Spherical codes with good separation, discrepancy and energy

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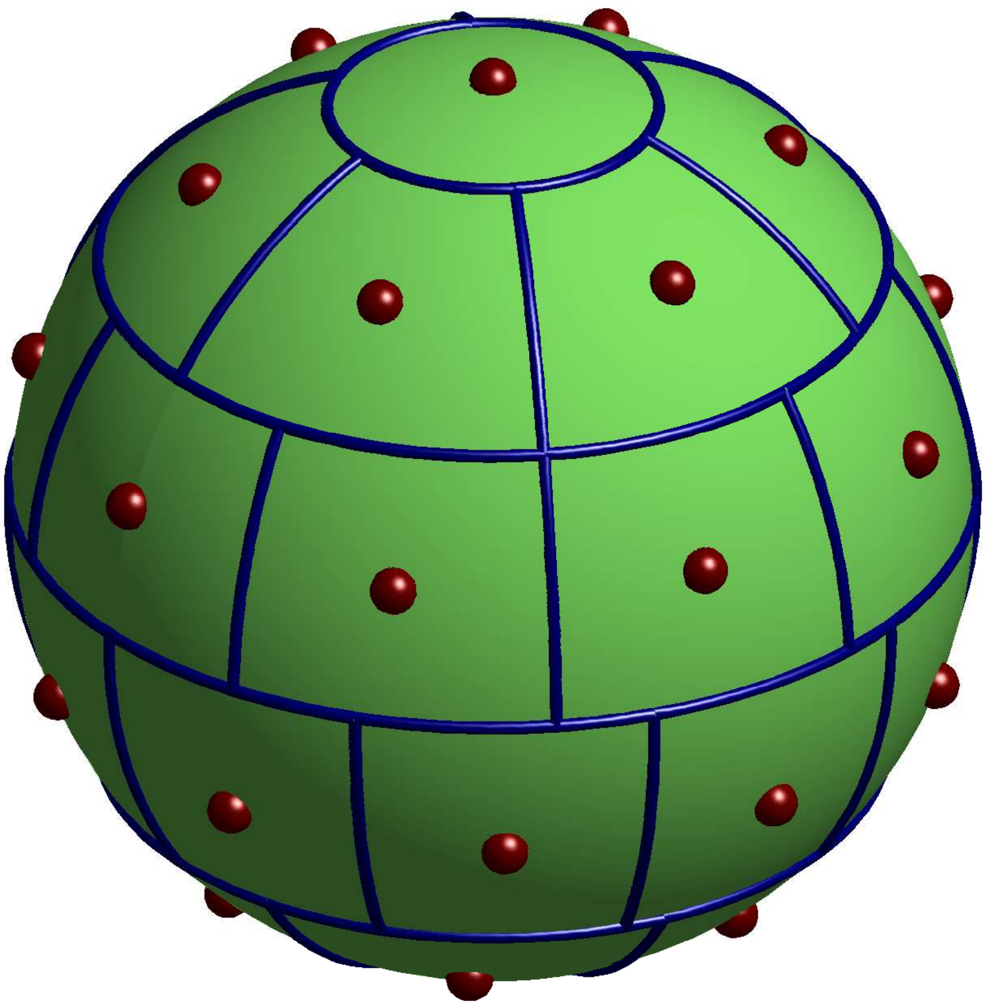
Outline of talk

EQ codes: The Recursive Zonal Equal Area spherical codes,

$\mathbf{EQP}(d, \mathcal{N}) \subset \mathbb{S}^d$, with $|\mathbf{EQP}(d, \mathcal{N})| = \mathcal{N}$.

- Overview of properties of the EQ codes
- Some precedents
- Definitions: coordinates, partitions, diameter bounds
- The Recursive Zonal Equal Area (EQ) partition
- Details of properties of the EQ codes
- Separation and discrepancy bounds imply energy bounds
- Separation and diameter bounds imply energy bounds
- More details of properties (if time permits)

The spherical code EQP(2,33) on $\mathbb{S}^2 \subset \mathbb{R}^3$



Geometric properties of the EQ codes

For $\mathbf{EQP}(d, \mathcal{N})$

Good:

- Centre points of regions of diameter $= \mathrm{O}(\mathcal{N}^{-1/d})$,
- Mesh norm (covering radius) $= \mathrm{O}(\mathcal{N}^{-1/d})$,
- Minimum distance and packing radius $= \Omega(\mathcal{N}^{-1/d})$.

Bad:

- Mesh ratio $= \Omega(\sqrt{d})$,
- Packing density $\leqslant \frac{\pi^{d/2}}{2^d \ \Gamma(d/2+1)}$ as $\mathcal{N} \to \infty$.

Approximation properties of the EQ codes

Not so bad?

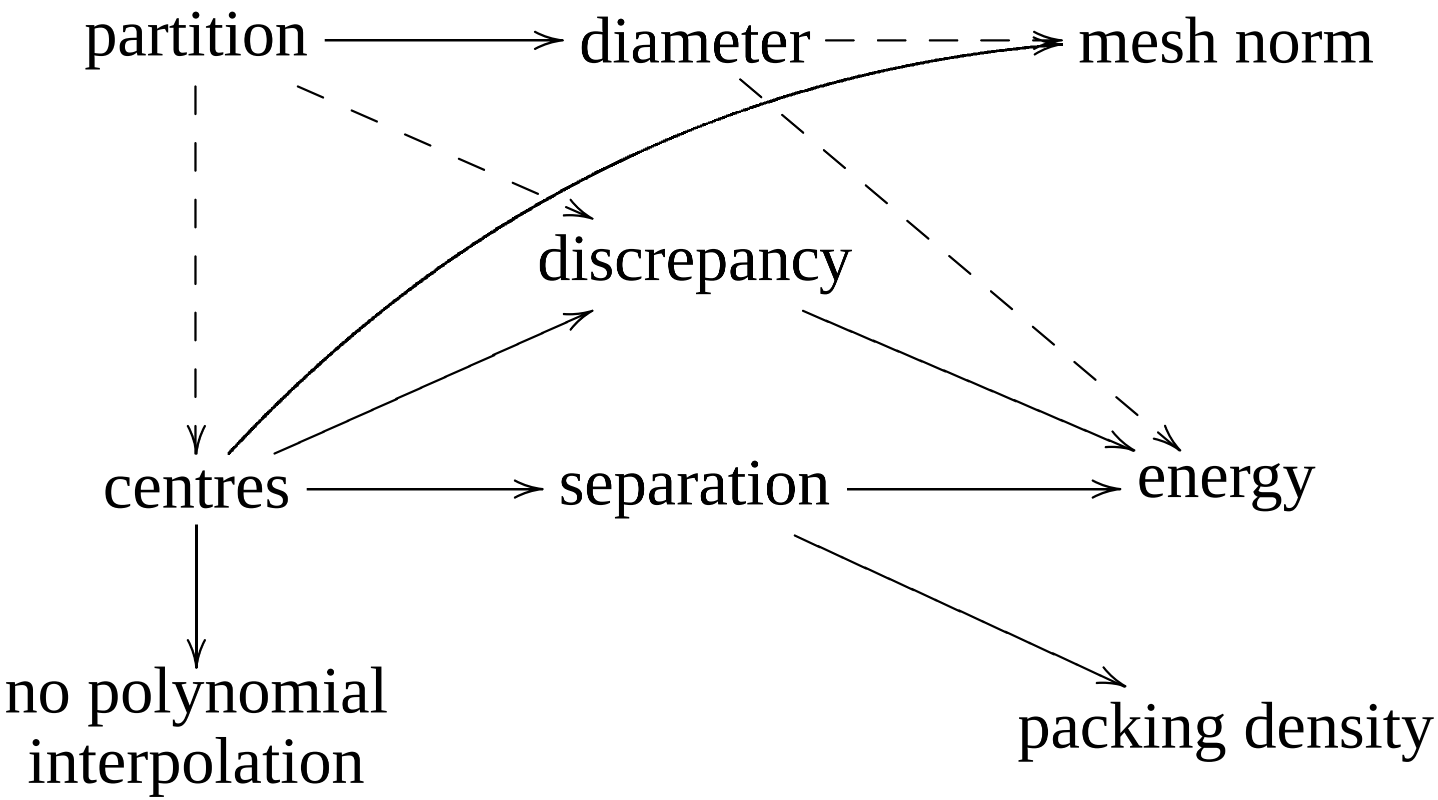
- Normalized spherical cap discrepancy $= \mathrm{O}(\mathcal{N}^{-1/d})$,
- Normalized s-energy

$$E_s = \begin{cases} I_s \pm \mathrm{O}(\mathcal{N}^{-1/d}) & 0 < s < d-1 \\ I_s \pm \mathrm{O}(\mathcal{N}^{-1/d} \log \mathcal{N}) & s = d-1 \\ I_s \pm \mathrm{O}(\mathcal{N}^{s/d-1}) & d-1 < s < d \\ \mathrm{O}(\log \mathcal{N}) & s = d \\ \mathrm{O}(\mathcal{N}^{s/d-1}) & s > d. \end{cases}$$

Ugly:

- Cannot be used for polynomial interpolation:
  proven for large enough $\mathcal{N}$, conjectured for small $\mathcal{N}$.

Relationships between properties of EQ codes



Some precedents

The $\mathbf{EQ}$ partition is based on Zhou's (1995) construction for $\mathbb{S}^2$ as modified by Saff, and on Sloan's sketch of a partition of $\mathbb{S}^3$ (2003).

Separation without equidistribution: Hamkins (1996) and Hamkins and Zeger (1997) constructed $\mathbb{S}^d$ codes with asymptotically optimal packing density.

Equidistibution without separation: Many constructions for $\mathbb{S}^2$, eg. mapped Hammersley, Halton, (t, s) etc. sequences.
Feige and Schechtman (2002) constructed a diameter bounded equal area partition of $\mathbb{S}^d$. Put one point in each region.

Equal-area partitions of $\mathbb{S}^d \subset \mathbb{R}^d$

An *equal area partition* of $\mathbb{S}^d \subset \mathbb{R}^d$ is a finite set $\mathcal{P}$ of Lebesgue measurable subsets of $\mathbb{S}^d$, such that

$$\bigcup_{R \in \mathcal{P}} R = \mathbb{S}^d,$$

and for each $R \in \mathcal{P}$,

$$\sigma(R) = \frac{\sigma(\mathbb{S}^d)}{|\mathcal{P}|},$$

where σ is the Lebesgue area measure on $\mathbb{S}^d$.

Diameter bounded sets of partitions

The *diameter* of a region $\boldsymbol{R} \subset \mathbb{R}^{\boldsymbol{d+1}}$ is defined by

$$\mathbf{diam}\,\boldsymbol{R} := \mathbf{sup}\{\|\mathrm{x} - \mathrm{y}\| \mid \mathrm{x}, \mathrm{y} \in \boldsymbol{R}\}.$$

A set $\boldsymbol{\Xi}$ of partitions of $\mathbb{S}^{\boldsymbol{d}} \subset \mathbb{R}^{\boldsymbol{d+1}}$ is *diameter-bounded* with *diameter bound* $\boldsymbol{K} \in \mathbb{R}_{+}$ if for all $\boldsymbol{\mathcal{P}} \in \boldsymbol{\Xi}$, for each $\boldsymbol{R} \in \boldsymbol{\mathcal{P}}$,

$$\mathbf{diam}\,\boldsymbol{R} \leqslant \boldsymbol{K}\,|\boldsymbol{\mathcal{P}}|^{\boldsymbol{-1/d}}\,.$$

Key properties of the EQ partition of $\mathbb{S}^d$

$\mathbf{EQ}(d, \mathcal{N})$ is the *recursive zonal equal area* partition of $\mathbb{S}^d$ into $\mathcal{N}$ regions.

The set of partitions $\mathbf{EQ}(d) := \{\mathbf{EQ}(d, \mathcal{N}) \mid \mathcal{N} \in \mathbb{N}_+\}$.

The $\mathbf{EQ}$ partition satisfies:

Theorem 1. *For* $d \geqslant 1$, $\mathcal{N} \geqslant 1$, $\mathbf{EQ}(d, \mathcal{N})$ *is an equal-area partition.*

Theorem 2. *For* $d \geqslant 1$, $\mathbf{EQ}(d)$ *is diameter-bounded.*

Spherical polar coordinates on $\mathbb{S}^d$

Spherical polar coordinates describe $\mathbf{x} \in \mathbb{S}^d \subset \mathbb{R}^{d+1}$ by one longitude, $\xi_1 \in \mathbb{R}$ (modulo 2π), and $d-1$ colatitudes, $\xi_j \in [0, \pi]$, for $j \in \{2, \ldots, d\}$.

The spherical polar to Cartesian coordinate map $\odot : \mathbb{R} \times [0, \pi]^{d-1} \longrightarrow \mathbb{S}^d \subset \mathbb{R}^{d+1}$ is

$$\odot(\xi_1, \xi_2, \ldots, \xi_d) = (x_1, x_2, \ldots, x_{d+1}),$$

$$\text{where} \quad x_1 := \cos \xi_1 \prod_{j=2}^{d} \sin \xi_j, \quad x_2 := \prod_{j=1}^{d} \sin \xi_j,$$

$$x_k := \cos \xi_{k-1} \prod_{j=k}^{d} \sin \xi_j, \quad k \in \{3, \ldots, d+1\}.$$

Spherical caps, zones, and collars

The *spherical cap* $S(\mathrm{p}, \theta) \subset \mathbb{S}^d$ is

$$S(\mathrm{p}, \theta) := \left\{\mathrm{q} \in \mathbb{S}^d \mid \mathrm{p} \cdot \mathrm{q} \geqslant \cos(\theta)\right\}.$$

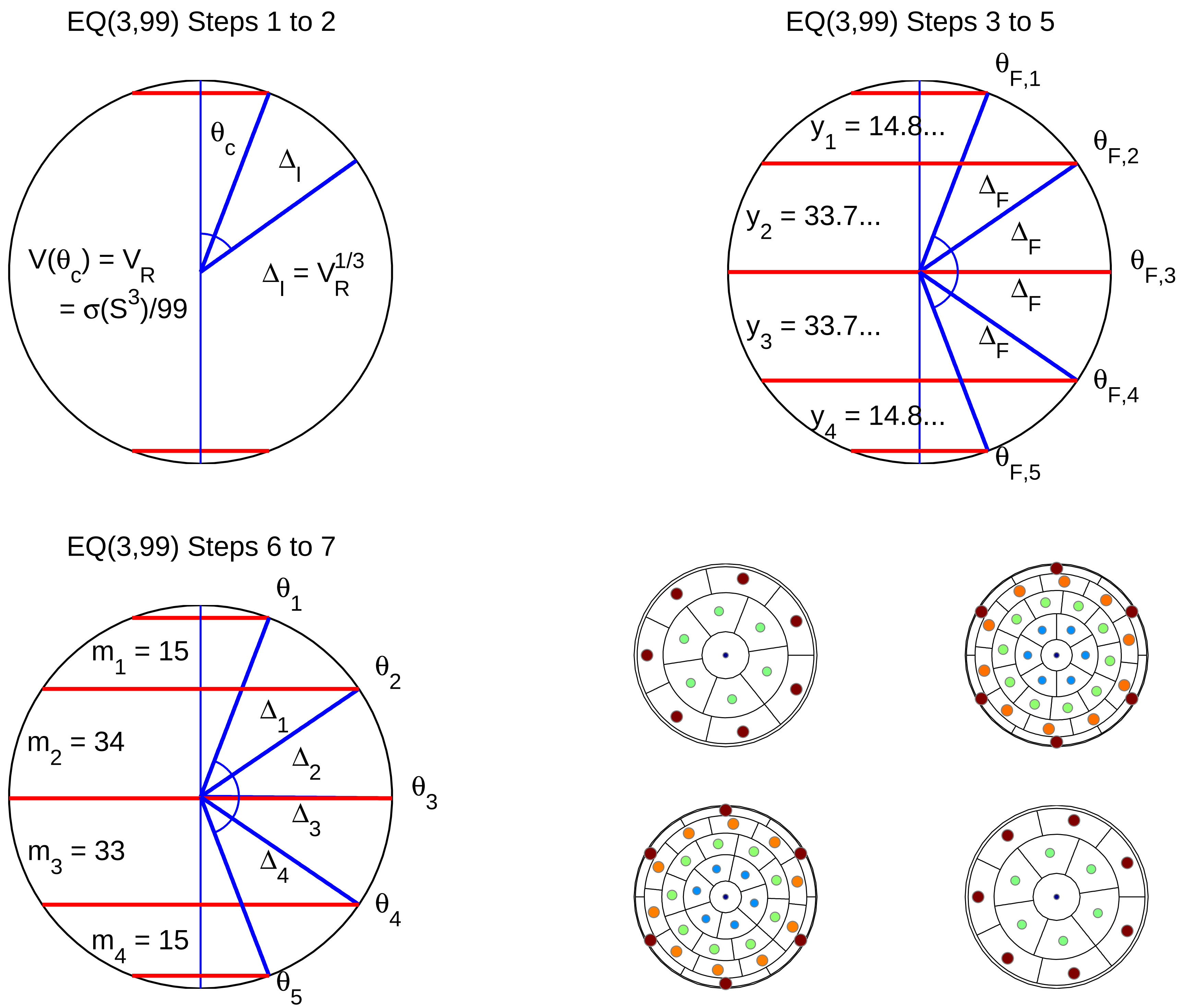
For $d > 1$, a *zone* can be described by

$$Z(\tau, \beta) := \left\{\odot(\xi_1, \ldots, \xi_d) \in \mathbb{S}^d \mid \xi_d \in [\tau, \beta]\right\},$$

where $0 \leqslant \tau < \beta \leqslant \pi$.

$Z(0, \beta)$ is a North polar cap and $Z(\tau, \pi)$ is a South polar cap.

If $0 < \tau < \beta < \pi$, $Z(\tau, \beta)$ is a *collar*.



EQ(3,99) Steps 1 to 2
θc
ΔI
V(θc) = VR
= σ(S3)/99
ΔI = VR1/3
EQ(3,99) Steps 3 to 5
θF,1
θF,2
θF,3
θF,4
θF,5
y1 = 14.8...
y2 = 33.7...
y3 = 33.7...
y4 = 14.8...
ΔF
EQ(3,99) Steps 6 to 7
θ1
θ2
θ3
θ4
θ5
m1 = 15
m2 = 34
m3 = 33
m4 = 15
Δ1
Δ2
Δ3
Δ4

Centre points of regions of $\mathrm{EQ}(d, \mathcal{N})$

The placement of the centre point $\boldsymbol{a} = \odot(\boldsymbol{\alpha})$ of a region

$$\boldsymbol{R} = \odot\left([\boldsymbol{\tau_1}, \boldsymbol{\beta_1}] \times \ldots \times [\boldsymbol{\tau_d}, \boldsymbol{\beta_d}]\right) \text{ is}$$

$$\boldsymbol{\alpha_1} := \begin{cases} \mathbf{0} & \boldsymbol{\beta_1} = \boldsymbol{\tau_1} \pmod{\mathbf{2}\boldsymbol{\pi}} \\ (\boldsymbol{\tau_1} + \boldsymbol{\beta_1})/\mathbf{2} \pmod{\mathbf{2}\boldsymbol{\pi}} & \text{otherwise}, \end{cases}$$

and for $\boldsymbol{j} > \mathbf{1}$,

$$\boldsymbol{\alpha_j} := \begin{cases} \mathbf{0} & \boldsymbol{\tau_j} = \mathbf{0} \\ \boldsymbol{\pi} & \boldsymbol{\beta_j} = \boldsymbol{\pi} \\ (\boldsymbol{\tau_j} + \boldsymbol{\beta_j})/\mathbf{2} & \text{otherwise}. \end{cases}$$

Minimum distance and packing radius

The *minimum distance* of $\boldsymbol{X} := \{\mathbf{x_1}, \ldots, \mathbf{x_{\mathcal{N}}}\} \subset \mathbb{S}^{\boldsymbol{d}}$ is

$$\text{min dist } \boldsymbol{X} := \min_{\mathbf{x} \neq \mathbf{y} \in \boldsymbol{X}} \|\mathbf{x} - \mathbf{y}\|,$$

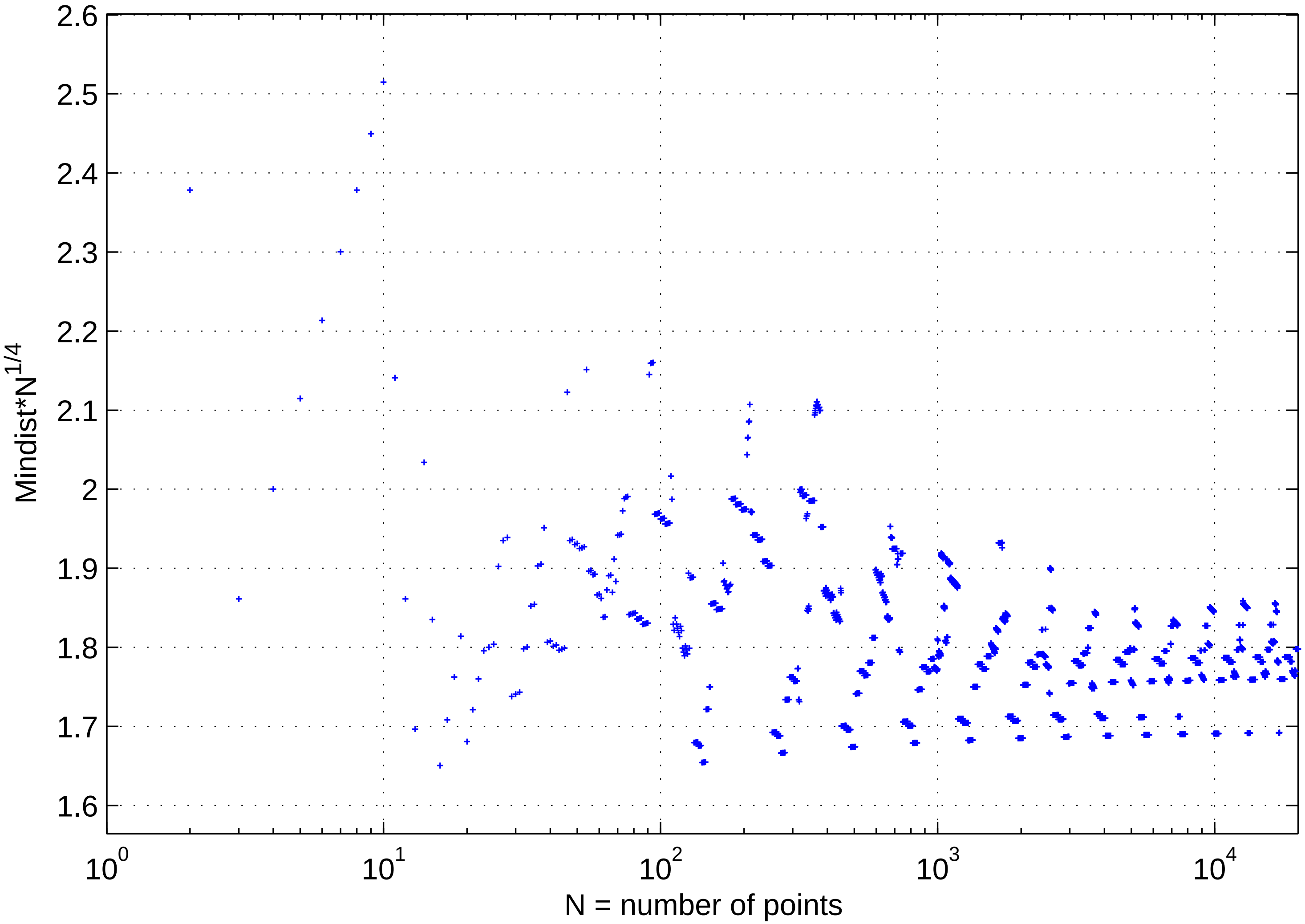
and the *packing radius* of $\boldsymbol{X}$ is

$$\text{prad } \boldsymbol{X} := \min_{\mathbf{x} \neq \mathbf{y} \in \boldsymbol{X}} \cos^{-1}(\mathbf{x} \cdot \mathbf{y})/2.$$

It can be shown that $\quad$ min dist $\mathrm{EQP}(\boldsymbol{d}, \mathcal{N}) = \Omega(\mathcal{N}^{-1/\boldsymbol{d}})$,

and therefore $\quad$ prad $\mathrm{EQP}(\boldsymbol{d}, \mathcal{N}) = \Omega(\mathcal{N}^{-1/\boldsymbol{d}})$.

Minimum distance of EQP(4) codes



Normalized spherical cap discrepancy

We use the probability measure $\overset{*}{\sigma} := \sigma / \sigma(\mathbb{S}^d)$.

For $X := \{x_1, \ldots, x_{\mathcal{N}}\} \subset \mathbb{S}^d$ the *normalized spherical cap discrepancy* is

$$\operatorname{disc} X := \sup_{y \in \mathbb{S}^d} \sup_{\theta \in [0,\pi]} \left| \frac{|X \cap S(y,\theta)|}{\mathcal{N}} - \overset{*}{\sigma}\big(S(y,\theta)\big) \right|.$$

It can be shown that

$$\operatorname{disc} \operatorname{EQP}(d, \mathcal{N}) = \operatorname{O}(\mathcal{N}^{-1/d}).$$

Normalized s-energy

For $X := \{\mathrm{x}_1, \ldots, \mathrm{x}_{\mathcal{N}}\} \subset \mathbb{S}^d$, $s \in \mathbb{R}$,
the *normalized s-energy* is

$$E_s(X) := \mathcal{N}^{-2} \sum_{i=1}^{\mathcal{N}} \sum_{\mathrm{x}_i \neq \mathrm{x}_j \in X} \|\mathrm{x}_i - \mathrm{x}_j\|^{-s},$$

and the *normalized energy double integral* for $0 < s < d$ is

$$I_s := \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \|\mathrm{x} - \mathrm{y}\|^{-s} \, d\overset{*}{\sigma}(\mathrm{x}) d\overset{*}{\sigma}(\mathrm{y}).$$

Separation and discrepancy imply energy

Theorem 3.

Let $(X_1, X_2, \ldots)$ *be a sequence of* $\mathbb{S}^d$ *codes for which there exist* $c_1, c_2 > 0$ *and* $0 < q < 1$ *such that each* $X_\mathcal{N} = \{\mathrm{x}_{\mathcal{N},1}, \ldots, \mathrm{x}_{\mathcal{N},\mathcal{N}}\}$ *satisfies*

$$\|\mathrm{x}_{\mathcal{N},i} - \mathrm{x}_{\mathcal{N},j}\| > c_1\, \mathcal{N}^{-1/d}, \quad (i \neq j)$$

$$\textit{disc}\ X_\mathcal{N} \leqslant c_2\, \mathcal{N}^{-q}.$$

Then for the normalized s *energy for* $0 < s < d$*, we have for some* $c_3 \geqslant 0$*,*

$$E_s(X_\mathcal{N}) \leqslant I_s + c_3\, \mathcal{N}^{(s/d-1)q}.$$

Separation and diameter imply energy

Theorem 4.

Let $((X_1, \mathcal{P}_1), (X_2, \mathcal{P}_2), \ldots)$ *be a sequence of pairs of* $\mathbb{S}^d$ *codes and equal area partitions such that* $|X_\mathcal{N}| = |\mathcal{P}_\mathcal{N}| = \mathcal{N}$*, each* $\mathrm{x}_{\mathcal{N},i} \in X_\mathcal{N}$ *lies in* $R_{\mathcal{N},i} \in \mathcal{P}_\mathcal{N}$*, and such that* $(X_1, X_2, \ldots)$ *is well separated and* $(\mathcal{P}_1, \mathcal{P}_2, \ldots)$ *is diameter bounded.*

Then for the normalized s *energy we have*

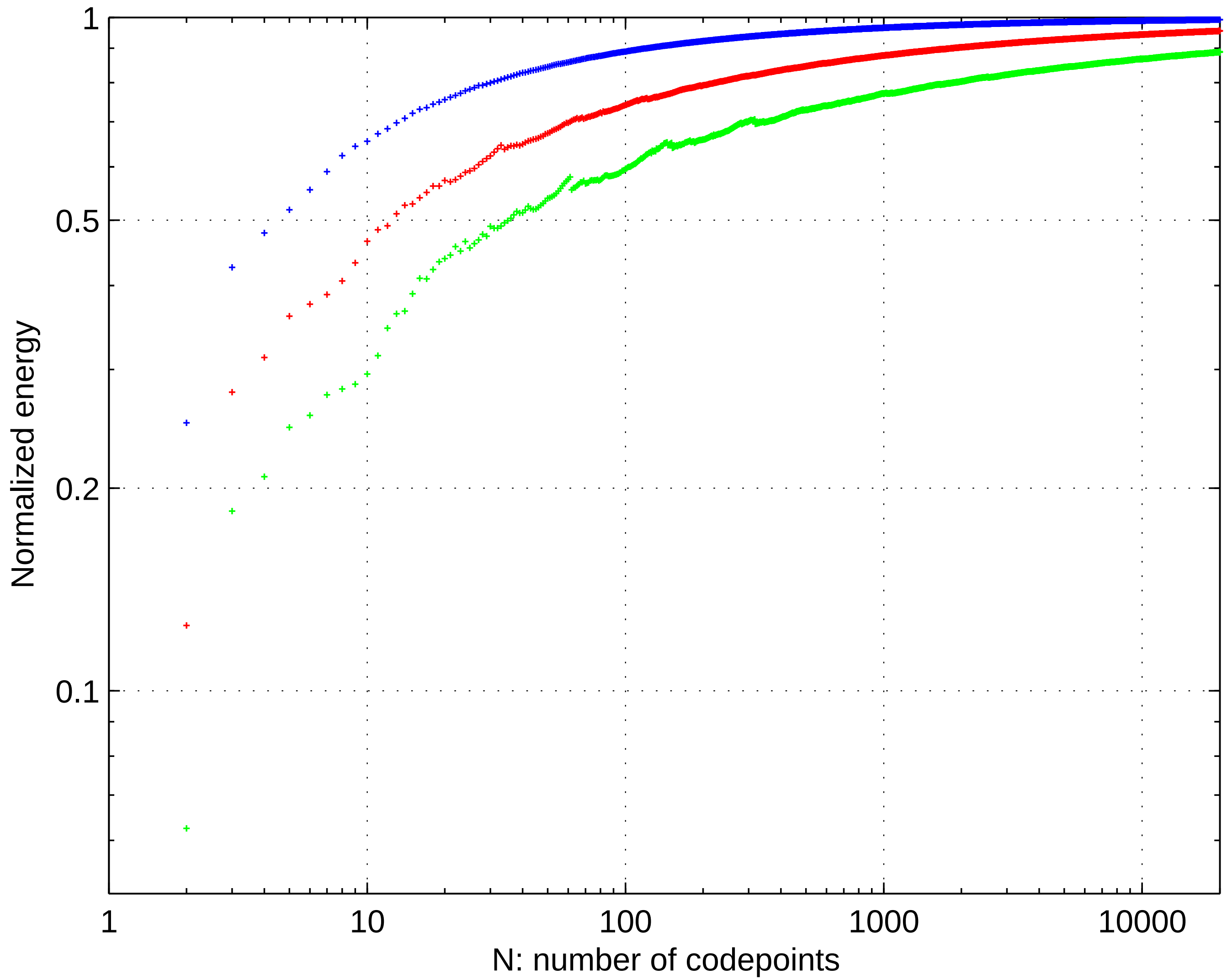
$$E_s(X_\mathcal{N}) = \begin{cases} I_s \pm \mathrm{O}(\mathcal{N}^{-1/d}) & 0 < s < d-1 \\ I_s \pm \mathrm{O}(\mathcal{N}^{-1/d} \log \mathcal{N}) & s = d-1 \\ I_s \pm \mathrm{O}(\mathcal{N}^{s/d-1}) & d-1 < s < d \\ \mathrm{O}(\log \mathcal{N}) & s = d \\ \mathrm{O}(\mathcal{N}^{s/d-1}) & s > d. \end{cases}$$

Comparison to minimum energy

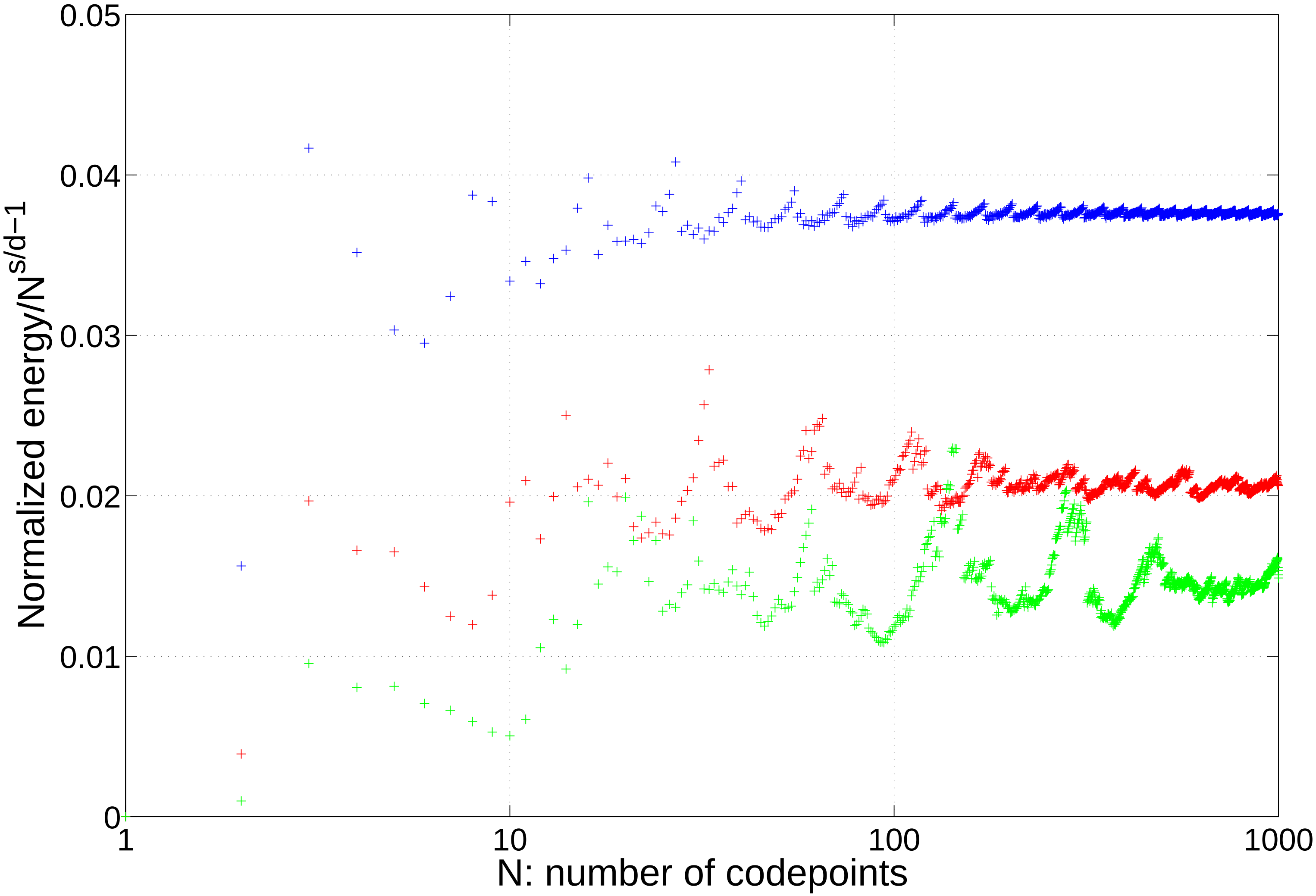
For $s > d - 1$, Theorem 4 yields energy bounds of the same order as $\mathcal{E}_s(\mathcal{N})$, the minimum normalized s energy for $\mathcal{N}$ points on $\mathbb{S}^d$.

$$\mathcal{E}_s(\mathcal{N}) = \begin{cases} I_s - \Theta(\mathcal{N}^{s/d-1}) & 0 < s < d \\ & \text{(Wagner;} \\ & \text{Rakhmanov, Saff \& Zhou;} \\ & \text{Brauchart)} \\ \mathrm{O}(\log \mathcal{N}) & s = d \quad \text{(Kuijlaars \& Saff)} \\ \mathrm{O}(\mathcal{N}^{s/d-1}) & s > d \quad \text{(Hardin \& Saff).} \end{cases}$$

$d - 1$ energy of EQP(2), EQP(3), EQP(4)



$2d$ energy of EQP(2), EQP(3), EQP(4)



Mesh norm (covering radius)

The *mesh norm* of $\boldsymbol{X} := \{\mathbf{x}_1, \ldots, \mathbf{x}_{\mathcal{N}}\} \subset \mathbb{S}^{\boldsymbol{d}}$ is

$$\text{mesh norm } \boldsymbol{X} := \sup_{\mathbf{y} \in \mathbb{S}^{\boldsymbol{d}}} \min_{\mathbf{x} \in \boldsymbol{X}} \cos^{-1}(\mathbf{x} \cdot \mathbf{y}).$$

Since $\mathbf{EQ}(\boldsymbol{d})$ is diameter bounded,

$$\text{mesh norm } \mathbf{EQP}(\boldsymbol{d}, \mathcal{N}) = \mathrm{O}(\mathcal{N}^{-1/\boldsymbol{d}}).$$

Mesh ratio and packing density

The *mesh ratio* of $X := \{\mathrm{x}_1, \ldots, \mathrm{x}_\mathcal{N}\} \subset \mathbb{S}^d$ is

$$\text{mesh ratio}\ X := \text{mesh norm}\ X \ /\ \text{prad}\ X.$$

The *packing density* of X is

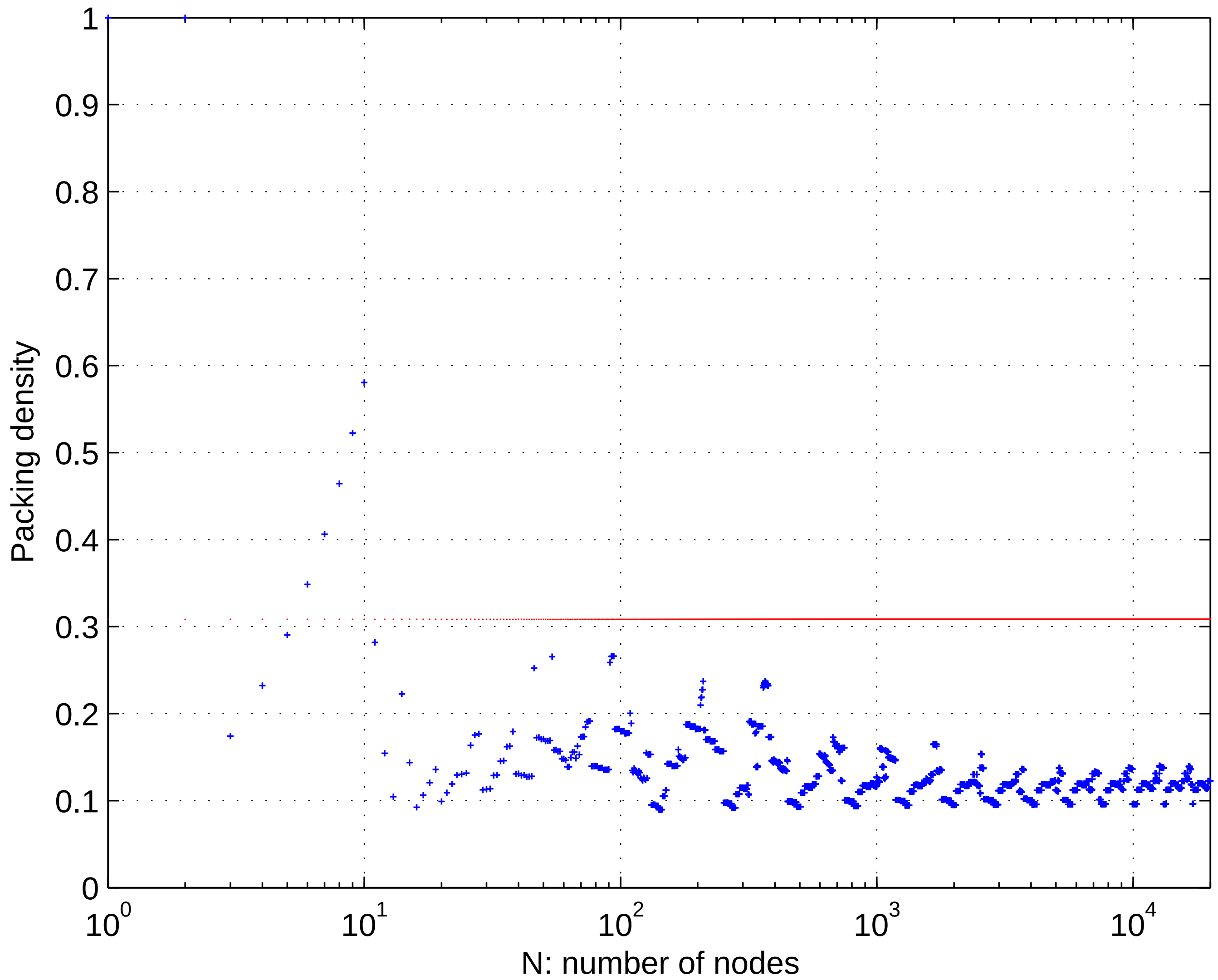
$$\text{pdens}\ X := \mathcal{N}\,\mathring{\sigma}\big(S\big(\mathrm{x}, \text{prad}\ X\big)\big).$$

Regions of $\mathrm{EQ}(d, \mathcal{N})$ near equators $\longrightarrow$ cubic as $\mathcal{N} \to \infty$, so

$$\text{mesh ratio}\ \mathrm{EQP}(d, \mathcal{N}) = \Omega(\sqrt{d}), \quad \text{and}$$

$$\text{pdens}\ \mathrm{EQP}(d, \mathcal{N}) \leqslant \frac{\pi^{d/2}}{2^d\ \Gamma(d/2+1)} \quad \text{as}\ \mathcal{N} \to \infty.$$

Packing density of EQP(4) codes



For EQSP Matlab code

See SourceForge web page for EQSP:

Recursive Zonal Equal Area Sphere Partitioning Toolbox:

`http://eqsp.sourceforge.net`