Spherical codes with good separation, discrepancy and energy

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Outline of talk

EQ codes: The Recursive Zonal Equal Area spherical codes,

$$\text{EQP}(d, \mathcal{N}) \subset S^d, \text{ with } |\text{EQP}(d, \mathcal{N})| = \mathcal{N}.$$ 

- Overview of properties of the EQ codes
- Some precedents
- Definitions: coordinates, partitions, diameter bounds
- The Recursive Zonal Equal Area (EQ) partition
- Details of properties of the EQ codes
- Separation and discrepancy bounds imply energy bounds
- Separation and diameter bounds imply energy bounds
- More details of properties (if time permits)
The spherical code EQP(2,33) on $S^2 \subset \mathbb{R}^3$
Geometric properties of the EQ codes

For \( \text{EQP}(d, \mathcal{N}) \)

Good:

- Centre points of regions of diameter \( = O(\mathcal{N}^{-1/d}) \),
- Mesh norm (covering radius) \( = O(\mathcal{N}^{-1/d}) \),
- Minimum distance and packing radius \( = \Omega(\mathcal{N}^{-1/d}) \).

Bad:

- Mesh ratio \( = \Omega(\sqrt{d}) \),
- Packing density \( \leq \frac{\pi^{d/2}}{2^d \Gamma(d/2+1)} \) as \( \mathcal{N} \to \infty \).
Not so bad?

- Normalized spherical cap discrepancy \( = O(N^{-1/d}) \),
- Normalized \( s \)-energy

\[
E_s = \begin{cases} 
I_s \pm O(N^{-1/d}) & 0 < s < d - 1 \\
I_s \pm O(N^{-1/d} \log N) & s = d - 1 \\
I_s \pm O(N^{s/d-1}) & d - 1 < s < d \\
O(\log N) & s = d \\
O(N^{s/d-1}) & s > d.
\end{cases}
\]

Ugly:

- Cannot be used for polynomial interpolation: proven for large enough \( N \), conjectured for small \( N \).
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Some precedents

The EQ partition is based on Zhou’s (1995) construction for $S^2$ as modified by Saff, and on Sloan’s sketch of a partition of $S^3$ (2003).


Equidistibution without separation: Many constructions for $S^2$, eg. mapped Hammersley, Halton, $(t, s)$ etc. sequences. Feige and Schechtman (2002) constructed a diameter bounded equal area partition of $S^d$. Put one point in each region.
Equal-area partitions of $\mathbb{S}^d \subset \mathbb{R}^d$

An *equal area partition* of $\mathbb{S}^d \subset \mathbb{R}^d$ is a finite set $\mathcal{P}$ of Lebesgue measurable subsets of $\mathbb{S}^d$, such that

$$\bigcup_{R \in \mathcal{P}} R = \mathbb{S}^d,$$

and for each $R \in \mathcal{P}$,

$$\sigma(R) = \frac{\sigma(\mathbb{S}^d)}{|\mathcal{P}|},$$

where $\sigma$ is the Lebesgue area measure on $\mathbb{S}^d$. 
Diameter bounded sets of partitions

The *diameter* of a region $R \subset \mathbb{R}^{d+1}$ is defined by

$$\text{diam } R := \sup\{\|x - y\| \mid x, y \in R\}.$$ 

A set $\Xi$ of partitions of $S^d \subset \mathbb{R}^{d+1}$ is *diameter-bounded* with *diameter bound* $K \in \mathbb{R}_+$ if for all $\mathcal{P} \in \Xi$, for each $R \in \mathcal{P}$,

$$\text{diam } R \leq K |\mathcal{P}|^{-1/d}.$$
Key properties of the EQ partition of $S^d$

$\text{EQ}(d, \mathcal{N})$ is the recursive zonal equal area partition of $S^d$ into $\mathcal{N}$ regions.

The set of partitions $\text{EQ}(d) := \{ \text{EQ}(d, \mathcal{N}) \mid \mathcal{N} \in \mathbb{N}_+ \}$.

The EQ partition satisfies:

**Theorem 1.** For $d \geq 1$, $\mathcal{N} \geq 1$, $\text{EQ}(d, \mathcal{N})$ is an equal-area partition.

**Theorem 2.** For $d \geq 1$, $\text{EQ}(d)$ is diameter-bounded.
Spherical polar coordinates on $\mathbb{S}^d$

*Spherical polar coordinates* describe $x \in \mathbb{S}^d \subset \mathbb{R}^{d+1}$ by one longitude, $\xi_1 \in \mathbb{R}$ (modulo $2\pi$), and $d - 1$ colatitudes, $\xi_j \in [0, \pi]$, for $j \in \{2, \ldots, d\}$.

The spherical polar to Cartesian coordinate map

$\circledast : \mathbb{R} \times [0, \pi]^{d-1} \to \mathbb{S}^d \subset \mathbb{R}^{d+1}$ is

$$\circledast(\xi_1, \xi_2, \ldots, \xi_d) = (x_1, x_2, \ldots, x_{d+1}),$$

where $x_1 := \cos \xi_1 \prod_{j=2}^{d} \sin \xi_j$, $x_2 := \prod_{j=1}^{d} \sin \xi_j$,

$$x_k := \cos \xi_{k-1} \prod_{j=k}^{d} \sin \xi_j, \quad k \in \{3, \ldots, d + 1\}.$$
Spherical caps, zones, and collars

The spherical cap $S(p, \theta) \subset \mathbb{S}^d$ is

\[ S(p, \theta) := \{ q \in \mathbb{S}^d \mid p \cdot q \geq \cos(\theta) \} . \]

For $d > 1$, a zone can be described by

\[ Z(\tau, \beta) := \{ \bigcirc(\xi_1, \ldots, \xi_d) \in \mathbb{S}^d \mid \xi_d \in [\tau, \beta] \} , \]

where $0 \leq \tau < \beta \leq \pi$.

$Z(0, \beta)$ is a North polar cap and $Z(\tau, \pi)$ is a South polar cap.

If $0 < \tau < \beta < \pi$, $Z(\tau, \beta)$ is a collar.
EQ(3,99) Steps 1 to 2

\[ V(\theta_c) = V_R = \sigma(S^3)/99 \]

\[ \Delta = V_R^{1/3} \]

EQ(3,99) Steps 3 to 5

\[ y_1 = 14.8... \]

\[ y_2 = 33.7... \]

\[ y_3 = 33.7... \]

\[ y_4 = 14.8... \]

EQ(3,99) Steps 6 to 7

\[ \theta_1 \]

\[ \theta_2 \]

\[ \theta_3 \]

\[ \theta_4 \]

\[ \theta_5 \]
Centre points of regions of $\text{EQ}(d, \mathcal{N})$

The placement of the centre point $a = \odot(\alpha)$ of a region

$$R = \odot ([\tau_1, \beta_1] \times \ldots \times [\tau_d, \beta_d])$$

is

$$\alpha_1 := \begin{cases} 0 & \beta_1 = \tau_1 \pmod{2\pi} \\ (\tau_1 + \beta_1)/2 \pmod{2\pi} & \text{otherwise}, \end{cases}$$

and for $j > 1$,

$$\alpha_j := \begin{cases} 0 & \tau_j = 0 \\ \pi & \beta_j = \pi \\ (\tau_j + \beta_j)/2 & \text{otherwise}. \end{cases}$$
The \textit{minimum distance} of \( X := \{x_1, \ldots, x_N\} \subset \mathbb{S}^d \) is

\[
\text{min dist } X := \min_{x \neq y \in X} \|x - y\|,
\]

and the \textit{packing radius} of \( X \) is

\[
\text{prad } X := \min_{x \neq y \in X} \cos^{-1}(x \cdot y)/2.
\]

It can be shown that \( \text{min dist } \text{EQP}(d, N) = \Omega(N^{-1/d}) \),

and therefore \( \text{prad } \text{EQP}(d, N) = \Omega(N^{-1/d}) \).
Minimum distance of EQP(4) codes

\[ \text{Min dist} \times N^{1/4} \]

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Normalized spherical cap discrepancy

We use the probability measure \( \sigma^* := \sigma / \sigma(\mathbb{S}^d) \).

For \( X := \{x_1, \ldots, x_N\} \subset \mathbb{S}^d \) the normalized spherical cap discrepancy is

\[
\text{disc } X := \sup_{y \in \mathbb{S}^d} \sup_{\theta \in [0, \pi]} \left| \frac{|X \cap S(y, \theta)|}{\mathcal{N}} - \sigma^*(S(y, \theta)) \right|.
\]

It can be shown that

\[
\text{disc } \text{EQP}(d, \mathcal{N}) = O(\mathcal{N}^{-1/d}).
\]
Normalized $s$-energy

For $X := \{x_1, \ldots, x_N\} \subset S^d$, $s \in \mathbb{R}$, the normalized $s$-energy is

$$E_s(X) := \mathcal{N}^{-2} \sum_{i=1}^{N} \sum_{x_i \neq x_j \in X} \|x_i - x_j\|^{-s},$$

and the normalized energy double integral for $0 < s < d$ is

$$I_s := \int_{S^d} \int_{S^d} \|x - y\|^{-s} \, d\sigma^*(x) d\sigma^*(y).$$
Theorem 3.

Let \((X_1, X_2, \ldots)\) be a sequence of \(S^d\) codes for which there exist \(c_1, c_2 > 0\) and \(0 < q < 1\) such that each \(X_N = \{x_{N,1}, \ldots, x_{N,N}\}\) satisfies

\[
\|x_{N,i} - x_{N,j}\| > c_1 N^{-1/d}, \quad (i \neq j)
\]

\[\text{disc } X_N \leq c_2 N^{-q}.\]

Then for the normalized \(s\) energy for \(0 < s < d\), we have for some \(c_3 \geq 0\),

\[
E_s(X_N) \leq I_s + c_3 N^{(s/d-1)q}.
\]
Separation and diameter imply energy

Theorem 4.

Let \(((X_1, \mathcal{P}_1), (X_2, \mathcal{P}_2), \ldots)\) be a sequence of pairs of \(\mathbb{S}^d\) codes and equal area partitions such that \(|X_N| = |\mathcal{P}_N| = N\), each \(x_{N,i} \in X_N\) lies in \(R_{N,i} \in \mathcal{P}_N\), and such that \((X_1, X_2, \ldots)\) is well separated and \((\mathcal{P}_1, \mathcal{P}_2, \ldots)\) is diameter bounded.

Then for the normalized \(s\) energy we have

\[
E_s(X_N) = \begin{cases} 
I_s \pm O(N^{-1/d}) & 0 < s < d - 1 \\
I_s \pm O(N^{-1/d} \log N) & s = d - 1 \\
I_s \pm O(N^{s/d-1}) & d - 1 < s < d \\
O(\log N) & s = d \\
O(N^{s/d-1}) & s > d.
\end{cases}
\]
Comparison to minimum energy

For \( s > d - 1 \), Theorem 4 yields energy bounds of the same order as \( \mathcal{E}_s(\mathcal{N}) \), the minimum normalized \( s \) energy for \( \mathcal{N} \) points on \( \mathbb{S}^d \).

\[
\mathcal{E}_s(\mathcal{N}) = \begin{cases} 
I_s - \Theta(\mathcal{N}^{s/d-1}) & 0 < s < d \\
0 & s = d \\
O(\log \mathcal{N}) & s > d \\
O(\mathcal{N}^{s/d-1}) & s > d
\end{cases}
\]

(Wagner; Rakhmanov, Saff & Zhou; Brauchart)

(Kuijlaars & Saff)

(Hardin & Saff).
$d - 1$ energy of EQP(2), EQP(3), EQP(4)
$2d$ energy of $\text{EQP}(2), \text{EQP}(3), \text{EQP}(4)$
The *mesh norm* of \( X := \{ x_1, \ldots, x_N \} \subset \mathbb{S}^d \) is

\[
\text{mesh norm } X := \sup_{y \in \mathbb{S}^d} \min_{x \in X} \cos^{-1}(x \cdot y).
\]

Since \( \text{EQ}(d) \) is diameter bounded,

\[
\text{mesh norm } \text{EQP}(d, \mathcal{N}) = O(\mathcal{N}^{-1/d}).
\]
Mesh ratio and packing density

The mesh ratio of $X := \{x_1, \ldots, x_N\} \subset S^d$ is

$$\text{mesh ratio } X := \text{mesh norm } X / \text{prad } X.$$

The packing density of $X$ is

$$\text{pdens } X := N^* \sigma(S(x, \text{prad } X)).$$

Regions of $\text{EQ}(d, N)$ near equators $\rightarrow$ cubic as $N \rightarrow \infty$, so

$$\text{mesh ratio } \text{EQP}(d, N) = \Omega(\sqrt{d}), \quad \text{and}$$

$$\text{pdens } \text{EQP}(d, N) \leq \frac{\pi^{d/2}}{2^d \Gamma(d/2 + 1)}$$

as $N \rightarrow \infty$. 
Packing density of EQP(4) codes
For EQSP Matlab code

See SourceForge web page for EQSP:

Recursive Zonal Equal Area Sphere Partitioning Toolbox:

http://eqsp.sourceforge.net