Visualization as its own reward
The mathematics of conformal chaos

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Topics

- Conformal geometric algebra
- Generation of conformal tori by exponentiation of bivectors
- Conformal chaos

Embedding of $\mathbb{R}^3$ in conformal geometry

The conformal geometry of conformal geometric algebra embeds $\mathbb{R}^{\mathbf{3}}$ into the space $\mathbb{R}^{\mathbf{4,1}}$.

If the basis elements of $\mathbb{R}^{\mathbf{4,1}}$ are denoted as $\mathbf{e_{-1}, e_1, e_2, e_3, e_4}$, we first form the null vectors $\boldsymbol{n_\infty} := \mathbf{e_{-1}} + \mathbf{e_4}$, $\boldsymbol{n_0} := \mathbf{e_{-1}} - \mathbf{e_4}$, and embed the point $\boldsymbol{x} \in \mathbb{R}^{\mathbf{3}}$ into $\mathbb{R}^{\mathbf{4,1}}$ as the null vector

$$\begin{aligned}\mathbf{cga3}(\boldsymbol{x}) &:= (\mathbf{e_4} - \boldsymbol{x})\boldsymbol{n_\infty}(\boldsymbol{x} - \mathbf{e_4}) \\ &= \boldsymbol{x}^{\mathbf{2}}\boldsymbol{n_\infty} + \mathbf{2}\boldsymbol{x} + \boldsymbol{n_0}.\end{aligned}$$

(Doran and Lasenby 2003)

Embedding of $\mathbb{R}^3$ in conformal geometry

The point $x \in \mathbb{R}^3$ is represented as any point on the null line $\lambda\, \mathbf{cga3}(x)$, with $\lambda \neq 0$. This is converted to standard form as

$$\mathbf{cga3std}(X) := \frac{-2}{X \cdot n_\infty} X.$$

For $X = \mathbf{cga3}(x)$, the point $x \in \mathbb{R}^3$ is recovered as

$$\mathbf{agc3}(X) := \mathbf{Proj}_{\mathbb{R}^3}\left(\mathbf{cga3std}(X)/2\right).$$

(Doran and Lasenby 2003)

Exponentiation of a bivector in $\mathbb{R}_{4,1}$

If $\boldsymbol{B}$ is a bivector in the Clifford algebra $\mathbb{R}_{\mathbf{4,1}}$, then $\boldsymbol{e^B}$ is in $\boldsymbol{Spin(4,1)}$ and

$$\boldsymbol{X \mapsto e^B X e^{-B}}$$

is a special orthogonal transformation of $\boldsymbol{R^{4,1}}$.

(Doran and Lasenby 2003, Dorst and Valkenburg 2011)

Orbits of the exponential of a bivector in $\mathbb{R}_{4,1}$

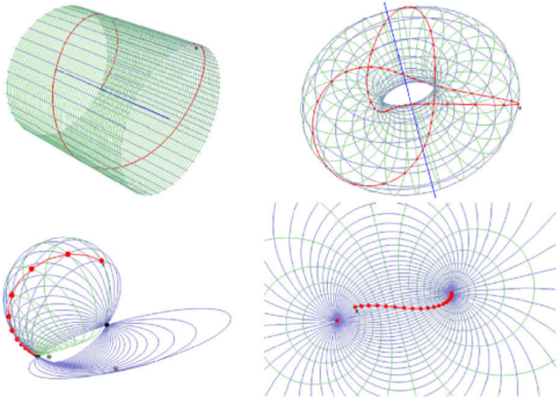

Fig. 5.3 Conformal coordinate grids induced by some rotors, with orbits for a point x indicated. See text for explanation

(Dorst and Valkenburg 2011)

Orbits of the exponential of a bivector in $\mathbb{R}_{4,0}$

If B is a bivector in the Clifford algebra $\mathbb{R}_{4,0} \subset \mathbb{R}_{4,1}$, but not a 2-blade, then the orbit

$$\{\mathbf{agc3}\left(e^{tB}\,\mathbf{cga3}(x)e^{-tB}\right) \mid t \in \mathbb{R}\}$$

is either closed, or it rules a conformal torus in $\mathbb{R}^3$.

(Dorst and Valkenburg 2011)

The reciprocal bivector $\mathbf{e_1 e_2 e_3 e_4} B^{-1}$ gives the same torus.

Conformal chaos

For $R := \exp(B)$, $S := \exp(\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3\mathrm{e}_4 B^{-1})$, the mappings

$$\phi_R : x \mapsto \mathrm{agc3}\left(R\,\mathrm{cga3}(x)R^{-1}\right), \text{ and}$$
$$\phi_S : x \mapsto \mathrm{agc3}\left(S\,\mathrm{cga3}(x)S^{-1}\right)$$

can be used in an algorithm inspired by Barnsley's *chaos game*:
Starting with a point $x_{(0)} := x \in \mathbb{R}^3$,
at step n choose $\phi_{(n)} = \phi_R$ or $\phi_{(n)} = \phi_S$
uniformly at random to obtain

$$x_{(n+1)} := \phi_{(n)}(x_{(n)}).$$

The resulting set is a subset of the torus ruled by B.

(Barnsley 1988)

References

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