A quick introduction to Clifford algebras

Paul Leopardi

paul.leopardi@unsw.edu.au

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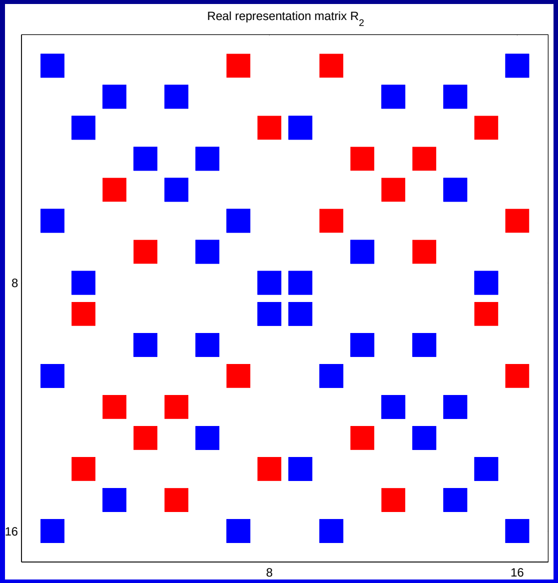


Quadratic forms

(Lounesto 1997)
For vector space $\mathbb{V}$ over field $\mathbb{F}$, characteristic $\neq 2$:

- Map $f : \mathbb{V} \to \mathbb{F}$, with

$$f(\lambda x) = \lambda^2 f(x), \forall \lambda \in \mathbb{F}, x \in \mathbb{V}$$

- $f(x) = b(x, x)$, where

$$b : \mathbb{V} \times \mathbb{V} \to \mathbb{F}, \text{given by}$$

$$b(x, y) := \frac{1}{2}\left(f(x + y) - f(x) - f(y)\right)$$

is a symmetric bilinear form

Quadratic spaces, Clifford maps

(Porteous 1995; Lounesto 1997)

- A *quadratic space* is the pair $(\mathbb{V}, f)$, where f is a quadratic form on $\mathbb{V}$
- A *Clifford map* is a vector space homomorphism

$$\varphi : \mathbb{V} \to \mathbb{A}$$

where $\mathbb{A}$ is an associative algebra, and

$$(\varphi v)^2 = f(v) \qquad \forall v \in \mathbb{V}$$

Universal Clifford algebras

(Lounesto 1997)

- The *universal Clifford algebra* $Cl(f)$ for the quadratic space $(\mathbb{V}, f)$ is the algebra generated by the image of the Clifford map φ_f such that $Cl(f)$ is the universal initial object such that $\forall$ suitable algebras $\mathbb{A}$ with Clifford map $\varphi_{\mathbb{A}}$ $\exists$ a homomorphism

$$P_{\mathbb{A}} : Cl(f) \to \mathbb{A}$$

$$\varphi_{\mathbb{A}} = P_{\mathbb{A}} \circ \varphi_f$$

Real Clifford algebras $\mathbb{R}_{p,q}$

(Porteous 1995)

- The real quadratic space $\mathbb{R}^{p,q}$ is $\mathbb{R}^{p+q}$ with

$$\phi(x) := -\sum_{k=-q}^{-1} {x_k}^2 + \sum_{k=1}^{p} {x_k}^2$$

- For each $p, q \in \mathbb{N}$, the real universal Clifford algebra for $\mathbb{R}^{p,q}$ is called $\mathbb{R}_{p,q}$.
- $\mathbb{R}_{p,q}$ is isomorphic to some matrix algebra over one of: $\mathbb{R}, \mathbb{R} \oplus \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{H} \oplus \mathbb{H}$
- For example, $\mathbb{R}_{1,1} \cong \mathbb{R}(2)$

Notation for integer sets

- For $S \subseteq \mathbb{Z}$, define

$$\sum_{k \in S} f_k := \sum_{\substack{k=\min S \\ k \in S}}^{\max S} f_k \qquad \prod_{k \in S} f_k := \prod_{\substack{k=\min S \\ k \in S}}^{\max S} f_k$$

$\mathbb{P}(S) :=$ the *power set* of S.

- For $m \leqslant n \in \mathbb{Z}$, define

$$\varsigma(m,n) := \{m, m+1, \ldots, n-1, n\} \setminus \{0\}$$

Frames for Clifford algebras

(Hestenes and Sobczyck 1984; Wene 1992; Ashdown)

- A *frame* is an ordered basis $(\gamma_{-q}, \ldots, \gamma_p)$ for $\mathbb{R}^{p,q}$ which puts a quadratic form into the canonical form ϕ.
- For $p, q \in \mathbb{N},$ embed the frame for $\mathbb{R}^{p,q}$ into $\mathbb{R}_{p,q}$ via the maps

$$\gamma : \varsigma(-q, p) \to \mathbb{R}^{p,q}$$
$$\varphi : \mathbb{R}^{p,q} \to \mathbb{R}_{p,q}$$
$$(\varphi\gamma_k)^2 = \phi\gamma_k = \operatorname{sgn} k.$$

Real frame groups

(Braden 1985; Lam and Smith 1989)
For $p, q \in \mathbb{N}$, define the real *frame group* $\mathbb{G}_{p,q}$ via the map

$$g : \varsigma(-q, p) \to \mathbb{G}_{p,q}$$

with generators and relations

$$\begin{aligned} \langle \mu, g_k \mid \mu g_k = g_k \mu,\ \ \mu^2 = 1, \\ (g_k)^2 = \begin{cases} \mu, & \text{if } k < 0, \\ 1, & \text{if } k > 0 \end{cases} \\ g_k g_m = \mu g_m g_k \ \ \forall k \neq m \rangle \end{aligned}$$

Canonical products

(Bergdolt 1996; Lounesto 1997; Dorst 2001)

- The real frame group $\mathbb{G}_{p,q}$ has order 2^{p+q+1}
- Each member w can be expressed as the canonically ordered product

$$w = \mu^a \prod_{k \in T} g_k$$

$$= \mu^a \prod_{k=-q,\ k \neq 0}^{p} {g_k}^{b_k}$$

where $T \subseteq \varsigma(-q, p),\ \ a, b_k \in \{0, 1\}$

Clifford algebra of frame group

(Braden 1985; Lam and Smith 1989; Lounesto 1997; Dorst 2001)

- For $p, q \in \mathbb{N}$ embed $\mathbb{G}_{p,q}$ into $\mathbb{R}_{p,q}$ via the map

$$\alpha : \mathbb{G}_{p,q} \to \mathbb{R}_{p,q}$$
$$\alpha 1 := 1, \qquad \alpha\mu := -1$$
$$\alpha g_k := \varphi\gamma_k, \quad \alpha(gh) := (\alpha g)(\alpha h).$$

- Define *basis elements* via the map

$$\mathbf{e} : \mathbb{P}\varsigma(-q,p) \to \mathbb{R}_{p,q}, \qquad \mathbf{e}_T := \alpha \prod_{k \in T} g_k,$$

Each $a \in \mathbb{R}_{p,q}$ can be expressed as

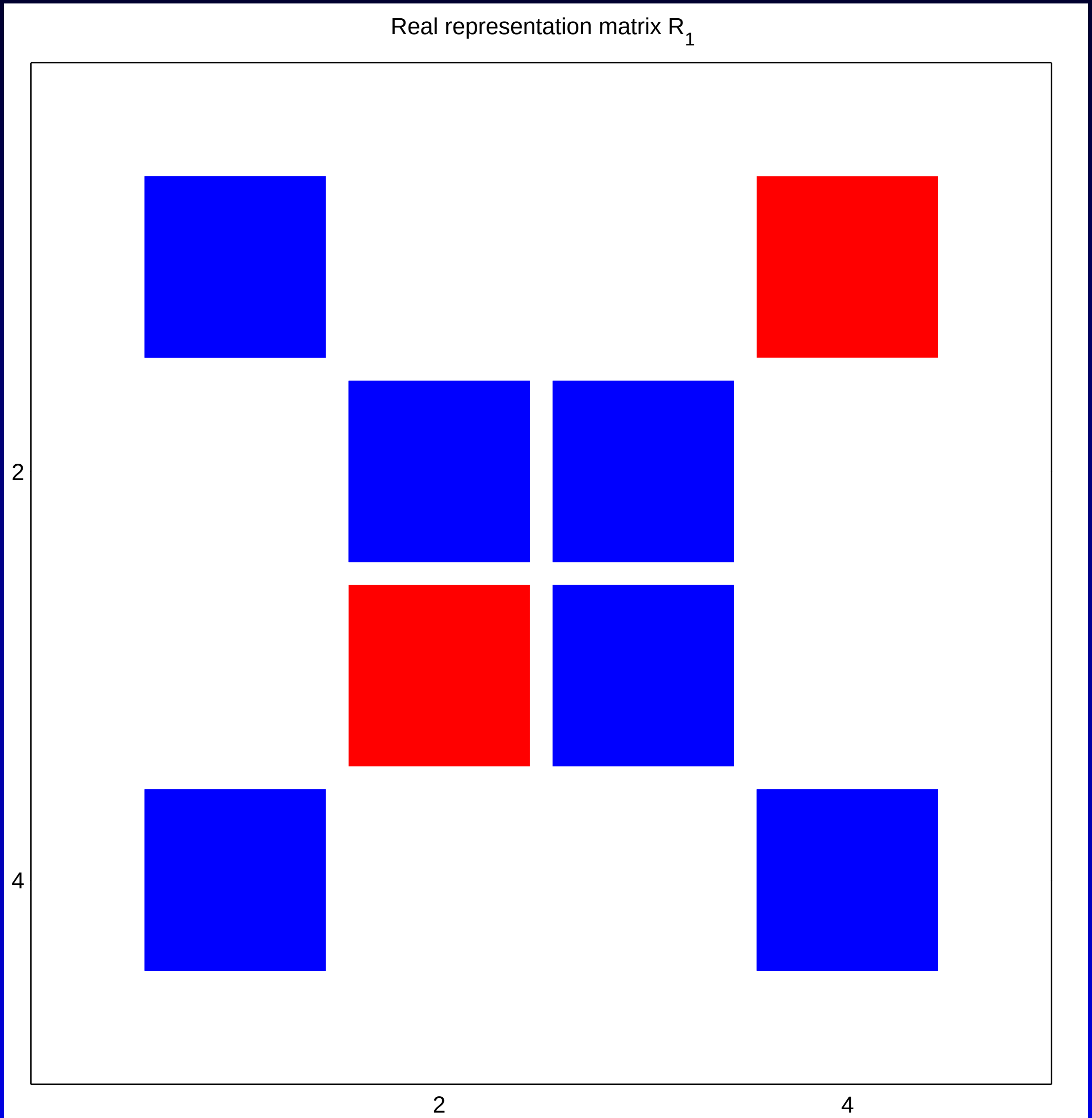
$$a = \sum_{T \subseteq \varsigma(-q,p)} a_T \mathbf{e}_T$$

Neutral matrix representations

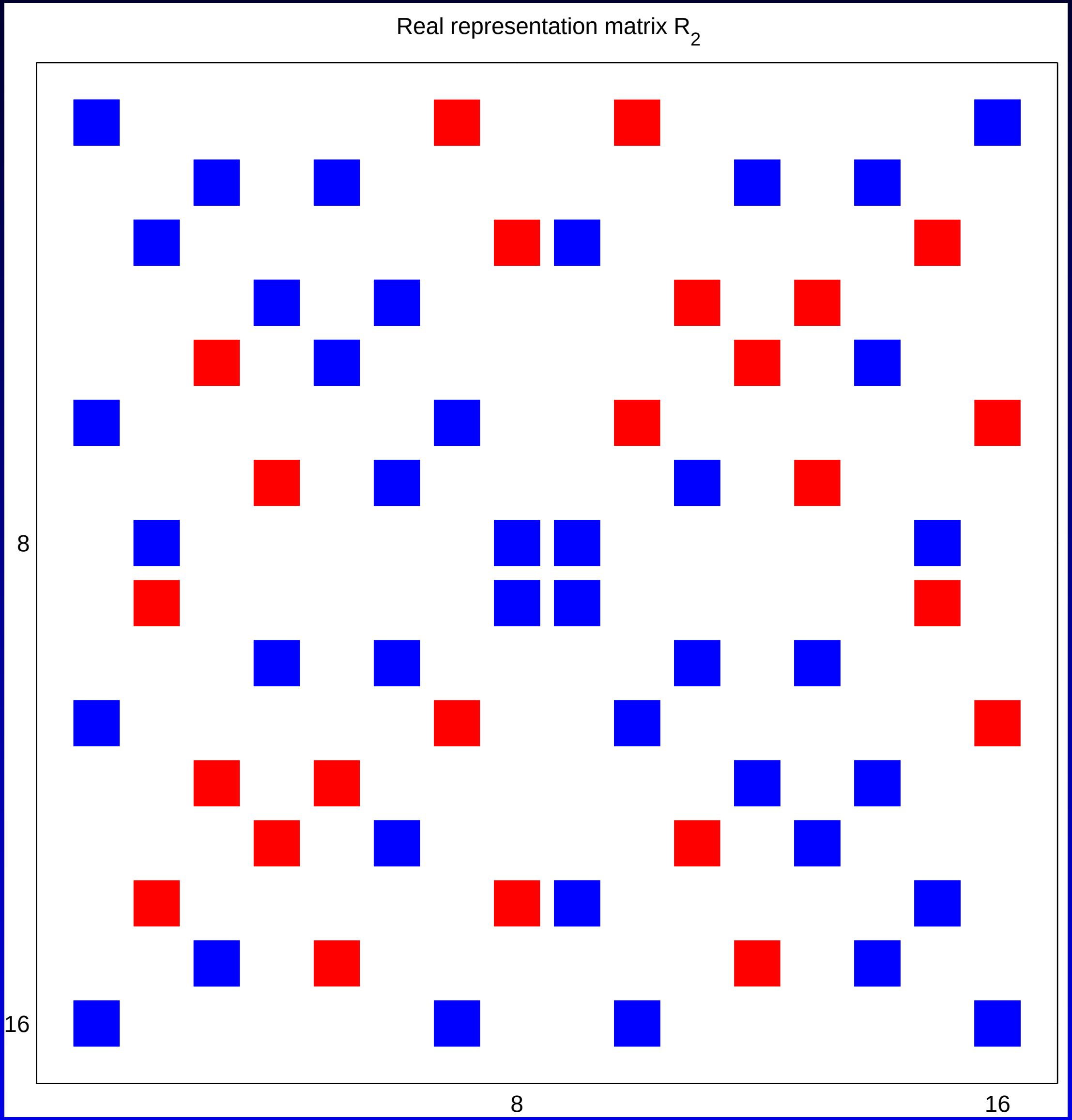
(Cartan and Study 1908; Porteous 1969; Lounesto 1997)
The *representation map* P_m and *representation matrix* R_m make the following diagram commute:

$$
\begin{array}{ccc}
\mathbb{R}_{m,m} & \xrightarrow{\text{coord}} & \mathbb{R}^{4^m} \\
P_m \big\downarrow & & \big\downarrow R_m \\
\mathbb{R}(2^m) & \xrightarrow[\text{reshape}]{} & \mathbb{R}^{4^m}
\end{array}
$$

Real representation matrix R_1



Real representation matrix R_2



References 1

REFERENCES

[Ablamowicz 1996] R. Ablamowicz, P. Lounesto, J. M. Parra (eds), *Clifford algebras with numeric and symbolic computations*, Birkhäuser, 1996.

[Ashdown] M. Ashdown, *GA Package for Maple V*, `http://www.mrao.cam.ac.uk/~clifford/software/GA/GAhelp5.html`

[Bergdolt 1996] G. Bergdolt, "Orthonormal basis sets in Clifford algebras", in [Ablamowicz 1996].

[Braden 1985] H. W. Braden, "N-dimensional spinors: Their properties in terms of finite groups", *J. Math. Phys*. 26 (4), April 1985. American Institute of Physics.

[Cartan] Elie Cartan, P. Montel, et al. (eds), *Oeveres Complètes*, Gauthier–Villars, 1953.

[Cartan & Study 1908] Elie Cartan, Eduard Study, "Nombres Complexes", *Encyclopaedia Sciences Mathématique*, édition française, 15, 1908, d'après l'article allemand de Eduard Study, pp329–468. Reproduced as pp107–246 of [Cartan].

References 2

REFERENCES

[Dorst 2001] Leo Dorst, “Honing geometric algebra for its use in the computer sciences”, pp127–152 of [Sommer 2001].

[Hestenes & Sobczyk 1984] David Hestenes, Garret Sobczyk, *Clifford algebra to geometric calculus : a unified language for mathematics and physics*, D. Reidel, 1984.

[Lam & Smith 1989] T. Y. Lam, Tara L. Smith, “On the Clifford-Littlewood-Eckmann groups: a new look at periodicity mod 8”, *Rocky Mountains Journal of Mathematics*, vol 19, no 3, Summer 1989.

[Lounesto 1997] P. Lounesto, *Clifford algebras and spinors*, 2nd edition, Cambridge University Press, 2001. 1st edition, 1997.

[Micali 1992] A. Micali, R. Boudet, J. Helmstetter, (eds), *Clifford algebras and their applications in mathematical physics : proceedings of second workshop held at Montpellier, France, 1989*, Kluwer Academic Publishers, 1992.

[Porteous 1969] I. Porteous, *Topological geometry*, Van Nostrand Reinhold, 1969.

References 3

REFERENCES

[Porteous 1995] I. Porteous, *Clifford algebras and the classical groups*, Cambridge University Press, 1995.

[Sommer 2001] G. Sommer (ed.), *Geometric Computing with Clifford Algebras*, Springer, 2001.

[Wene 1995] G. P. Wene, "The Idempotent stucture of an infi nite dimensional Clifford algebra", pp161–164 of [Micali 1992].