A generic library of universal Clifford algebra templates

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ABSTRACT The deal.II and POOMA libraries provide frameworks for the numerical solution of partial differential equations. Their use of C++ templates creates an opportunity to explore the Clifford numerical solution of PDEs, if Clifford algebras can be used as numeric data types in C++.

The GluCat library is just such an implementation of Clifford algebras, designed for use with deal.II, POOMA and similar libraries. This poster describes some aspects of GluCat: notation, morphisms, matrix generators and functions of multivectors. **Keywords:** C++, Clifford algebras, deal.II, GluCat, partial differential equations, POOMA, template library.

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1 GluCat

GluCat [12] is a Generic library of universal Clifford algebra templates: a C++ template library for use with other libraries such as *deal.II* [3] and *POOMA* [10]. The library is based on a prototype by Arvind Raja [18] and previous work by Lounesto [13] [14] and others. ***GluCat*** is the result of a coursework masters project at the University of New South Wales, supervised by Bill McLean. For full credits and more details, see [12].

GluCat provides floating point approximations of two isomorphic representations of real Clifford algebras, the *framed* representation and the *matrix* representation. The descriptions below of these representations draw heavily on the work of Porteous [17], Lounesto [15], Braden [5] and others, and contain essentially no new results.

2 Framed representations

First, some notation:

$$\mathbb{B} := \{0,1\}, \quad \mathbb{P}(S) := \{T : T \subseteq S\}, \quad A^B := \{f : B \to A\}$$

For $S \subseteq \mathbb{Z}$, define

$$S^- := \{k \in S : k < 0\}, \quad S^+ := \{k \in S : k > 0\}, \quad S^\pm := S^- \cup S^+,$$
$$\sum_{k \in S} f_k := \sum_{k=\min S \atop k \in S}^{\max S} f_k, \quad \prod_{k \in S} f_k := \prod_{k=\min S \atop k \in S}^{\max S} f_k.$$

For $m \leqslant n \in \mathbb{Z}$, define $[[m,n]] := \{m, m+1, \ldots, n-1, n\}^\pm$.

This gives $\mathbb{R}^{\mathbb{P}(S)}$ a canonical $2^{\sharp S}$ element basis.

$$x = \sum_{T \subseteq S} x_T \mathbf{e}_T$$

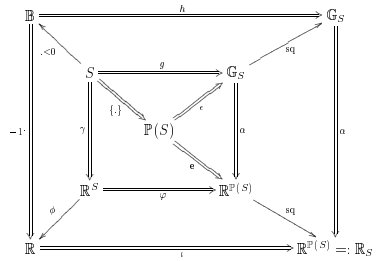
Here, addition is the usual addition of functions in $\mathbb{R}^{\mathbb{P}(S)}$.

- Define $\gamma : S \to \mathbb{R}^S$ by $(\gamma_k)_m = \delta_{k,m}$.
- Define $\varphi : \mathbb{R}^S \to \mathbb{R}^{\mathbb{P}(S)}$ by
$$\varphi \sum_{k \in S} v_k \gamma_k := \sum_{k \in S} v_k \mathbf{e}_{\{k\}}$$
- Define $\phi : \mathbb{R}^S \to \mathbb{R}$ by
$$\phi(x) := -\sum_{k \in S^-} {x_k}^2 + \sum_{k \in S^+} {x_k}^2$$
- Define $\iota : \mathbb{R} \to \mathbb{R}^{\mathbb{P}(S)}$ by $\iota(r) := r\mathbf{e}_\emptyset$.
- Define $\alpha : \mathbb{G}_S \to \mathbb{R}^{\mathbb{P}(S)}$ by
$$\alpha(\epsilon_T) := \mathbf{e}_T \quad \forall T \subseteq S$$
$$\alpha(h_1 a) := -\alpha(a) \quad \forall a \in \mathbb{G}_S$$
- Define $\times : \mathbb{R}^{\mathbb{P}(S)} \times \mathbb{R}^{\mathbb{P}(S)} \to \mathbb{R}^{\mathbb{P}(S)}$ by
$$\mathbf{e}_T \times \mathbf{e}_U := \alpha(\epsilon_T \epsilon_U)$$
$$\sum_{T \subseteq S} x_T \mathbf{e}_T \times \sum_{U \subseteq S} y_U \mathbf{e}_U := \sum_{T \subseteq S} \sum_{U \subseteq S} x_T y_U (\mathbf{e}_T \times \mathbf{e}_U)$$

Proposition. $\alpha(ab) = \alpha(a)\alpha(b) \quad \forall a, b \in \mathbb{G}_S$.

2.4 Summary diagram

Proposition. *The following diagram commutes:*



2.1 Framed representation

The *framed* representation uses a construction which builds a lattice of real Clifford algebras corresponding to the lattice of subsets of a finite set of non-zero integers, $S \subset \mathbb{Z}^\pm$ [2] [6]. Each multivector is represented as a map $x : \mathbb{P}(S) \to \mathbb{R}$. The subsets of S are called *index sets* [18] [7].

The framed representation of the real Clifford algebra $\mathbb{R}_S$ can be constructed by defining a multiplication on $\mathbb{R}^{\mathbb{P}(S)}$ such that $\mathbb{R}_S \cong \mathbb{R}_{\sharp S^+, \sharp S^-}$, where $\mathbb{R}_{p,q}$ is the universal Clifford algebra for $\mathbb{R}^{p,q}$ as per Porteous ([17], Chapters 5 and 15) and Lounesto ([15], Chapters 14, 15 and 16).

The multiplication can be defined in three stages:

1. Construct the *real frame group* $\mathbb{G}_S$ [5]
2. Construct a natural injection from $\mathbb{G}_S$ to $\mathbb{R}^{\mathbb{P}(S)}$.
3. Define the algebra multiplication on $\mathbb{R}_S$ as a bilinear map which preserves the group multiplication.

2.2 Real frame groups

To each finite $S \subset \mathbb{Z}^\pm$ there corresponds the real *frame group* $\mathbb{G}_S$. This group can be defined using generators and relations and the maps $g : S \to \mathbb{G}_S$ and $h : \mathbb{B} \to \mathbb{G}_S$ by

$$\begin{aligned}
\mathbb{G}_S := \big\langle h_1, g_k \mid k \in S, \quad {h_1}^2 &= 1, \\
h_1 g_k &= g_k h_1, && \forall k \in S \\
g_k g_m &= h_1 g_m g_k, && \forall k \neq m \\
{g_k}^2 &= 1, && \forall k > 0 \\
{g_k}^2 &= h_1, && \forall k < 0 \big\rangle \\
\mathbb{G}_{p,q} &:= \mathbb{G}_{[[-q,p]]}
\end{aligned}$$

Groups of this type have been extensively studied by Braden [5], Lam and Smith [11], Bergdolt [4] and others, but there is no generally accepted name for them.

The real frame group $\mathbb{G}_S$ has order $2^{\sharp S+1}$. Each member w can be expressed as the canonically ordered product [15]:

$$w = h_1^a \prod_{k \in S}^{p} g_k^{b_k} \quad \text{where } a, b_k \in \mathbb{B}.$$

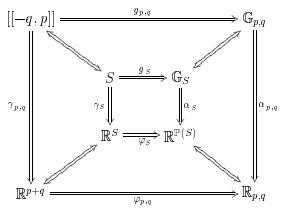
2.3 Construction via canonical maps

For finite $S \subset \mathbb{Z}^\pm$,

- Define $\epsilon : \mathbb{P}(S) \to \mathbb{G}_S$ by
$$\epsilon_T = \prod_{k \in S} g_k^{(\chi_T)_k} = \prod_{k \in T} g_k \quad \forall T \subseteq S$$
where χ_T is the characteristic function of T;
- Define $\mathbf{e} : \mathbb{P}(S) \to \mathbb{R}^{\mathbb{P}(S)}$ by $(\mathbf{e}_T)_U = \delta_{T,U}$.

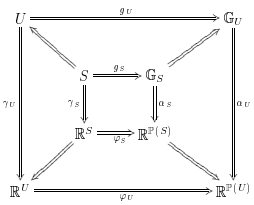
2.5 Isomorphisms

$$\sharp S^+ = p = \sharp T^+, \quad \sharp S^- = q = \sharp T^- \quad \Rightarrow \quad \mathbb{R}^S \equiv \mathbb{R}^T, \mathbb{G}_S \equiv \mathbb{G}_T, \mathbb{R}_S \equiv \mathbb{R}_T$$



2.6 Subalgebras

$$S \subseteq \text{ finite } U \subset \mathbb{Z}^\pm \Rightarrow \mathbb{R}^S \sqsubseteq \mathbb{R}^U, \mathbb{G}_S \sqsubseteq \mathbb{G}_U, \mathbb{R}_S \sqsubseteq \mathbb{R}_U$$



([17], Proposition 15.1)

2.7 Framed inner product, norm

We can define an inner product on $\mathbb{R}_S$.

For $x = \sum_{T \subseteq S} x_T \mathbf{e}_T, \quad y = \sum_{T \subseteq S} y_T \mathbf{e}_T \in \mathbb{R}_S$,

- The *real framed inner product* is $x \bullet y := \sum_{T \subseteq S} x_T y_T$
- The *real framed norm* [8] is $\|x\|_{RF} := (x \bullet x)^{\frac{1}{2}}$

2.8 Framed notation

For finite $S \subset \mathbb{Z}^\pm$ define the *framed notation* [2] $\{\mathbf{k}_1, \mathbf{k}_2, \ldots\} := \mathbf{e}_{\{k_1, k_2, \ldots\}}$.

3 Matrix representations

Each real Clifford algebra $\mathbb{R}_{p,q}$ can be *represented* by an endomorphism algebra over a finite dimensional module over one of: $\mathbb{R}, {}^2\mathbb{R}, \mathbb{C}, \mathbb{H}, {}^2\mathbb{H}$ ([17], Table 15.27).

3.1 Real representations of $\mathbb{R}_{p,q}$

p \ q	0	1	2	3	4	5	6	7
0	**1**	2	4	8	8	8	**8**	16
1	2	**2**	4	8	16	16	16	**16**
2	**2**	4	**4**	8	16	32	32	32
3	4	**4**	8	**8**	16	32	64	64
4	8	8	**8**	16	**16**	32	64	128
5	16	16	16	**16**	32	**32**	64	128
6	16	32	32	32	**32**	64	**64**	128
7	16	32	64	64	64	**64**	128	**128**

The table, based on Porteous ([17], Table 15.27), lists N for $\mathbb{R}(N) \equiv \mathbb{R}^{N\times N}$.

GluCat chooses generators for $\mathbb{R}_{p,q}$ from the generator set for $\mathbb{R}_{P,Q} \cong \mathbb{R}(N)$ with **bold** N.

3.2 Explicit calculation of the matrix representation

The mapping between the framed representation and the real matrix representation is an algebra isomorphism and so is linear and invertible. In the case of the *neutral* Clifford algebra $\mathbb{R}_{m,m}$ we have the following diagram:

$$\begin{array}{ccc} \mathbb{R}_{[|-m,m|]} & \xrightarrow{\text{coord}} & \mathbb{R}^{4^m} \\ \Big\downarrow{\scriptstyle \rho_m} & & \Big\downarrow{\scriptstyle R_m} \\ \mathbb{R}(2^m) & \xrightarrow{\text{reshape}} & \mathbb{R}^{4^m} \end{array}$$

where

- coord maps a multivector $x \in \mathbb{R}_{[|-m,m|]}$ to a coordinate vector in $\mathbb{R}^{4^m}$ with respect to a canonical ordering of the index sets $T \in [[-m,m]]$.
- reshape reshapes a matrix $X \in \mathbb{R}(2^m)$ column-by-column into a coordinate vector in $\mathbb{R}^{4^m}$.
- $\rho_m : \mathbb{R}_{[|-m,m|]} \to \mathbb{R}(2^m)$ is a *representation map*.
- $R_m : \mathbb{R}^{4^m} \to \mathbb{R}^{4^m}$ is a *representation matrix*.

3.3 Matrix inner product, norm

We can define an inner product on $\mathbb{R}(N)$.

For $A, B \in \mathbb{R}(N)$,

- The *normalised Frobenius inner product* is:

$$A \bullet B := \frac{1}{N} \sum_{j=1}^{N} \sum_{k=1}^{N} A_{j,k} B_{j,k}$$

- The *normalised Frobenius norm* is:

$$\|A\|_{NF} := \sqrt{A \bullet A} = \frac{1}{\sqrt{N}} \|A\|_F$$

where $\|A\|_F$ is the *Frobenius norm* [9] of A.

3.4 Basis matrices are orthonormal

Lemma. *The basis matrices for $\mathbb{R}_{m,m}$ are orthonormal with respect to the normalised Frobenius inner product $\bullet$:*

$$\rho(\mathbf{e}_T) \bullet \rho(\mathbf{e}_U) = \delta_{T,U} \qquad \forall T, U \subseteq S$$

where $\rho : \mathbb{R}_{[|-m,m|]} \to \mathbb{R}(2^m)$ is the representation map.

Corollary. *The real representation matrix R_m is 2^m times an orthogonal matrix.*

3.5 Inverse representation maps

Lemma. *For $a, b \in \mathbb{R}_{m,m}$*

$$\rho(a) \bullet \rho(\mathbf{e}_T) = a \bullet \mathbf{e}_T, \quad \rho(a) \bullet \rho(b) = a \bullet b$$

Proposition. *The normalized Frobenius inner product can be used to determine the coordinates a_T of the framed representation of $a \in \mathbb{R}_S$ from the real matrix representation $\rho(a)$ as follows: $a_T = \rho(a) \bullet \rho(\mathbf{e}_T)$*

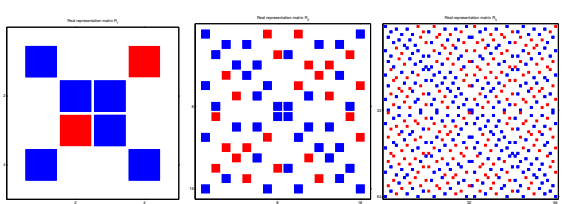


FIGURE 1. Representation matrices R_1, R_2, R_3 with red $= -1$, blue $= 1$

4 Functions of multivectors

Consider the real Clifford algebra $\mathbb{R}_S$ to be a subalgebra of the some Clifford algebra $\mathbb{R}_U$ isomorphic to some complex matrix algebra $\mathbb{C}(N)$ for some N ([17], Table 15.27).

Define i to be the unit volume element $\mathrm{i} := \mathbf{e}_U$, of $\mathbb{R}_U$.

We have $\mathrm{i}^2 = -1, \mathrm{i}x = x\mathrm{i} \quad \forall x \in \mathbb{R}_U$ [5].

Define a function of $x \in \mathbb{R}_S$ in terms of the corresponding function in $\mathbb{C}$ [19] [9].

$$f(x) := \frac{1}{2\pi\mathrm{i}} \int_{\partial\Omega} f(z)(z-x)^{-1} dz$$

with Ω an open subset of $\mathbb{C}$ with smooth boundary $\partial\Omega$ and $\sigma(\rho(x)) \subset \Omega$ where $\rho : \mathbb{R}_S \to \mathbb{C}(N)$ and $\sigma(X)$ is the spectrum of X.

Use norm equality between $\mathbb{R}_S$, $\mathbb{R}_U$ and $\mathbb{C}(N)$ to go on to define Taylor series, Pade approximations, etc.

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