigma_k - t} = 1 + \frac{t}{\sigma_k - t}. \]
Since \( \sigma_k \to \infty \) and \( t \) is bounded the claim follows. \( \square \)

It is now possible, using techniques based on the Arzelà-Ascoli theorem, to show that the sequence \( \mathcal{X}_k \) converges (in a certain local-uniform-Gromov-Hausdorff sense) to some limit \( \mathcal{X}_\infty \), which is a (possibly non-compact) smooth solution of MCF whose maximum curvature occurs at the origin of \( \mathbb{R}^{n+1} \) (see [HS99a, RS, Ba10, Br11]). Moreover, the following theorem shows that this limit is (weakly) convex:

**Theorem 2.3** (Huisken-Sinestrari [HS99a, HS99b, ALM12]). Let \( \mathcal{X} \) be a compact mean convex solution of MCF. Then \( \mathcal{X} \) becomes convex at any point where the curvature blows-up. More precisely, for any \( \epsilon > 0 \) there exists \( C_\epsilon > 0 \) such that
\[ -\kappa_1(x, t) \leq \epsilon H(x, t) + C_\epsilon. \]

**Proof.** The idea is to try to construct a smooth, homogeneous (degree zero, say) function \( G \) of the principal curvatures which is non-negative and vanishes if and only if all of the curvatures are non-negative. Once this is done, we want to obtain an estimate on \( G \) analogous to the one above for \( \kappa_1 \). It suffices to bound the function \( G_{\epsilon, \sigma} := (G - \epsilon)H^\sigma \) by some constant depending on \( \epsilon \) (but not \( \sigma \)) for some \( \sigma > 0 \). Unfortunately, the maximum principle is not sufficient, and some rather delicate integral estimates are required. See [HS99a, HS99b, ALM12] (c.f. [Hu84]). \( \square \)

**Corollary 2.4.** For any \( \epsilon > 0 \), there exists \( C_\epsilon \) such that:
\[ -\kappa_1^k(x, t) \leq \epsilon H_k(x, t) + \frac{C_\epsilon}{\sqrt{L_k}} \] (2.2)
for all \((x, t) \in M \times \{ \alpha_k, \sigma_k \}\). In particular, \( \mathcal{X}_\infty \) is convex.
Proof. We have
\[ \kappa_i(x,t) = \frac{1}{\sqrt{L_k}} \kappa_i(x, L_k^{-1} t + t_k), \]
so the result follows from theorem 2.3. □

In the type-II case, it is possible to split out lines, \( \mathbb{R} \), from \( \mathcal{X}_\infty \) using the Frobenius theorem (similar to the application in theorem 1.1), until we are left with a strictly convex component (see [HS99a]). By Hamilton’s Harnack estimate, this component must be a translating solution. The result is as follows:

**Theorem 2.5.** Any (type-II) blow-up limit of a mean convex mean curvature flow about a type-II singularity is a translating solution of the mean curvature flow of the form \( \mathcal{X}_\infty : \Gamma^k \times \mathbb{R}^{n-k} \times \mathbb{R} \to \mathbb{R}^{n+1} \), where \( \mathcal{X}_\infty |_{\Gamma^k} \) is strictly convex.

We remark that there is not yet any analogue of theorem 1.1 for translating solutions (even in the convex case).

2.2. Type-I Singularities. The main tool for studying type-I singularities is Huisken’s monotonicity formula:

**Theorem 2.6** (The Monotonicity Formula, Huisken [Hu90]). Let \( \mathcal{X} : M \times (\alpha, \sigma) \to \mathbb{R}^{n+1} \) be a compact solution of MCF. Then
\[ \frac{d}{dt} \int_M \rho(\mathcal{X}(x,t), t) \, d\mu_t = - \int_M \rho(\mathcal{X}(x,t), t) \left\| \mathcal{X}^\perp(x,t) \right\|^2 \left\| - \frac{\rho(x)}{2(\Sigma - t)} + H(x,t) \right\|^2 d\mu_t \]
for any \( \Sigma > b \), where
\[ \rho(y,t) := \left( \frac{1}{4\pi(\Sigma - t)} \right)^n \exp \left( - \frac{\|y\|^2}{4(\Sigma - t)} \right), \]
is the backwards heat kernel on \( \mathbb{R}^{n+1} \) centred at \((0, \Sigma)\), and \( \mathcal{X}^\perp := \langle \mathcal{X}, \nu \rangle \nu \) is the normal component of \( \mathcal{X} \).

**Proof.** The hard part is again knowing which quantity to study. Other monotonicity formulae are known for semilinear heat equations in Euclidean space. Analogous monotone quantities are also known for minimal surfaces and other harmonic maps (where the monotonicity is with respect to a sphere over which the integration is to be performed). Once a worthy candidate is found, one confidently grinds out the computation. See [Hu90, RS]. □

**Theorem 2.7** (Huisken [Hu93], see also [RS]). Any (type-I) blow-up limit of a mean convex flow about a type-I singularity satisfies
\[ \frac{1}{2(\Sigma - t)} \langle \mathcal{X}_\infty, \nu \rangle = H. \]

**Proof.** The idea is to integrate the monotonicity formula over the blow-up sequence to obtain an expression whose right hand side involves the space-time measure of \( \frac{1}{2(\Sigma - t)} \langle \mathcal{X}_k, \nu \rangle - H \) and whose left hand side vanishes in the limit \( k \to \infty \) for a type-I singularity. See [Hu93]. □

It follows that \( \mathcal{X}_\infty \) is a dilating solution, and therefore one of the shrinking cylinders of theorem 1.1.

**References**


