Positive quadrature on the sphere and conjectures on monotonicities of Jacobi polynomials

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Outline of talk

- Some definitions,
- Property (R) and Reimer’s proofs,
- Conjectures on Jacobi polynomials,
- Partial results in \([-1/2, 1/2]^2\),
- Weaker result for \(\alpha \geq \beta > -1/2\),
- Application to Property (R).
Some definitions: 1 – notation

\[ S^d := \left\{ x \in \mathbb{R}^{d+1} \mid \sum_{k=1}^{d+1} x_k^2 = 1 \right\}, \]

\[ \omega_d := \sigma(S^d), \]

\[ \tilde{P}_n^{(\alpha, \beta)} := P_n^{(\alpha, \beta)} / P_n^{(\alpha, \beta)}(1), \]

\[ \Theta_n^{(\alpha, \beta)} := \text{smallest zero in } \theta \text{ of } P_n^{(\alpha, \beta)}(\cos \theta), \]

\[ Z_\alpha(z) := \Gamma(\alpha + 1) \left( \frac{2}{z} \right)^\alpha J_\alpha(z). \]
Some definitions: 2 – polynomial spaces

We use $\mathbb{P}_n(\mathbb{S}^d)$ to denote the real polynomials on $\mathbb{R}^{d+1}$, of maximum total degree $n$, restricted to $\mathbb{S}^d$, with dimension

$$\mathcal{D}(d, n) := \dim \mathbb{P}_n(\mathbb{S}^d) = \binom{n + d}{d} + \binom{n + d - 1}{d}$$

and reproducing kernel $\Phi^{(d+1)}_n(x, y) := \Phi^{(d+1)}_n(x \cdot y)$, where

$$\Phi^{(d+1)}_n := \frac{2}{\omega_d} \frac{(d + 1)^{n-1}}{(d/2 + 1)^{n-1}} P^{(d/2, d/2 - 1)}_n$$

$$= \frac{\mathcal{D}(d, n)}{\omega_d} \tilde{P}^{(d/2, d/2 - 1)}_n.$$
Property (R)


An admissible sequence of quadrature rules \((Q_1, \ldots)\) on \(S^d \subset \mathbb{R}^{d+1}\), has rule \(Q_t = (X_t, W_t)\) with strength \(t\) and cardinality \(|X_t| = N_t\), with all weights \(W_{t,k}\) positive.

An admissible sequence of quadrature rules has property (R) (Hesse and Sloan, 2003, 2004) if and only if, given \(\phi \in [0, \frac{\pi}{2}]\), there exists positive constants \(\gamma\) and \(t_0\) such that for all \(y \in S^d\) and each rules \(Q_t\) in the sequence, if \(t \geq t_0\) then

\[
\sum_{x_{t,k} \in S(y, \frac{\phi}{t})} w_{t,k} \leq \gamma \sigma \left( S \left( y, \frac{\phi}{t} \right) \right).
\]
Reimer (2000, 2003) proved that any admissible sequence of quadrature rules is quadrature regular and satisfies Property (R).

The (2000) proof uses \( P_{n}^{\left(\frac{d}{2}, \frac{d}{2} - 1\right)} \), and the following limit theorem (Szegö 1939 – 1975).

**Theorem 1.** For \( \alpha, \beta > -1 \), \( z \in \mathbb{C} \),

\[
\lim_{n \to \infty} \tilde{P}_{n}^{(\alpha, \beta)} \left( \cos \frac{z}{n} \right) = Z_{\alpha}(z).
\]

*The formula holds uniformly in every bounded region of the complex \( z \) plane.*
From Reimer’s proofs (2000, 2003) immediately follows:

**Lemma 1.** Let $Q := (X, W)$ be a positive weight quadrature rule on $\mathbb{S}^d$ of strength $2n$.

Let $K := \Phi_n^{(d+1)}$.

Then for $\theta \in ]0, n\Theta_{\frac{d}{2}, \frac{d}{2} - 1}[$, for any $y \in \mathbb{S}^d$,

$$
\sum_{x_k \in S(y, \frac{\theta}{n})} \omega_k \leq \frac{K(1)}{K^2 \left( \cos \frac{\theta}{n} \right)}
$$

$$
= \frac{\omega_d}{\mathcal{D}(d, n)} \left( \tilde{P}_{\frac{d}{2}, \frac{d}{2} - 1} \left( \cos \frac{\theta}{n} \right) \right)^{-2}.
$$
Monotonicity of $\tilde{P}_{n}^{(1,0)}(\cos \theta / n)$?

Sequence of $\tilde{P}_{n}^{(1,0)}(\cos \theta / n)$ seems monotonic to the first zero:
Conjectures on Jacobi polynomials

Conjecture 1. For $\alpha > -1, \beta > -1$, if for $\theta \in ]0, \Theta_1^{(\alpha,\beta)}]$ we have

$$\tilde{P}_1^{(\alpha,\beta)}(\cos \theta) < \tilde{P}_2^{(\alpha,\beta)} \left(\cos \frac{\theta}{2}\right)$$

(1)

then for $n \geq 1, \theta \in ]0, n\Theta_n^{(\alpha,\beta)}]$, we have

$$\tilde{P}_n^{(\alpha,\beta)} \left(\cos \frac{\theta}{n}\right) < \tilde{P}_{n+1}^{(\alpha,\beta)} \left(\cos \frac{\theta}{n + 1}\right)$$

(2)

and therefore

$$n\Theta_n^{(\alpha,\beta)} < (n + 1)\Theta_{n+1}^{(\alpha,\beta)}.$$ 

(3)
Where does premise (1) hold?

\[(3\alpha^2 + 2\alpha\beta - \beta^2 + 9\alpha + \beta + 4) \sqrt{\frac{\beta + 1}{\alpha + \beta + 2}} + (\alpha + \beta)^2 + 3\alpha + 7\beta + 4 = 0.\]
Partial results in $[-1/2, 1/2]^2$

Previously known (Gegenbauer polynomials):

$$(n + \frac{1}{2} + \alpha) \Theta_n^{(\alpha, \alpha)} < (n + \frac{3}{2} + \alpha) \Theta_{n+1}^{(\alpha, \alpha)}$$

for $n \geq 1$, $\alpha \in \left]-\frac{1}{2}, \frac{1}{2}\right[$ (Szegö (1939)).

So far proved:

$$n \Theta_n^{(\alpha, \beta)} < (n + 1) \Theta_{n+1}^{(\alpha, \beta)}$$

for $n \geq 1$, $(\alpha, \beta) \in \left]-\frac{1}{2}, \frac{1}{2}\right[^2$ (Sturm comparison or Gatteschi (1987)),

$$\tilde{P}_n^{(\alpha, \beta)} \left(\cos \frac{\theta}{n}\right) < \tilde{P}_{n+1}^{(\alpha, \beta)} \left(\cos \frac{\theta}{n+1}\right)$$

for $n \geq 1$, $\theta \in ]0, \pi[\ (\alpha, \beta) \in \left\{\left(-\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right\}$ (Koumandos 2005).
Weaker result for $\alpha \geq \beta > -\frac{1}{2}$

**Theorem 2.** For $n \geq 1$, $\alpha \geq \beta > -\frac{1}{2}$, $\theta \in [0, \frac{\pi}{2}]$, we have

\[
\left( 2n \sin \frac{\theta}{2n} \right)^{\alpha-\beta} \left( n \sin \frac{\theta}{n} \right)^{\beta+\frac{1}{2}} P_n^{(\alpha, \beta)} \left( \cos \frac{\theta}{n} \right) < \\
\left( (2n + 2) \sin \frac{\theta}{2n+2} \right)^{\alpha-\beta} \left( (n+1) \sin \frac{\theta}{n+1} \right)^{\beta+\frac{1}{2}} P_{n+1}^{(\alpha, \beta)} \left( \cos \frac{\theta}{n+1} \right).
\]

Proved by Sturm comparison using

\[
F_n^{(\alpha, \beta)}(\theta) := \frac{1}{n^2} \left( \frac{\frac{1}{4} - \alpha^2}{4 \sin^2 \frac{\theta}{2n}} + \frac{\frac{1}{4} - \beta^2}{4 \cos^2 \frac{\theta}{2n}} \right) + \left( 1 + \frac{\alpha + \beta + 1}{2n} \right)^2,
\]

\[
V_n^{(\alpha, \beta)}(\theta) := \left( 2n \sin \frac{\theta}{2n} \right)^{\alpha+\frac{1}{2}} \left( \cos \frac{\theta}{2n} \right)^{\beta+\frac{1}{2}} P_n^{(\alpha, \beta)} \left( \cos \frac{\theta}{n} \right),
\]

\[
\frac{\partial^2}{\partial \theta^2} V_n^{(\alpha, \beta)}(\theta) + F_n^{(\alpha, \beta)}(\theta) V_n^{(\alpha, \beta)}(\theta) = 0.
\]
From Lemma 1 and Conjecture 1 immediately follows:

**Conjecture 2.** Let $Q := (X, W)$ be a positive weight quadrature rule on $S^d$ of strength $2n$.

Then for $\theta \in ]0, \Theta^{(\frac{d}{2}, \frac{d}{2}, \frac{1}{2})}[$, for any $y \in S^d$,

$$
\sum_{x_k \in S(y, \frac{\theta}{n})} w_k \leq \frac{\omega_d}{D(d, n)} \left( \tilde{P}^{(\frac{d}{2}, \frac{d}{2}, \frac{1}{2})}(\cos \theta) \right)^{-2}.
$$
Conjecture 3. For \( t \geq t_0 \geq 2 \), let \( Q = (X, W) \) be a positive weight quadrature rule on \( \mathbb{S}^d \) which is exact on \( \mathbb{P}_t(\mathbb{S}^d) \).

Then for \( \phi \in ]0, \pi[ \), for any \( y \in \mathbb{S}^d \), we have

\[
\sum_{x_k \in S(y, \frac{\phi}{t})} w_k \leq c_1 \ t^{-d} \leq c_1 \ c_2 \ \sigma \left( S \left( y, \frac{\phi}{t} \right) \right),
\]

where

\[
c_1 := 2^{d-1} \ \omega_d \ d! \ \left( \tilde{P}_1^{(\frac{d}{2}, \frac{d}{2} - 1)} \left( \cos \frac{\phi}{2} \right) \right)^{-2},
\]

\[
c_2 := \frac{d}{\omega_{d-1}} \left( \text{sinc} \frac{\phi}{t_0} \right)^{-d+1} \phi^{-d}.
\]
Lemma 1 and our weaker result, Theorem 2, give us only:

**Theorem 3.** With the same conditions and notation as Conjecture 3, for $\phi \in ]0, \pi[$, for any $y \in \mathbb{S}^{d}$, we have

$$
\sum_{x_{k} \in \mathbb{S}(y, \phi \frac{t}{c})} w_{k} \leq c_{3} t^{-d} \leq c_{3} c_{2} \sigma \left(S \left(y, \frac{\phi}{t}\right)\right),
$$

where

$$
c_{3} := c_{1} \left(\frac{\text{sinc} \frac{\phi}{2}}{2}\right)^{-d-1},
$$

$c_{1}, c_{2}$ as per Conjecture 3.