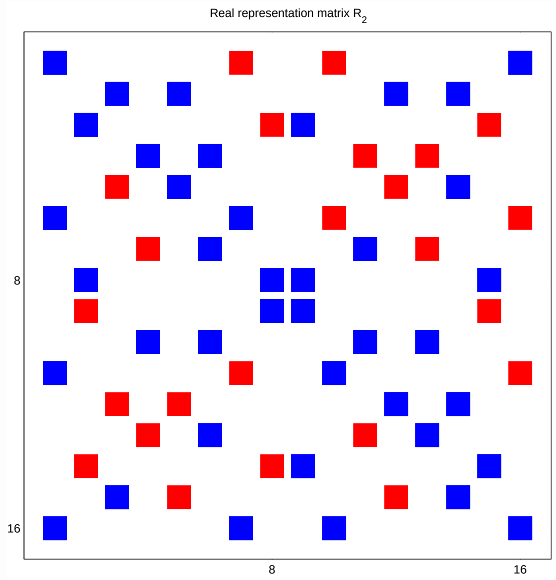
A generalized FFT for Clifford algebras

Paul Leopardi

paul.leopardi@unsw.edu.au

School of Mathematics, University of New South Wales.

For presentation at ICIAM, Sydney, 2003-07-07.



GFT for finite groups

(*Beth 1987, Diaconis and Rockmore 1990; Clausen and Baum 1993*)

The generalized Fourier transform (GFT) for a finite group $\mathbb{G}$ is the representation map D from the group algebra $\mathbb{C}\mathbb{G}$ to a faithful complex matrix representation of $\mathbb{G}$, such that D is the direct sum of a complete set of irreducible representations.

$$D : \mathbb{C}\mathbb{G} \rightarrow \mathbb{C}(M), \quad D = \bigoplus_{k=1}^{n} D_k,$$

$$\text{where} \quad D_k : \mathbb{C}\mathbb{G} \rightarrow \mathbb{C}(m_k), \quad \text{and} \sum_{k=1}^{n} m_k = M.$$

The generalized FFT is any fast algorithm for the GFT.

GFT for supersolvable groups

(*Baum 1991; Clausen and Baum 1993*)

Definition 1.

The 2-linear complexity $\boldsymbol{L_2(X)}$, *of a linear operator* $\boldsymbol{X}$ *counts non-zero additions* $\mathbb{A}\boldsymbol{(X)}$, *and non-zero multiplications, except multiplications by* $\mathbf{1}$ *or* $\mathbf{-1}$.

The GFT $\boldsymbol{D}$, for supersolvable groups has

$$\boldsymbol{L_2(D) = \mathrm{O}(|G| log_2 |G|)}.$$

GFT for Clifford algebras

(*Hestenes and Sobczyck 1984; Wene 1992; Felsberg et al. 2001*)

The GFT for a real universal Clifford algebra, $\mathbb{R}_{p,q}$, is the representation map $P_{p,q}$ from the real framed representation to the real matrix representation of $\mathbb{R}_{p,q}$.

$P_{p,q} : \mathbb{R}^{\mathbb{P}\varsigma(-q,p)} \rightarrow \mathbb{R}(2^{N(p,q)})$, where

- $\varsigma(a,b) := \{a, a+1, \ldots, b\} \setminus \{0\}$.
- $\mathbb{P}\varsigma(-q,p)$ is the power set of $\varsigma(-q,p)$, a set of index sets with cardinality 2^{p+q}.
- The real framed representation $\mathbb{R}^{\mathbb{P}\varsigma(-q,p)}$ is the set of maps from $\mathbb{P}\varsigma(-q,p)$ to $\mathbb{R}$, isomorphic as a real vector space to the set of 2^{p+q} tuples of real numbers indexed by subsets of $\varsigma(-q,p)$.

This is *not* the “discrete Clifford Fourier transform” of Felsberg, et al.

Clifford algebras and supersolvable groups

(*Braden 1985; Lam and Smith 1989*)

The real universal Clifford algebra $\mathbb{R}_{p,q}$, is a quotient of the group algebra $\mathbb{R}\mathbb{G}_{p,q}$, by an ideal, where $\mathbb{G}_{p,q}$ is a 2-group, here called the frame group. $\mathbb{G}_{p,q}$ is supersolvable.
The GFT for Clifford algebras is related to that for supersolvable groups:

$$\begin{array}{ccc}
\mathbb{C}\mathbb{G}_{p,q} & \xrightarrow{D} & D\left(\mathbb{C}\mathbb{G}_{p,q}\right) \subseteq \mathbb{C}(M) \\
\text{project}\big\downarrow & & \big\downarrow\text{project} \\
\mathbb{R}\mathbb{G}_{p,q} & \xrightarrow{D} & D\left(\mathbb{R}\mathbb{G}_{p,q}\right) \subseteq \mathbb{C}(M) \\
\text{quotient}\big\downarrow & & \big\downarrow\text{quotient} \\
\mathbb{R}_{p,q} & \xrightarrow[P_{p,q}]{} & P_{p,q}\left(\mathbb{R}_{p,q}\right) \subseteq \mathbb{R}(2^{N(p,q)})
\end{array}$$

GFT for the neutral Clifford algebra $\mathbb{R}_{n,n}$

(*Braden 1985; Lam and Smith 1989*)

The GFT for the neutral Clifford algebra $\mathbb{R}_{n,n}$ is a map:
$P_n : \mathbb{R}^{\mathbb{P}\varsigma(-n,n)} \rightarrow \mathbb{R}(2^n)$. $|\mathbb{P}\varsigma(-n,n)| = 4^n$.

The frame group $\mathbb{G}_{n,n}$ is an extraspecial 2-group. $|\mathbb{G}_{n,n}| = 2^{2n+1}$.

For $\mathbb{R}_{n,n}$ we might expect $L_2(P_n) = \mathrm{O}(n4^n)$ this way:

$$\begin{array}{ccc} \mathbb{R}_{n,n} & \xrightarrow{P_{n,n}} & \mathbb{R}(2^n) \\ \downarrow & & \uparrow \\ \mathbb{C}\mathbb{G}_{n,n} & \xrightarrow[D]{} & D\left(\mathbb{C}\mathbb{G}_{n,n}\right) \end{array}$$

but there are explicit algorithms for both forward and inverse GFT.

Kronecker product

Definition 2.
If $\boldsymbol{A} \in \mathbb{R}(\boldsymbol{r})$ *and* $\boldsymbol{B} \in \mathbb{R}(\boldsymbol{s})$, *then*

$$(\boldsymbol{A} \otimes \boldsymbol{B})_{\boldsymbol{j},\boldsymbol{k}} = \boldsymbol{A}_{\boldsymbol{j},\boldsymbol{k}}\boldsymbol{B}$$

if $\boldsymbol{A} \otimes \boldsymbol{B}$ *is treated as an* $\boldsymbol{r} \times \boldsymbol{r}$ *block matrix with* $\boldsymbol{s} \times \boldsymbol{s}$ *blocks.*

A well known property of the Kronecker product is:

Lemma 3.
If $\boldsymbol{A}, \boldsymbol{C} \in \mathbb{R}(\boldsymbol{r})$ *and* $\boldsymbol{B}, \boldsymbol{D} \in \mathbb{R}(\boldsymbol{s})$, *then*

$$(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{C} \otimes \boldsymbol{D}) = \boldsymbol{AC} \otimes \boldsymbol{BD}$$

Generating set for $\mathbb{R}_{n,n}$

(*Porteous 1969*)

Definition 4. *Here and in what follows, define:*

$$I_n := \text{ unit matrix of dimension } 2^n, \quad I := I_1$$

$$J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad K := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Lemma 5.
If S *is an orthonormal generating set for* $\mathbb{R}(2^{n-1})$, *then*

$$\{-JK \otimes A \mid A \in S\} \cup \{J \otimes I_{n-1}, K \otimes I_{n-1}\}$$

is an orthonormal generating set for $\mathbb{R}(2^n)$.

Generalized Fourier transform

Definition 6. *Use Lemma* **5** *to define the GFT of each generator of* $\mathbb{R}_{n,n}$

$$P_n \mathrm{e}_{\{-n\}} := J \otimes I_{n-1}, \quad P_n \mathrm{e}_{\{n\}} := K \otimes I_{n-1},$$

$$\text{for} \quad 1 - n \leqslant k \leqslant n - 1, \quad P_n \mathrm{e}_{\{k\}} := -JK \otimes P_{n-1} \mathrm{e}_{\{k\}}.$$

We can now compute $P_n \mathrm{e}_T$ *as:* $\quad P_n \mathrm{e}_T = \prod_{k \in T} P_n \mathrm{e}_{\{k\}}$

Each basis matrix is monomial *and each non-zero is* -1 *or* 1.
We can now define $P_n : \mathbb{R}^{\mathbb{P}\varsigma(-n,n)} \rightarrow \mathbb{R}(2^n)$, *by:*

$$P_n a = \sum_{T \subseteq \varsigma(-n,n)} a_T P_n \mathrm{e}_T, \quad \text{for} \quad a = \sum_{T \subseteq \varsigma(-n,n)} a_T \mathrm{e}_T.$$

Bound for linear complexity of GFT

Theorem 7.
$L_2(P_n)$ *is bounded by* $d^{3/2}$,
where d *is the dimension of* $\mathbb{R}(2^n) \cong \mathbb{R}_{n,n}$.

Proof
Since $P_n e_T$ is of size $2^n \times 2^n$ and is monomial, it has 2^n non-zeros.

$\mathbb{R}(2^n)$ has 4^n basis elements.

$\mathbb{A}(P_n)$ is therefore bounded by

$$4^n \times 2^n = (4^n)^{3/2} = d^{3/2},$$

where d is the dimension of $\mathbb{R}(2^n) \cong \mathbb{R}_{n,n}$.

There are no non-trivial multiplications. $\square$

$\mathbb{Z}_2$ grading and $\otimes$ are keys to GFT

(*Lam 1973*)
The algebras $\mathbb{R}_{p,q}$ are $\mathbb{Z}_2$-*graded*. Each $a \in \mathbb{R}_{p,q}$ can be split into odd and even parts, $a = a^+ + a-$, with $odd \times odd = even$, etc. Scalars are even and the generators are odd.

We can express P_n in terms of its actions on the even and odd parts of a multivector: $P_n a = P_n a^+ + P_n a^-$, $a = a^+ + a^- \in \mathbb{R}_{n,n}$.
Lemma 8. *For all* $b \in \mathbb{R}_{n-1,n-1}$, *we have*

$$P_n b^+ = I \otimes P_{n-1} b^+, \quad P_n b^- = -JK \otimes P_{n-1} b^-,$$

so that

$$\begin{aligned}(P_n a^-)(P_n b^-) &= (-JK \otimes P_{n-1} a^-)(-JK \otimes P_{n-1} b^-) \\ &= I \otimes (P_{n-1} a^-)(P_{n-1} b^-) \\ &= I \otimes \left(P_{n-1}(a^- b^-)\right) = P_n(a^- b^-).\end{aligned}$$

Recursive expression for P_n

Theorem 9. *For* $n > 0$, *for the GFT* P_n *as per Definition 6, for* $a \in \mathbb{R}_{n,n}$, $a = a^+ + a^-$, *with*

$$a^+ = a^+_{\underline{\emptyset}} + e_{\{-n\}} a^+_{\underline{-n}} + a^+_{\underline{n}} e_{\{n\}} + e_{\{-n\}} a^+_{\underline{-n,n}} e_{\{n\}},$$

$$a^- = a^-_{\underline{\emptyset}} + e_{\{-n\}} a^-_{\underline{-n}} + a^-_{\underline{n}} e_{\{n\}} + e_{\{-n\}} a^-_{\underline{-n,n}} e_{\{n\}},$$

we have $P_n a = P_n a^+ + P_n a^-$,

$$P_n a^+ = I \otimes P_{n-1} a^+_{\underline{\emptyset}} + K \otimes P_{n-1} a^+_{\underline{-n}} + \\ - J \otimes P_{n-1} a^+_{\underline{n}} + JK \otimes P_{n-1} a^+_{\underline{-n,n}},$$ *and*

$$P_n a^- = -JK \otimes P_{n-1} a^-_{\underline{\emptyset}} + J \otimes P_{n-1} a^-_{\underline{-n}} + \\ K \otimes P_{n-1} a^-_{\underline{n}} + I \otimes P_{n-1} a^-_{\underline{-n,n}}.$$

Base cases and linear complexity

Theorem 10.

$$P_0 a^+ = [a^+], \quad P_0 a^- = 0.$$

Proof If $a \in \mathbb{R}_{0,0}$ then a is even, so $a^- = 0$. □

Theorem 11. *For* $n \geq 0$,

$$L_2(P_n) \leqslant n4^n = \frac{1}{2} d \log_2 d,$$

where $d = 4^n$ *is the dimension of* $\mathbb{R}_{n,n}$.

Proof (Sketch)
Count non-zero additions at each level of recursion.

You will obtain at most 4^n additions at each of n levels. □

Real framed inner product

Recall that if $a \in \mathbb{R}_{n,n}$, then a can be expressed as

$$a = \sum_{T \subseteq \varsigma(-n,n)} a_T \mathrm{e}_T$$

The basis $\{\mathrm{e}_T \mid T \subseteq \varsigma(-n,n)\}$ is orthornormal with respect to the real framed inner product

$$a \bullet b := \sum_{T \subseteq \varsigma(-n,n)} a_T b_T.$$

We have

$$\mathrm{e}_S \bullet \mathrm{e}_T = \delta_{S,T} \quad \text{and} \quad a_T = a \bullet \mathrm{e}_T.$$

Normalized Frobenius inner product

Since the GFT P_n is an isomorphism, it preserves this inner product. That is, there is an inner product $\bullet : \mathbb{R}(2^n) \times \mathbb{R}(2^n) \longrightarrow \mathbb{R},$ such that, for $a, b \in \mathbb{R}_{n,n},$

$$P_n a \bullet P_n b = a \bullet b,$$

$$\text{so } P_n a \bullet P_n \mathrm{e}_T = a \bullet \mathrm{e}_T = a_T.$$

This is the *normalized Frobenius* inner product.

Lemma 12.

For $A, B \in \mathbb{R}(2^n),$ *the normalized Frobenius inner product:*

$$A \bullet B := 2^{-n} \operatorname{tr} A^T B \quad = 2^{-n} \sum_{j,k=1}^{2^n} A_{j,k} B_{j,k},$$

satisfies $P_n a \bullet P_n b = a \bullet b,$ *for* $a, b \in \mathbb{R}_{n,n}.$

Inverse GFT

Since $P_n a \bullet P_n e_T = a_T$, we can define $Q_n := P_n^{-1}$ by

Definition 13.

$$Q_n : \mathbb{R}(2^n) \rightarrow \mathbb{R}^{\mathbb{P}\varsigma(-n,n)}$$

$$\textit{For} \quad A \in \mathbb{R}(2^n), \quad T \subseteq \varsigma(-n,n),$$

$$(Q_n A)_T := A \bullet P_n e_T.$$

Naive algorithm for Q_n evaluates $A \bullet P_n e_T$ for each $T \subseteq \varsigma(-n,n)$.

Theorem 14. $L_2(Q_n) \leqslant d^{3/2} + d \log d$, *where* $d = 4^n$.

Proof (Sketch)

$\mathbb{A}(Q_n) \leqslant 2^n \times 4^n$, and the naive algorithm also needs at most 4^n

divisions by 2^n. $\square$

Left Kronecker quotient

The *left Kronecker quotient* is a binary operation which is an inverse operation to the Kronecker matrix product.

Definition 15.

$$\oslash : \mathbb{R}(r) \times \mathbb{R}(rs) \rightarrow \mathbb{R}(s),$$

$$\textit{for} \;\; A \in \mathbb{R}(r), \;\; \mathrm{nnz}(A) \neq 0, \;\; C \in \mathbb{R}(rs),$$

$$(A \oslash C)_{j,k} := \frac{1}{\mathrm{nnz}(A)} \sum_{A_{j,k} \neq 0} \frac{C_{j,k}}{A_{j,k}},$$

where C *is treated as an* $r \times r$ *block matrix with* $s \times s$ *blocks, ie. as if* $C \in \mathbb{R}(s)(r)$.

Left Kronecker quotient

Theorem 16.
The left Kronecker quotient is an inverse operation to the Kronecker matrix product, when applied from the left, as follows:

For $\boldsymbol{A} \in \mathbb{R}(\boldsymbol{r}),\ \mathbf{nnz}(\boldsymbol{A}) \neq \mathbf{0},\ \boldsymbol{B} \in \mathbb{R}(\boldsymbol{s}),$
we have $\boldsymbol{A} \oslash (\boldsymbol{A} \otimes \boldsymbol{B}) = \boldsymbol{B}.$

Proof

$$\begin{aligned} \boldsymbol{A} \oslash (\boldsymbol{A} \otimes \boldsymbol{B}) &= \frac{1}{\mathbf{nnz}(\boldsymbol{A})} \sum_{\boldsymbol{A}_{j,k} \neq 0} \frac{\boldsymbol{A}_{j,k}\boldsymbol{B}}{\boldsymbol{A}_{j,k}} \\ &= \frac{1}{\mathbf{nnz}(\boldsymbol{A})} \sum_{\boldsymbol{A}_{j,k} \neq 0} \boldsymbol{B} \\ &= \boldsymbol{B} \end{aligned}$$

□

Left Kronecker quotient and orthogonality

Lemma 17. *For* $A \in \mathbb{R}(2^n)$, $B \in \mathbb{R}(2^n)$, $C \in \mathbb{R}(2^n s)$, *if* $\mathrm{nnz}(A) = 2^n$ *then* $A \oslash (B \otimes C) = (A' \bullet B)C$, *where*

$$A'_{j,k} = \frac{1}{A_{j,k}}, \text{ if } A_{j,k} \neq 0, \ 0 \text{ otherwise.}$$

Proof

$$A \oslash (B \otimes C) = \frac{1}{\mathrm{nnz}(A)} \sum_{A_{j,k} \neq 0} \frac{B_{j,k} C}{A_{j,k}}$$

$$= \frac{1}{2^n} \sum_{j,k=1}^{2^n} A'_{j,k} B_{j,k} C$$

$$= (A' \bullet B)C$$

□

Left Kronecker quotient and orthogonality

Lemma 18. *If* $n > 0$ *and*

$$A \in \mathbb{R}(2^{n+m}) = \sum_{T \subseteq \varsigma(-n,n)} (P_n e_T) \otimes A_T, \quad \textit{where}$$

$A_T \in \mathbb{R}(2^m)$, *then* $(P_n e_T) \oslash A = A_T$, *for* $T \subseteq \varsigma(-n,n)$.

Corollary 19.

$$\begin{aligned}
&\textit{If } n > 0, \quad A_I, A_J, A_K, A_{JK} \in \mathbb{R}(2^{n-1}),\\
&\textit{and } A = I \otimes A_I + J \otimes A_J + K \otimes A_K + JK \otimes A_{JK},\\
&\textit{then } I \oslash A = A_I, \quad J \oslash A = A_J,\\
&\qquad K \oslash A = A_K, \quad JK \oslash A = A_{JK}.
\end{aligned}$$

Recursive expression for Q_n

Theorem 20.
For $n > 0$, $A \in \mathbb{R}(2^n)$, Q_n *as per Definition 13,*

$$
\begin{aligned}
Q_n(A) = {} & Q_{n-1}(I \oslash A)^+ - Q_{n-1}(JK \oslash A)^- \\
& + \mathrm{e}_{\{-n\}} \left(Q_{n-1}(JK \oslash A)^+ + Q_{n-1}(I \oslash A)^-\right) \mathrm{e}_{\{n\}} \\
& + \mathrm{e}_{\{-n\}} \left(Q_{n-1}(K \oslash A)^- + Q_{n-1}(J \oslash A)^+\right) \\
& + \left(-Q_{n-1}(J \oslash A)^- + Q_{n-1}(K \oslash A)^+\right) \mathrm{e}_{\{n\}}.
\end{aligned}
$$

For $n = 0$, *we have* $Q_0[a] = a$.

Proof (Sketch)

Start with Theorems 9 and 10 and apply Corollary 19. □

Linear complexity of inverse GFT

Theorem 21.
$L_2(Q_n) \leqslant 2n4^n = d\log_2 d$,
where $d = 4^n$ *is the dimension of* $\mathbb{R}_{n,n}$.

Proof

Q_n uses $\circledbslash$ four times. Each time needs at most 4^{n-1} additions.

Q_n also uses Q_{n-1} four times. So,

$$\mathbb{A}(Q_n) \leqslant 4^n + 4\mathbb{A}(Q_{n-1}) \quad \leqslant n4^n = \frac{1}{2} d\log_2 d.$$

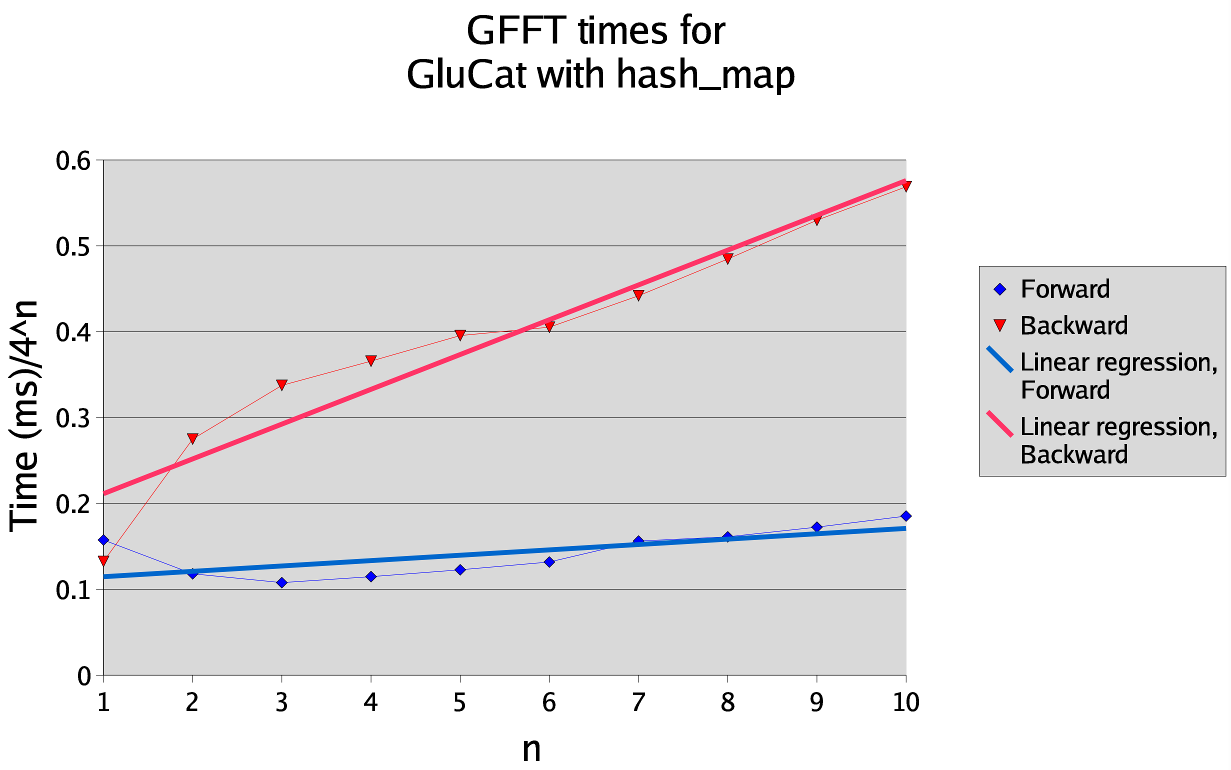
For Q_n, each of the four uses of $\circledbslash$ needs 4^{n-1} divisions by 2.

So $L_2(Q_n) \leqslant 2n4^n = d\log_2 d$. $\square$

Benchmark for GluCat implementation

(*Lounesto et al. 1987; Lounesto 1992; Raja 1996*)

- Generic library of universal Clifford algebra templates
- For details, see `http://glucat.sf.net`



References 1

References

[Ablamowicz 1996] R. Ablamowicz, P. Lounesto, J. M. Parra (eds), *Clifford algebras with numeric and symbolic computations*, Birkhäuser, 1996.

[Baum 1991] Ulrich Baum, "Existence and effi cient construction of fast Fourier transforms on supersolvable groups", *Computational Complexity* 1 (1991), no. 3, 235–256

[Bayro Corrochano & Sobczyk] E. Bayro Corrochano, G. Sobczyk, *Geometric algebra with applications in science and engineering*. Birkhäuser, 2001.

[Beth 1987] Thomas Beth, "On the computational complexity of the general discrete Fourier transform", *Theoretical Computer Science*, 51 (1987), no. 3, 331–339.

[Braden 1985] H. W. Braden, "N-dimensional spinors: Their properties in terms of fi nite groups", *J. Math. Phys*. 26 (4), April 1985. American Institute of Physics.

References 2

References

[Bulow et al. 2001] Thomas Bülow, Michael Felsberg, Gerald Sommer, "Non-commutative hypercomplex Fourier transforms of multidimensional signals", pp187–207 of [Sommer 2001].

[Chernov 2001] Vladimir M. Chernov, "Clifford algebras as projections of group algebras", pp461–476 of [Bayro Corrochano & Sobczyk].

[Clausen & Baum 1993] Michael Clausen, Ulrich Baum, *Fast Fourier transforms*. Bibliographisches Institut, Mannheim, 1993.

[Diaconis & Rockmore 1990] Persi Diaconis, Daniel Rockmore, "Effi cient computation of the Fourier transform on fi nite groups", *J. Amer. Math. Soc.* 3 (1990), no. 2, 297–332.

[Felsberg et al. 2001] Michael Felsberg, Thomas Bülow, ; Gerald Sommer, Vladimir M. Chernov, "Fast algorithms of hypercomplex Fourier transforms", pp231–254 of [Sommer 2001].

References 3

References

[Hestenes & Sobczyk 1984] David Hestenes, Garret Sobczyk, *Clifford algebra to geometric calculus : a unified language for mathematics and physics*, D. Reidel, 1984.

[Lam 1973] T. Y. Lam, *The algebraic theory of quadratic forms*, W. A. Benjamin, Inc., 1973.

[Lam & Smith 1989] T. Y. Lam, Tara L. Smith, “On the Clifford-Littlewood-Eckmann groups: a new look at periodicity mod 8”, *Rocky Mountains Journal of Mathematics*, vol 19, no 3, Summer 1989.

[Lounesto 1987] P. Lounesto, R. Mikkola, V. Vierros, *CLICAL User Manual: Complex Number, Vector Space and Clifford Algebra Calculator for MS-DOS Personal Computers*, Institute of Mathematics, Helsinki University of Technology, 1987.

[Lounesto 1992] P. Lounesto, “Clifford algebra calculations with a microcomputer”, pp39–55 of [Micali et al. 1989].

References 4

REFERENCES

[Micali et al. 1989] A. Micali, R. Boudet, J. Helmstetter, (eds), *Clifford algebras and their applications in mathematical physics : proceedings of second workshop held at Montpellier, France, 1989*, Kluwer Academic Publishers, 1992.

[Porteous 1969] I. Porteous, *Topological geometry*, Van Nostrand Reinhold, 1969.

[Raja 1996] A. Raja, “Object-oriented implementations of Clifford algebras in C++: a prototype”, in [Ablamowicz 1996].

[Sommer 2001] G. Sommer (ed.), *Geometric Computing with Clifford Algebras*, Springer, 2001.

[Wene 1995] G. P. Wene, “The Idempotent stucture of an infi nite dimensional Clifford algebra”, pp161–164 of [Micali et al. 1989].