LOOP REALIZATIONS OF QUANTUM AFFINE ALGEBRAS

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Abstract. We give a simplified description of quantum affine algebras in their loop presentation. This description is related to Drinfeld's new realization via halves of vertex operators. We also define an idempotent version of the quantum affine algebra which is suitable for categorification.

Contents

1. Introduction

Let $\mathfrak{g}$ be a finite dimensional simple simply-laced complex Lie algebra. There are two well-known ways to add a parameter to $\mathfrak{g}$. The first is q-deformation, where one deforms the enveloping algebra $U(\mathfrak{g})$ to a new algebra $U_q(\mathfrak{g})$, commonly known as a quantum group. The second is affinization, where one replaces the Lie algebra $\mathfrak{g}$ by the affine Lie algebra $\widehat{\mathfrak{g}} := \mathfrak{g} \otimes \Bbbk[t, t^{-1}] \oplus \Bbbk c$. The Lie algebra $\widehat{\mathfrak{g}}$ is a central extension of the loop algebra of $\mathfrak{g}$. Both affine Lie algebras and quantum groups arise in representation theory, low dimensional topology, algebraic geometry, and many other parts of mathematics and mathematical physics.

The quantum affine algebra $U_q(\widehat{\mathfrak{g}})$, which is a deformation of the enveloping algebra of $\widehat{\mathfrak{g}}$, combines both ideas. However, the description of $U_q(\widehat{\mathfrak{g}})$ is much more complicated than that of either $U_q(\mathfrak{g})$ or $\widehat{\mathfrak{g}}$. Perhaps on account of connections to physics, the relations in $U_q(\widehat{\mathfrak{g}})$ are typically encoded in generating functions (see Section 2.3), resulting in a presentation known as Drinfeld's realization [D]. A central role in this presentation is played by the quantum Heisenberg algebra $\widehat{\mathfrak{h}} \subset U_q(\widehat{\mathfrak{g}})$, since many of the relations in $U_q(\widehat{\mathfrak{g}})$ are expressed using generating functions whose terms are in $\widehat{\mathfrak{h}}$.

The main motivation for the current note comes from categorification in the representation theory of quantum affine algebras. In the accompanying paper [CL2], we construct a 2-representation of the quantum affine (and quantum toroidal) algebra on the derived categories of Hilbert schemes of points on the surface $\widehat{\mathbb{C}^2/\Gamma}$, where $\Gamma \subset SL_2(\mathbb{C})$ is the finite subgroup associated to $\mathfrak{g}$ under the McKay correspondence. After passing to equivariant K-theory, this gives a representation of $U_q(\widehat{\mathfrak{g}})$. However, the presentation of $U_q(\widehat{\mathfrak{g}})$ which appears naturally in this construction is not identical to Drinfeld's new realization (though it is very similar). Thus, in the process of producing an action of $U_q(\widehat{\mathfrak{g}})$ on equivariant K-theory, we were forced to show an equivalence between various presentations of the quantum affine algebra. Since these results seems to be of some independent interest and since they

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do not require 2-categories or algebraic geometry, we decided to write them down independently from [CL2]. We emphasize, however, that this paper, and Theorem 2.1 in particular, has application to the proof that quantum affine algebras act on the equivariant K-theory of Hilbert schemes.

We begin the current paper by rewriting the Drinfeld realization using halves of vertex operators. This gives a different set of generators of the quantum Heisenberg subalgebra $\widehat{\mathfrak{h}}$ (Section 2.4). Next we describe an idempotent modification $\dot{U}_q(\widehat{\mathfrak{g}})$ of the quantum affine algebra in Section 2.5. The idempotent version $\dot{U}_q(\widehat{\mathfrak{g}})$ is not a unital algebra anymore, but it has a somewhat simplified presentation. Moreover, any representation of $\dot{U}(\widehat{\mathfrak{g}})$ automatically gives rise to a representation of the quantum affine algebra. The presentation of $\dot{U}_q(\widehat{\mathfrak{g}})$ which we use in [CL2] is, up to a slight renomalization, the presentation in Section 2.5. We describe this renormalized realization in Section 5.3.

Much of the content of this paper is known to experts working on quantum affine algebras. Our main contribution is to explain how the relations in Drinfeld's new realization (and also in our idempotent modification) are far from being minimal. More precisely, in Section 2.6 we give a smaller set of relations in $\dot{U}_q(\widehat{\mathfrak{g}})$ and prove (Theorem 2.1) that all other relations are a consequence of these relations. In the last section we speculate on a minimal realization of $\dot{U}_q(\widehat{\mathfrak{g}})$ (Conjecture 1) which is generated only by Es and Fs and contains just four kinds of relations.

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2. Main definitions and results

2.1. **Dynkin data.** We work over a base field $\Bbbk$ of characteristic zero. Fix a simply-laced Dynkin diagram of finite type and denote its vertex set by I and the associated Lie algebra by $\mathfrak{g}$. We denote the weight lattice of $\mathfrak{g}$ by X and the root lattice of $\mathfrak{g}$ by Y. Thus Y is a sublattice of X. We equip X with the standard pairing $\langle \cdot, \cdot \rangle$. For $i \in I$, $\alpha_i \in Y$ and $\Lambda_i \in X$ will denote the simple roots and fundamental weights. We will often write i instead of α_i, especially in the pairing; thus for $\lambda \in X$ we will write $\langle \lambda, i \rangle$ instead of $\langle \lambda, \alpha_i \rangle$.

The pairing of simple roots satisfies $\langle i, j \rangle = C_{i,j}$, where $C_{i,j}$ is the Cartan matrix of $\mathfrak{g}$. In particular, we have:

- $\langle i, i \rangle = 2$ for all $i \in I$,
- $\langle i, j \rangle = -1$ when $i \neq j \in I$ are joined by an edge,
- $\langle i, j \rangle = 0$ when $i \neq j \in I$ are not joined by an edge, and
- $\langle \Lambda_i, j \rangle = \delta_{i,j}$ for all $i, j \in I$.

Let $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \Bbbk[t, t^{-1}] \oplus \Bbbk c$ be the affine Lie algebra associated to $\mathfrak{g}$, and denote by $\widehat{X}$ the affine weight lattice. The affine root lattice is denoted $\widehat{Y}$. Note that $\widehat{Y} = Y \oplus \mathbb{Z}\delta$ where $\langle i, \delta \rangle = 0$, $\langle \Lambda_i, \delta \rangle = 1$ for all $i \in I$, and $\langle \delta, \delta \rangle = 0$.

2.2. **Graded vector spaces.** Consider a $\mathbb{Z}$-graded finite dimensional vector space $V = \oplus_i V(i)$. One can associate to V the polynomial $f_V := \sum_i q^i \dim V(i)$. This gives a bijection between isomorphism classes of finite dimensional graded vector spaces and elements $f \in \mathbb{N}[q, q^{-1}]$.

From V one can construct the associated $\mathbb{Z}$-graded vector spaces $\mathrm{Sym}^n(V)$ and $\Lambda^n(V)$. If V has graded dimension $f \in \mathbb{N}[q, q^{-1}]$ then we denote by $\mathrm{Sym}^n(f)$ and $\Lambda^n(f)$ the graded dimensions of the $\mathbb{Z}$-graded vector spaces $\mathrm{Sym}^n(V)$ and $\Lambda^n(V)$. For example, if $f = q + q^{-1}$ then $\mathrm{Sym}^n(f) = q^n + q^{n-2} + \cdots + q^{-n+2} + q^{-n}$ which is just the quantum integer $[n+1]$. On the other hand, $\Lambda^n(f)$ is $1, [2], 1$ if $n = 0, 1, 2$, and is zero otherwise.

2.3. **The Drinfeld realization.** We begin with the Drinfeld realization of the quantum affine algebra as defined in [N1, Sec. 1.2]. It has generators $e_{i,r}, f_{i,r}$ ($i \in I$, $r \in \mathbb{Z}$), q^h ($h \in X^*$) and $h_{i,m}$, $q^{\pm d}$, $q^{\pm c/2}$ where $i \in I$ and $m \in \mathbb{Z} \setminus \{0\}$. The set of relations are as follows:

(1) $q^{\pm c/2}$ is central
(2) $q^0 = 1$, $q^h q^{h'} = q^{h+h'}$, $[q^h, h_{i,m}] = 0$, $q^d q^{-d} = 1$, $q^{c/2} q^{-c/2} = 1$
(3) $\psi_i^\pm(z)\psi_j^\pm(w) = \psi_j^\pm(w)\psi_i^\pm(z)$
(4) $\psi_i^-(z)\psi_j^+(w) = \frac{(z-q^{-\langle i,j\rangle}q^c w)(z-q^{\langle i,j\rangle}q^{-c}w)}{(z-q^{\langle i,j\rangle}q^c w)(z-q^{-\langle i,j\rangle}q^{-c}w)}\psi_j^+(w)\psi_i^-(z)$
(5) $[q^d, q^h] = 0$, $q^d h_{i,m} q^{-d} = q^m h_{i,m}$ and $q^d e_{i,r} q^{-d} = q^r e_{i,r}$, $q^d f_{i,r} q^{-d} = q^r f_{i,r}$
(6) $(q^{\pm sc/2}z - q^{\pm\langle i,j\rangle}w)\psi_j^s(z)x_i^\pm(w) = (q^{\pm\langle i,j\rangle}q^{\pm sc/2}z - w)x_i^\pm(w)\psi_j^s(z)$ where $s = \pm$
(7) $[x_i^+(z), x_j^-(w)] = \delta_{ij}\frac{1}{q-q^{-1}}\big(\delta(q^c w z^{-1})\psi_i^+(q^{\frac{c}{2}}w) - \delta(q^c z w^{-1})\psi_i^-(q^{\frac{c}{2}}z)\big)$
(8) $(z - q^{\pm 2}w)x_i^\pm(z)x_i^\pm(w) = (q^{\pm 2}z - w)x_i^\pm(w)x_i^\pm(z)$
(9) $(z - q^{\mp 1}w)x_i^\pm(z)x_j^\pm(w) = x_j^\pm(w)x_i^\pm(z)(q^{\mp 1}z - w)$ if $\langle i,j\rangle = -1$
(10) if $\langle i,j\rangle \le 0$ then

$$\sum_{\sigma \in S_N}\sum_{s=0}^{N=1-\langle i,j\rangle}(-1)^s \begin{bmatrix} N \\ s \end{bmatrix} x_i^\pm(z_{\sigma(1)})\dots x_i^\pm(z_{\sigma(s)})x_j^\pm(w)x_i^\pm(z_{\sigma(s+1)})\dots x_i^\pm(z_{\sigma(N)}) = 0.$$

In the above relations we have

$$\begin{aligned} x_i^+(z) &= \sum_{n\in\mathbb{Z}} e_{i,n}z^{-n} \qquad x_i^-(z) = \sum_{n\in\mathbb{Z}} f_{i,n}z^{-n} \\ \psi_i^\pm(z) &= \sum_{n\ge 0}\psi_i^\pm(\pm n)z^{\mp n} = q^{\pm h_i}\exp\big(\pm(q-q^{-1})\sum_{n>0}h_{i,\pm n}z^{\mp n}\big) \\ \delta(z) &= \sum_{n\in\mathbb{Z}} z^n \end{aligned}$$

Notice that we only deal with the case when the affine Lie algebra is simply laced.

2.4. **The vertex realization.** We now rewrite Drinfeld's realization in terms of halves of vertex operators. This means that instead of h's we will use Ps and Qs defined as homogeneous components in z of generating functions:

$$\sum_{n\ge 0} Q_i^{(n)} z^n := \exp\left(\sum_{n\ge 1}\frac{h_{i,n}}{[n]}z^n\right) \text{ and } \sum_{n\ge 0}(-1)^n Q_i^{(1^n)} z^n := \exp\left(-\sum_{n\ge 1}\frac{h_{i,n}}{[n]}z^n\right)$$

$$\sum_{n\ge 0} P_i^{(n)} z^n := \exp\left(\sum_{n\ge 1}\frac{h_{i,-n}}{[n]}z^n\right) \text{ and } \sum_{n\ge 0}(-1)^n P_i^{(1^n)} z^n := \exp\left(-\sum_{n\ge 1}\frac{h_{i,-n}}{[n]}z^n\right).$$

We will also define

$$E_{i,r} := (q^{-c/2})^r e_{i,r} \text{ and } F_{i,r} := (q^{-c/2})^r f_{i,r}. \tag{1}$$

For convenience we also define

$$Q_i^{[1^n]} := \sum_{m=0}^n (-q)^m [m] Q_i^{(1^{n-m})} Q_i^{(m)} \quad \text{and} \quad Q_i^{[n]} := \sum_{m=0}^n (-q)^m [m] Q_i^{(n-m)} Q_i^{(1^m)}$$

$$P_i^{[1^n]} := \sum_{m=0}^n (-q)^{-m} [m] P_i^{(1^{n-m})} P_i^{(m)} \quad \text{and} \quad P_i^{[n]} := \sum_{m=0}^n (-q)^{-m} [m] P_i^{(n-m)} P_i^{(1^m)}$$

Note that $P_i^{[1]} = -q^{-1}P_i$ and $Q_i^{[1]} = -qQ_i$.

Thus, we have generators $E_{i,r}, F_{i,r}, q^h$ $(h \in X^*)$, $q^{\pm d}$, $q^{\pm c/2}$,$P_i^{(n)}, P_i^{(1^n)}$ $Q_i^{(n)}$ and $Q_i^{(1^n)}$ where $i \in I, r \in \mathbb{Z}$ and $n \in \mathbb{N}$. The set of relations between these generators are now given by

(1) $q^{\pm c/2}$ is central
(2) $q^0 = 1$, $q^h q^{h'} = q^{h+h'}$, $[q^h, P^{(n)}] = 0 = [q^h, Q^{(n)}]$, $q^d q^{-d} = 1$, $q^{c/2} q^{-c/2} = 1$
(3) The generators $\{P_i^{(n)}, P_i^{(1^n)}\}_{i \in I}$ commute among each other and likewise $\{Q_i^{(n)}, Q_i^{(1^n)}\}_{i \in I}$ commute among each other.
(4) We have

$$Q_j^{(n)} P_i^{(m)} = \begin{cases} \sum_{k \geq 0} \mathrm{Sym}^k([2][c]) P_i^{(m-k)} Q_i^{(n-k)} \text{ if } i = j \\ \sum_{k \geq 0} (-1)^k \Lambda^k([c]) P_i^{(m-k)} Q_j^{(n-k)} \text{ if } \langle i,j \rangle = -1 \\ P_i^{(m)} Q_j^{(n)} \text{ if } \langle i,j \rangle = 0 \end{cases} \tag{2}$$

$$Q_j^{(1^n)} P_i^{(m)} = \begin{cases} \sum_{k \geq 0} \Lambda^k([2][c]) P_i^{(m-k)} Q_i^{(1^{n-k})} \text{ if } i = j \\ \sum_{k \geq 0} (-1)^k \mathrm{Sym}^k([c]) P_i^{(m-k)} Q_j^{(1^{n-k})} \text{ if } \langle i,j \rangle = -1 \\ P_i^{(m)} Q_j^{(1^n)} \text{ if } \langle i,j \rangle = 0 \end{cases} \tag{3}$$

and likewise if you exchange (a) and (1^a) everywhere.
(5) $[q^d, q^h] = 0$, $q^d Q_i^{(m)} q^{-d} = q^m Q_i^{(m)}$ $q^d P_i^{(m)} q^{-d} = q^{-m} P_i^{(m)}$ and $q^d(E_{i,r}) q^{-d} = q^r(E_{i,r})$, $q^d(F_{i,r}) q^{-d} = q^r(F_{i,r})$
(6) We have

$$\begin{aligned} q^c[Q_i^{[1^{a+1}]}, E_{i,b}] &= \begin{cases} q^2 Q_i^{[1^a]} E_{i,b+1} - q^{-2} E_{i,b+1} Q_i^{[1^a]} \text{ if } a > 0 \\ [2] E_{i,b+1} \text{ if } a = 0. \end{cases} \\ [Q_i^{[1^{a+1}]}, F_{i,b}] &= \begin{cases} q^{-2} Q_i^{[1^a]} F_{i,b+1} - q^2 F_{i,b+1} Q_i^{[1^a]} \text{ if } a > 0 \\ -[2] F_{i,b+1} \text{ if } a = 0. \end{cases} \\ [P_i^{[1^{a+1}]}, E_{i,b+1}] &= \begin{cases} q^2 E_{i,b} P_i^{[1^a]} - q^{-2} P_i^{[1^a]} E_{i,b} \text{ if } a > 0 \\ [2] E_{i,b} \text{ if } a = 0 \end{cases} \\ q^{-c}[P_i^{[1^{a+1}]}, F_{i,b+1}] &= \begin{cases} q^{-2} F_{i,b} P_i^{[1^a]} - q^2 P_i^{[1^a]} F_{i,b} \text{ if } a > 0 \\ -[2] F_{i,b} \text{ if } a = 0. \end{cases} \end{aligned}$$

while if $\langle i,j \rangle = -1$ we have

$$\begin{aligned} q^c[Q_j^{[1^{a+1}]}, E_{i,b}] &= \begin{cases} -q E_{i,b+1} Q_j^{[1^a]} + q^{-1} Q_j^{[1^a]} E_{i,b+1} \text{ if } a > 0 \\ -E_{i,b+1} \text{ if } a = 0. \end{cases} \\ [Q_j^{[1^{a+1}]}, F_{i,b}] &= \begin{cases} -q^{-1} F_{i,b+1} Q_j^{[1^a]} + q Q_j^{[1^a]} F_{i,b+1} \text{ if } a > 0 \\ F_{i,b+1} \text{ if } a = 0 \end{cases} \\ [P_j^{[1^{a+1}]}, E_{i,b+1}] &= \begin{cases} -q^{-1} E_{i,b} P_j^{[1^a]} + q P_j^{[1^a]} E_{i,b} \text{ if } a > 0 \\ -E_{i,b} \text{ if } a = 0 \end{cases} \\ q^{-c}[P_j^{[1^{a+1}]}, F_{i,b+1}] &= \begin{cases} -q F_{i,b} P_j^{[1^a]} + q^{-1} P_j^{[1^a]} F_{i,b} \text{ if } a > 0 \\ F_{i,b} \text{ if } a = 0; \end{cases} \end{aligned}$$

if $\langle i,j \rangle = 0$ we have that $P_j^{[1^a]}$ and $Q_j^{[1^a]}$ both commute with $E_{i,b}$ and $F_{i,b}$.

(7) We have

$$[E_{i,a}, F_{i,b}] = \begin{cases} q^{ac} q^{h_i} Q_i^{[1^{a+b}]} & \text{if } a+b > 0 \\ q^{bc} q^{-h_i} P_i^{[1^{-a-b}]} & \text{if } a+b < 0 \\ \frac{q^{ac} q^{h_i} - q^{-ac} q^{-h_i}}{q-q^{-1}} & \text{if } a+b = 0. \end{cases}$$

while if $i \neq j$ then $[E_{i,a}, F_{j,b}] = 0$.

(8) For any $m, n \in \mathbb{Z}$ we have

$$\begin{aligned} E_{i,m}E_{i,n-1} + E_{i,n}E_{i,m-1} &= q^2 \left(E_{i,m-1}E_{i,n} + E_{i,n-1}E_{i,m}\right) \\ F_{i,n-1}F_{i,m} + F_{i,m-1}F_{i,n} &= q^2 \left(F_{i,n}F_{i,m-1} + F_{i,m}F_{i,n-1}\right). \end{aligned}$$

(9) For any $m, n \in \mathbb{Z}$, if $\langle i, j \rangle = -1$ we have

$$\begin{aligned} E_{i,m}E_{j,n-1} + E_{j,n}E_{i,m-1} &= q^{-1} \left(E_{j,n-1}E_{i,m} + E_{i,m-1}E_{j,n}\right) \\ F_{i,m-1}F_{j,n} + F_{j,n-1}F_{i,m-1} &= q^{-1} \left(F_{j,n}F_{i,m-1} + F_{i,m}F_{j,n-1}\right) \end{aligned}$$

while if $\langle i, j \rangle = 0$ then

$$E_{i,m}E_{j,n} = E_{j,n}E_{i,m} \text{ and } F_{i,m}F_{j,n} = F_{j,n}F_{i,m}.$$

(10) If $\langle i, j \rangle = -1$ then

$$\sum_{\sigma \in S_2} \left(E_{j,n}E_{i,m_{\sigma(1)}}E_{i,m_{\sigma(2)}} + E_{i,m_{\sigma(1)}}E_{i,m_{\sigma(2)}}E_{j,n}\right) = \sum_{\sigma \in S_2} [2] E_{i,m_{\sigma(1)}}E_{j,n}E_{i,m_{\sigma(2)}}$$

and similarly if we replace all Es by Fs.

Proposition 1. *The Drinfeld and vertex realizations of quantum affine algebras are equivalent.*

Proof. All but one of the relations in the vertex realization are obtained directly from the Drinfeld realization by writing out the condition. The only exception is condition (4) involving the commutation of Ps and Qs. The fact that it is equivalent to condition (4) in the Drinfeld realization was checked in [CL1] when $c = 1$ (i.e. in the level one case). The same proof extends without complications to an arbitrary $c \in \mathbb{N}$ using the relation

$$[h_{i,m}, h_{j,n}] = \delta_{m,-n}[\langle i, j \rangle n] \frac{[nc]}{n}.$$

□

2.5. **The idempotent realization.** Any representation $V = \oplus_{\lambda \in \widehat{X}} V(\lambda)$ of $U_q(\widehat{\mathfrak{g}})$ with a weight space decomposition has a natural collection of idempotent endomorphisms, namely, for each $\lambda \in \widehat{X}$ there is the endomorphism given by projection onto the weight space $V(\lambda)$. It is therefore natural to consider an idempotent modification $\dot{U}_q(\widehat{\mathfrak{g}})$ of the quantum affine algebra, with the unit replaced by this collection of idempotent endomorphisms, one for each weight $\lambda \in \widehat{X}$. Then giving a representation of $\dot{U}_q(\widehat{\mathfrak{g}})$ will be equivalent to giving a representation of $U_q(\widehat{\mathfrak{g}})$ together with a weight space decomposition. This point of view is used frequently in the literature on Kac-Moody categorification, since, so far at least, all categorified representations have a weight space decomposition.

For any $\lambda \in \widehat{X}$ denote by 1_λ the idempotent which projects onto this weight space λ. We also fix c to be a positive integer. We define the algebra $\dot{U}_q(\widehat{\mathfrak{g}})$ via generators and relations as follows.

The generators are

$$E_{i,r}1_\lambda, F_{i,r}1_\lambda, Q_i^{(n)}1_\lambda, P_i^{(n)}1_\lambda, Q_i^{(1^n)}1_\lambda, P_i^{(1^n)}1_\lambda, \text{ where } i \in I \text{ and } r, k \in \mathbb{Z}.$$

Note that we no longer have generators q^h, $q^{\pm d}$ or $q^{\pm c/2}$.

(1) This condition is redundant

(2) $\{1_\lambda : \lambda \in \widehat{X}\}$ are mutually orthogonal idempotents, moreover

$$E_{i,r}1_\lambda = 1_\mu E_{i,r}1_\lambda = 1_\mu E_{i,r}$$
$$F_{i,r}1_\mu = 1_\lambda E_{i,r}1_\mu = 1_\lambda E_{i,r}$$

where $\mu = \lambda + \alpha_i + rc\delta$

(3) Same as the corresponding vertex realization relation.

(4) Same as the corresponding vertex realization relations (equations (2) and (3)).

(5) We have

$$P_i^{(n)}1_\lambda = 1_\mu P_i^{(n)}1_\lambda = 1_\mu P_i^{(n)} \quad \text{and} \quad P_i^{(1^n)}1_\lambda = 1_\mu P_i^{(1^n)}1_\lambda = 1_\mu P_i^{(1^n)}$$
$$Q_i^{(n)}1_\mu = 1_\lambda Q_i^{(n)}1_\mu = 1_\lambda Q_i^{(n)} \quad \text{and} \quad Q_i^{(1^n)}1_\mu = 1_\lambda Q_i^{(1^n)}1_\mu = 1_\lambda Q_i^{(1^n)}$$

where $\mu = \lambda + nc\delta$.

(6) Same as the corresponding vertex realization relation.

(7) We have

$$[E_{i,a}, F_{i,b}]1_\lambda = \begin{cases} q^{ac}q^{\langle\lambda,i\rangle}Q_i^{[1^{a+b}]}1_\lambda \text{ if } a+b>0 \\ q^{bc}q^{-\langle\lambda,i\rangle}P_i^{[1^{-a-b}]}1_\lambda \text{ if } a+b<0 \\ [\langle\lambda,i\rangle + ac]1_\lambda \text{ if } a+b=0 \end{cases}$$

while if $i \neq j$ then $[E_{i,a}, F_{j,b}]1_\lambda = 0$.

(8) Same as the corresponding vertex realization relation.

(9) Same as the corresponding vertex realization relation.

(10) Same as the corresponding vertex realization relation.

Now suppose that V is a representation of $\widehat{\mathfrak{g}}$ with weight space decomposition $V = \oplus_{\lambda \in \widehat{X}} V(\lambda)$. We say that V is an **integrable representation** if for any $\lambda \in \widehat{X}$ and root $\alpha \in \widehat{Y}$ the weight space $V(\lambda + n\alpha)$ is zero for $n \gg 0$ and $n \ll 0$. In an integrable highest weight representation of $\widehat{\mathfrak{g}}$, the central element c will act as $\langle\lambda,\delta\rangle$id, where λ is the highest weight. The integer $\langle\lambda,\delta\rangle$ is called the **level** of the representation.

2.6. **Redundancy in relations.** Many of the relations in the presentation of $\dot{U}_q(\widehat{\mathfrak{g}})$ above turn out to be redundant. We now summarize a smaller set of relations.

The generators are the same as those of the previous section. However, it suffices to consider the following smaller set of relations.

(1) $\{1_\lambda : \lambda \in \widehat{X}\}$ are mutually orthogonal idempotents with

$$E_{i,r}1_\lambda = 1_\mu E_{i,r}1_\lambda = 1_\mu E_{i,r}$$
$$F_{i,r}1_\mu = 1_\lambda E_{i,r}1_\mu = 1_\lambda E_{i,r}$$

where $\mu = \lambda + \alpha_i + rc\delta$

(2) The Ps and Qs satisfy the same relations as before (conditions (4) and (5) above).

(3) We have

$$[E_{i,a}, F_{i,b}]1_\lambda = \begin{cases} q^{ac}q^{\langle\lambda,i\rangle}Q_i^{[1^{a+b}]}1_\lambda \text{ if } a+b>0 \\ q^{bc}q^{-\langle\lambda,i\rangle}P_i^{[1^{-a-b}]}1_\lambda \text{ if } a+b<0 \\ [\langle\lambda,i\rangle + ac]1_\lambda \text{ if } a+b=0 \end{cases}$$

while if $i \neq j$ then $[E_{i,a}, F_{j,b}]1_\lambda = 0$.

(4) For any $m, n \in \mathbb{Z}$ we have $E_{i,n}E_{i,n-1}1_\lambda = q^2 E_{i,n-1}E_{i,n}1_\lambda$ and $F_{i,n-1}F_{i,n}1_\lambda = q^2 F_{i,n}F_{i,n-1}1_\lambda$.

(5) For any $m, n \in \mathbb{Z}$ we have

$$E_{i,1}E_j 1_\lambda + E_{j,1}E_i 1_\lambda = q^{-1}\left(E_j E_{i,1}1_\lambda + E_i E_{j,1}1_\lambda\right) \quad \text{if} \quad \langle i,j\rangle = -1$$
$$E_{i,m}E_{j,n}1_\lambda = E_{j,n}E_{i,m}1_\lambda \quad \text{if} \quad \langle i,j\rangle = 0$$

and similarly

$$F_{i,-1}F_j 1_\lambda + F_{j,-1}F_i 1_\lambda = q^{-1}\left(F_j F_{i,-1}1_\lambda + F_i F_{j,-1}1_\lambda\right) \quad \text{if} \quad \langle i,j\rangle = -1$$
$$F_{i,m}F_{j,n}1_\lambda = F_{j,n}F_{i,m}1_\lambda \quad \text{if} \quad \langle i,j\rangle = 0.$$

(6) If $\langle i,j\rangle = -1$ then

$$(E_{j,n})(E_{i,m})^2 1_\lambda + (E_{i,m})^2(E_{j,n})1_\lambda = [2](E_{i,m})(E_{j,n})(E_{i,m})1_\lambda.$$

and similarly if we replace all Es by Fs.

Theorem 2.1. *The relations above imply all of the relations in Drinfeld's realization. Moreover, when acting on an integrable representation, condition (6) is not necessary, since it follows formally from the other relations.*

3. Proof of theorem 2.1

We need to show that the relatons in $\dot{U}_q(\widehat{\mathfrak{g}})$ follow from the relations in Section 2.6. Conditions (1), (2), (3) and (5) are easy to check. We will verify the rest of the relations. For simplicity we will often omit the projectors 1_λ and write q^{h_i} instead of $q^{\langle\lambda,i\rangle}1_\lambda$.

3.1. **Proof of (7).** If $i \neq j$ then this condition states that $[x_i^+(z), x_j^-(w)] = 0$ which means that $E_{i,a}$ and $F_{j,b}$ commute (as claimed).

We now deal with the case $i = j$. To simplify notation we drop the i subscripts everywhere. So $h_{i,n}$ becomes h_n, $\psi_i^\pm(z)$ is just $\psi^\pm(z)$ and so on. Condition (5) then becomes

$$[x^+(z), x^-(w)] = \frac{1}{q-q^{-1}}\left(\delta(q^c w z^{-1})\psi^+(q^{\frac{c}{2}}w) - \delta(q^c z w^{-1})\psi^-(q^{\frac{c}{2}}z)\right) \tag{4}$$

where

$$\psi^+(z) = \sum_{n\geq 0}\psi^+(n)z^{-n} = q^h \exp\left((q-q^{-1})\sum_{n\geq 1}h_n z^{-n}\right)$$
$$\psi^-(z) = \sum_{n\geq 0}\psi^-(-n)z^{n} = q^{-h} \exp\left(-(q-q^{-1})\sum_{n\geq 1}h_{-n} z^{n}\right).$$

Now, the coefficient of $z^{-a}w^{-b}$ of left side of equation (4) is $(-q)^{-a-b}q^{-(a+b)c/2}[E_a, F_b]$ (recall that we have rescaled using equation (1)). On the other hand, the coefficient of $z^{-a}w^{-b}$ on the right side of (4) is

$$\frac{1}{q-q^{-1}}\left(q^{ac}(q^{\frac{c}{2}})^{-a-b}\psi^+(a+b) - q^{bc}(q^{\frac{c}{2}})^{-a-b}\psi^-(a+b)\right).$$

Notice that if $a+b<0$ then the first term above vanishes while if $a+b>0$ then the second term vanishes.

Case $a+b=0$**.** As expected, we end up with

$$[E_a, F_{-a}] = \frac{1}{q-q^{-1}}\left(q^{ac}q^h - q^{-ac}q^{-h}\right).$$

Case $a+b>0$**.** Expanding out everything we get

$$[E_a, F_b] = \frac{(-q)^{a+b}}{q-q^{-1}}q^{ac}\psi^+(a+b).$$

Now consider

$$\begin{aligned}\phi^+(z) &:= \sum_{n\geq 0} Q^{(n)} z^n = \exp\left(\sum_{n\geq 1} \frac{h_n}{[n]} z^n\right)\\ \phi^-(z) &:= \sum_{n\geq 0} (-1)^n Q^{(1^n)} z^n = \exp\left(-\sum_{n\geq 1} \frac{h_n}{[n]} z^n\right)\end{aligned}$$

First notice that

$$q^h \phi^+(qz^{-1})\phi^-(q^{-1}z^{-1}) = q^h \exp\left(\sum_{n\geq 1} \frac{h_n}{[n]}(q^n - q^{-n})z^{-n}\right) = \psi^+(z).$$

On the other hand, we also have

$$\begin{aligned}\phi^+(qz^{-1}) - \phi^+(q^{-1}z^{-1}) &= \sum_{n\geq 0}(q^n - q^{-n})Q^{(n)}z^{-n}\\ &= (q-q^{-1})\sum_{n\geq 0}[n]Q^{(n)}z^{-n}.\end{aligned}$$

If we multiply this by $\phi^-(q^{-1}z^{-1})$ and use that $\phi^+\phi^- = 1$ we get

$$\begin{aligned}\phi^+(qz^{-1})\phi^-(q^{-1}z^{-1}) - 1 &= (q-q^{-1})\sum_{n\geq 0}\sum_{m=0}^{n}[m]Q^{(m)}(-q)^{m-n}Q^{(1^{n-m})}z^{-n}\\ &= (q-q^{-1})\sum_{n\geq 0}(-q)^{-n}Q^{[1^n]}z^{-n}\end{aligned}$$

which means that if $a+b>0$ we have

$$\frac{(-q)^{a+b}}{q-q^{-1}}\psi^+(a+b) = q^h Q^{[1^{a+b}]}. \tag{5}$$

Thus we get that

$$[E_a, F_b] = q^{ac} q^h Q^{[1^{a+b}]}.$$

Case $a+b<0$. Expanding out we get

$$[E_a, F_b] = -\frac{(-q)^{a+b}}{q-q^{-1}} q^{-ac}\psi^-(a+b).$$

Now consider

$$\begin{aligned}\phi^+(z) &:= \sum_{n\geq 0} P^{(n)} z^n = \exp\left(\sum_{n\geq 1} \frac{h_{-n}}{[n]} z^n\right)\\ \phi^-(z) &:= \sum_{n\geq 0} (-1)^n P^{(1^n)} z^n = \exp\left(-\sum_{n\geq 1} \frac{h_{-n}}{[n]} z^n\right)\end{aligned}$$

First we have

$$q^{-h}\phi^+(q^{-1}z)\phi^-(qz) = q^{-h}\exp\left(\sum_{n\geq 1}\frac{h_{-n}}{[n]}(q^{-n}-q^n)z^n\right) = \psi^-(z).$$

On the other hand, we also have

$$\begin{aligned}\phi^+(q^{-1}z)-\phi^+(qz) &= \sum_{n\geq 0}(q^{-n}-q^n)P^{(n)}z^n\\ &= -(q-q^{-1})\sum_{n\geq 0}[n]P^{(n)}z^n.\end{aligned}$$

If we multiply this by $\phi^-(qz)$ and use that $\phi^+\phi^-=1$ we get

$$\begin{aligned}\phi^+(q^{-1}z)\phi^-(qz)-1 &= -(q-q^{-1})\sum_{n\geq 0}\sum_{m=0}^{n}[m]P^{(m)}(-q)^{n-m}P^{(1^{n-m})}z^n\\ &= -(q-q^{-1})\sum_{n\geq 0}(-q)^nP^{[1^n]}z^n\end{aligned}$$

which means that if $a+b<0$ we have

$$-\frac{(-q)^{a+b}}{q-q^{-1}}\psi^-(a+b)=q^{-h}P^{[1^{-a-b}]}. \tag{6}$$

Thus we get that

$$[E_a,F_b]=q^{bc}q^{-h}P^{[1^{-a-b}]}.$$

3.2. **Proof of (6).** If we write out relation (6) from section 2.3 when $s=+$ and $i=j$ we get two relations. In the first relation (where we take ψ_i^+) consider the coefficient of $z^{-a}w^{-b}$ to obtain

$$q^{c/2}\psi_i^+(a+1)e_{i,b}-q^2\psi_i^+(a)e_{i,b+1}=q^2q^{c/2}e_{i,b}\psi_i^+(a)-e_{i,b+1}\psi_i^+(a).$$

We show in the proof of relation (7) that if $\ell>0$ then

$$\psi_i^+(\ell)=(q-q^{-1})q^hQ_i^{[1^\ell]}$$

(see equation (5)) while $\psi_i^+(0)=q^h$. Thus, if $a>0$ then we get (after simplifying and using (1))

$$q^cQ_i^{[1^{a+1}]}E_{i,b}-q^2Q_i^{[1^a]}(E_{i,b+1})=q^cE_{i,b}Q_i^{[1^{a+1}]}-q^{-2}E_{i,b+1}Q_i^{[1^a]}$$

while if $a=0$ we get

$$q^c(q-q^{-1})Q_iE_{i,b}-q^2E_{i,b+1}=q^c(q-q^{-1})E_{i,b}Q_i-q^{-2}E_{i,b+1}.$$

Thus, we end up with

$$q^c[Q_i^{[1^{a+1}]},E_{i,b}]=\begin{cases}q^2Q_i^{[1^a]}(E_{i,b+1})-q^{-2}(E_{i,b+1})Q_i^{[1^a]} \text{ if } a>0\\ [2](E_{i,b+1}) \text{ if } a=0.\end{cases}$$

Now we will show that this is a consequence of the other relations. To simplify notation we will temporarily write E_m for $E_{i,m}$ and F_m for $F_{i,m}$ and Q instead of Q_i.

Case $a=0$**.** First we prove the case $a=0$ which says that $q^c[Q,E_b]=[2]E_{b+1}$. To do this we only use the two relations

$$[E_{b+1},F_{-b}]=q^{(b+1)c}q^hQ \text{ and } E_{b+1}E_b=q^2E_bE_{b+1}.$$

We have

$$\begin{aligned}
q^h q^c[Q, E_b] &= q^c(q^h Q E_b - q^2 E_b q^h Q) \\
&= q^c q^{-(b+1)c}\big(E_{b+1}F_{-b}E_b - F_{-b}E_{b+1}E_b - q^2 E_b E_{b+1} F_{-b} + q^2 E_b F_{-b} E_{b+1}\big) \\
&= q^{-bc}\big(E_{b+1}E_bF_{-b} - E_{b+1}[\ell + bc] - q^2 F_{-b}E_bE_{b+1} - E_{b+1}E_bF_{-b} \\
&\quad + q^2 F_{-b}E_bE_{b+1} + q^2 E_{b+1}[\ell + bc + 2]\big) \\
&= q^{-bc}\big(E_{b+1}(q^{\ell+bc+3} + q^{\ell+bc+1}\big) \\
&= q^h[2]E_{b+1}
\end{aligned}$$

where $\ell := \langle \lambda, \alpha_i \rangle$ and λ is the weight space on the far right (*i.e.* the domain). Cancelling the q^h completes the proof.

Case $a > 0$**.** Here we need to show that

$$q^c[Q^{[1^{a+1}]}, E_b] = q^2 Q^{[1^a]} E_{b+1} - q^{-2} E_{b+1} Q^{[1^a]}. \tag{7}$$

We begin by computing q^h times the left side of (7). We have

$$\begin{aligned}
q^h \cdot (LS) &= q^c(q^h Q^{[1^{a+1}]} E_b - q^2 E_b q^h Q^{[1^{a+1}]}) \\
&= q^c q^{-(a+b+1)c}\big([E_{a+b+1}, F_{-b}]E_b - q^2 E_b [E_{a+b+1}, F_{-b}]\big) \\
&= q^{-(a+b)c}\big(E_{a+b+1}E_bF_{-b} - E_{a+b+1}[\ell + bc] - F_{-b}E_{a+b+1}E_b \\
&\quad - q^2 E_b E_{a+b+1} F_{-b} + q^2 F_{-b} E_b E_{a+b+1} + q^2 E_{a+b+1}[\ell + 2 + bc]\big) \\
&= q^{-(a+b)c}\big((-E_{b+1}E_{a+b} + q^2 E_{a+b}E_{b+1})F_{-b} + F_{-b}(E_{b+1}E_{a+b} - q^2 E_{a+b}E_{b+1}) \\
&\quad + q^{\ell+2+bc}[2]E_{a+b+1}\big)
\end{aligned}$$

where ℓ is as above. Here we used that $[E_m, F_n] = q^{mc} q^h Q^{[1^{m+n}]}$ to obtain the second equality (where we take $m = a+b+1$ and $n = -b$), we use the standard relation for $[E_b, F_{-b}]$ to get the third equality, and we use the relation

$$E_m E_{n-1} + E_n E_{m-1} = q^2(E_{m-1}E_n + E_{n-1}E_m) \tag{8}$$

to conclude the last equality.

Remark 3.1. This relation appears as condition (8) in the vertex or idempotent realizations and is proved in the next subsection. The argument we employ is not circular because in that proof we only use the fact that $[P_i, E_{i,n}] = -[2](E_{i,n-1})$ which is the case $a = 0$ proved above. To deal with this case we only use that $E_{b+1}E_b = q^2 E_b E_{b+1}$ which is one of the relations included in the definition in section 2.6.

Similarly, we compute q^h times the right side of (7). We have

$$\begin{aligned}
q^h \cdot (RS) &= q^2 q^h Q^{[1^a]} E_{b+1} - E_{b+1} q^h Q^{[1^a]} \\
&= q^{-(a+b)c}\big[q^2(E_{a+b}F_{-b} - F_{-b}E_{a+b})E_{b+1} - E_{b+1}(E_{a+b}F_{-b} - F_{-b}E_{a+b})\big] \\
&= q^{-(a+b)c}\big[q^2(E_{a+b}E_{b+1}F_{-b} - E_{a+b}q^{(b+1)c}q^h Q - F_{-b}E_{a+b}E_{b+1}) \\
&\quad (-E_{b+1}E_{a+b}F_{-b} + F_{-b}E_{b+1}E_{a+b} + q^{(b+1)c}q^h Q E_{a+b})\big].
\end{aligned}$$

Now, using the case $a = 0$ relation proved above we get

$$-q^2 E_{a+b} q^h Q + q^h Q E_{a+b} = q^h [Q, E_{a+b}] = q^{-c} q^h [2] E_{a+b+1} = q^{\ell+2-c}[2]E_{a+b+1}.$$

Substituting we get that

$$\begin{aligned} q^h \cdot (RS) &= q^{-(a+b)c}\big[(q^2 E_{a+b}E_{b+1} - E_{b+1}E_{a+b})F_{-b} + F_{-b}(E_{b+1}E_{a+b} - q^2 E_{a+b}E_{b+1}) \\ &\quad + q^{\ell+2+bc}[2]E_{a+b+1}\big] \\ &= q^h \cdot (LS) \end{aligned}$$

and we are done.

There are three other cases to prove, namely:

$$\begin{aligned} q^{-c/2}\psi_i^+(a+1)f_{i,b} - q^{-2}\psi_i^+(a)f_{i,b+1} &= q^{-2}q^{-c/2}f_{i,b}\psi_i^+(a) - f_{i,b+1}\psi_i^+(a) \\ q^{-c/2}\psi_i^-(-a)e_{i,b} - q^2\psi_i^-(-a-1)e_{i,b+1} &= q^2 q^{-c/2}e_{i,b}\psi_i^-(-a) - e_{i,b+1}\psi_i^+(-a-1) \\ q^{c/2}\psi_i^-(-a)f_{i,b} - q^{-2}\psi_i^-(-a-1)f_{i,b+1} &= q^{-2}q^{c/2}f_{i,b}\psi_i^-(-a) - f_{i,b+1}\psi_i^-(-a-1). \end{aligned}$$

These follow in precisely the same way as the proof above.

3.2.1. *Case 2:* $\langle i, j\rangle = -1$. If we write out relation (6) when $s = +$ with ψ_j^+ and consider the coefficient of $[z^{-a}][w^{-b}]$ we get

$$q^{c/2}\psi_j^+(a+1)e_{i,b} - q^{-1}\psi_j^+(a)e_{i,b+1} = q^{-1}q^{c/2}e_{i,b}\psi_j^+(a+1) - e_{i,b+1}\psi_j^+(a).$$

Substituting and simplifying leads to

$$q^c[Q_j^{[1^{a+1}]}, E_{i,b}] = \begin{cases} -q(E_{i,b+1})Q_j^{[1^a]} + q^{-1}Q_j^{[1^a]}(E_{i,b+1}) \text{ if } a > 0 \\ -(E_{i,b+1}) \text{ if } a = 0. \end{cases}$$

Case $a = 0$**.** We need to show that $q^c[Q_j, E_{i,b}] = -E_{i,b+1}$. To do this we will use

$$[E_{j,b+1}, F_{j,-b}] = q^{bc}q^{h_j}Q_j \tag{9}$$

$$E_{i,b+1}E_{j,b} + E_{j,b+1}E_{i,b} = q^{-1}E_{j,b}E_{i,b+1} + q^{-1}E_{i,b}E_{j,b+1}. \tag{10}$$

and that E_i's and F_j's commute. We have

$$\begin{aligned} q^{h_j} \cdot (LS) &= q^c q^{-bc-c}\big(E_{j,b+1}F_{j,-b}E_{i,b} - F_{j,-b}E_{j,b+1}E_{i,b} \\ &\quad - q^{-1}E_{i,b}E_{j,b+1}F_{j,-b} + q^{-1}E_{i,b}F_{j,-b}E_{j,b+1}\big) \\ &= q^{-bc}\big(-E_{i,b+1}E_{j,b}F_{j,-b} + F_{j,-b}E_{i,b+1}E_{j,b} \\ &\quad + q^{-1}E_{j,b}E_{i,b+1}F_{j,-b} - q^{-1}F_{j,-b}E_{j,b}E_{i,b+1}\big) \\ &= q^{-bc}\big(-E_{i,b+1}[\ell_j + bc] + q^{-1}E_{i,b+1}[\ell_j - 1 + bc]\big) \\ &= -q^{-bc}E_{i,b+1}q^{\ell_j+bc-1} \\ &= -q^{h_j}E_{i,b+1} \\ &= q^{h_j} \cdot (RS) \end{aligned}$$

where $\ell_j := \langle \lambda, j\rangle$ and λ is the weight space on the far right (*i.e.* the domain). Here we used (9) to get the first equality and (10) to get the second.

Case $a > 0$**.** This proof is similar to the one in case 1 so we omit it.

There are also three other cases to consider, namely:

$$\begin{aligned} q^{-c/2}\psi_j^+(a+1)f_{i,b} - q\psi_j^+(a)f_{i,b+1} &= qq^{c/2}f_{i,b}\psi_j^+(a+1) - f_{i,b+1}\psi_j^+(a) \\ q^{-c/2}\psi_j^-(-a)e_{i,b} - q^{-1}\psi_j^-(-a-1)e_{i,b+1} &= q^{-1}q^{-c/2}e_{i,b}\psi_j^-(-a) - e_{i,b+1}\psi_j^-(-a-1) \\ q^{c/2}\psi_j^-(-a)f_{i,b} - q\psi_j^-(-a-1)f_{i,b+1} &= qq^{c/2}f_{i,b}\psi_j^-(-a) - f_{i,b+1}\psi_j^-(-a-1). \end{aligned}$$

These follow in the same way as the proof above.

3.2.2. *Case 3:* $\langle i,j\rangle = 0$. Relation (6) immediately simplifies to $\psi_j^s(z)x_i^{\pm}(w) = x_i^{\pm}(w)\psi_j^s(z)$. This implies that any $P_i^{[1^a]}$ or $Q_i^{[1^a]}$ commutes with any $E_{j,b}$ or $F_{j,b}$.

3.3. **Proof of (8).** Writing out the condition gives:

$$(E_{i,m})(E_{i,n-1}) + (E_{i,n})(E_{i,m-1}) - q^2\left[E_{i,m-1})(E_{i,n}) + (E_{i,n-1})(E_{i,m})\right] = 0. \tag{11}$$

Let us denote the left side of this equation by $f(m,n)$. Note that it is symmetric in that $f(m,n) = f(n,m)$. Now, from condition (3) we know that

$$[P_i, E_{i,n}] = -[2](E_{i,n-1}).$$

Multiplying (11) on the left by P_i and using this relation repeatedly we get

$$[P_i, f(m,n)] = -[2]\left[f(m-1,n) + f(m,n-1)\right].$$

Thus $f(m,n) = 0 \Rightarrow f(m-1,n) + f(m,n-1) = 0$. Applying Q_i instead of P_i likewise gives that $f(m,n) = 0 \Rightarrow f(m+1,n) + f(m,n+1) = 0$.

Thus, if you know that $f(n,n) = 0$ then $f(n-1,n) = 0$ since $f(n-1,n) = f(n,n-1)$ and then $f(n,n) + f(n-1,n+1) = 0$ which means $f(n-1,n+1) = 0$. Continuing in this way one finds that

$$f(n,n) = 0 \Rightarrow f(n+1,n) = 0 \Rightarrow f(n+1,n-1) = 0 \Rightarrow f(n+2,n-1) = 0 \Rightarrow f(n+2,n-2) = 0 \Rightarrow \dots$$

which means that $f(n,n) = 0 \Rightarrow f(n+k,n-k) = 0$ and $f(n+k+1,n-k) = 0$ for any $k \in \mathbb{Z}$. Thus it suffices to know that $f(n,n) = 0$ for all $n \in \mathbb{Z}$ which is condition (4) from 2.4.

3.4. **Proof of (9).** Now suppose $\langle i,j\rangle = -1$. Writing out condition (9) from section 2.3 gives

$$(E_{i,m})(E_{j,n-1}) + (E_{j,n})(E_{i,m-1}) - q^{-1}\left[(E_{j,n-1})(E_{i,m}) + (E_{i,m-1})(E_{j,n}\right] = 0.$$

Again, let us denote the left hand side $f(m,n)$. This time we will use that

$$[P_i, E_{i,m}] = [2](E_{i,m-1}) \text{ and } [P_i, E_{j,n}] = -E_{j,n-1}.$$

Now, multiplying $f(m,n)$ by P_i and using these relations gives

$$[P_i, f(m,n)] = [2]f(m-1,n) + q^{-1}f(m,n-1)$$

while multiplying by P_j gives

$$[P_j, f(m,n)] = -q^{-1}f(m-1,n) + [2]f(m,n-1).$$

Putting this together gives

$$f(m,n) = 0 \Rightarrow f(m-1,n) = 0 \text{ and } f(m,n-1) = 0.$$

Using Q_i and Q_j instead of P_i and P_j also gives us

$$f(m,n) = 0 \Rightarrow f(m+1,n) = 0 \text{ and } f(m,n+1) = 0.$$

From this it follows that we only need to know $f(m,n) = 0$ for one pair (m,n). Taking $(m,n) = (1,1)$ is precisely condition (9) from section 2.4.

Finally, if $\langle i,j\rangle = 0$ then writing out the condition in (9) from section 2.3 immediately simplifies to condition (9) in section 2.4.

3.5. **Proof of (10).** The Drinfeld condition is equivalent to

$$\sum_{\sigma \in S_2} \begin{array}{c}(E_{j,n})(E_{i,m_{\sigma(1)}})(E_{i,m_{\sigma(2)}})+ \\ (E_{i,m_{\sigma(1)}})(E_{i,m_{\sigma(2)}})(E_{j,n})\end{array} = \sum_{\sigma \in S_2} [2](E_{i,m_{\sigma(1)}})(E_{j,n})(E_{i,m_{\sigma(2)}}) \tag{12}$$

if $\langle i,j \rangle = -1$.

Notice that we have

$$\begin{aligned} & q^{-1}(E_{i,m_1})(E_{j,n})(E_{i,m_2}) \\ = \; & \left[(E_{i,m_1+1})(E_{j,n-1}) + (E_{j,n})(E_{i,m_1}) - q^{-1}(E_{j,n-1})(E_{i,m_1+1})\right](E_{i,m_2}) \end{aligned}$$

and likewise

$$\begin{aligned} & q(E_{i,m_1})(E_{j,n})(E_{i,m_2}) \\ = \; & (E_{i,m_1})\left[(E_{j,n-1})(E_{i,m_2+1}) + (E_{i,m_2})(E_{j,n}) - q(E_{i,m_2+1})(E_{j,n-1})\right]. \end{aligned}$$

Adding and symmetrizing with respect to m_1 and m_2 we get the Drinfeld cond ition above but only if we can show that

$$\begin{aligned} & \sum_{\sigma \in S_2} (E_{i,m_{\sigma(1)+1}})(E_{j,n-1})(E_{i,m_{\sigma(2)}}) + (E_{i,m_{\sigma(1)}})(E_{j,n-1})(E_{i,m_{\sigma(2)}+1}) \\ = \; & \sum_{\sigma \in S_2} q^{-1}(E_{j,n-1})(E_{i,m_{\sigma(1)}+1})(E_{i,m_{\sigma(2)}}) + q(E_{i,m_{\sigma(1)}})(E_{i,m_{\sigma(2)}+1})(E_{j,n-1}). \end{aligned}$$

Now multiply both sides of this equation by $q+q^{-1}$. If we collect the terms on the right side with a coefficient of q^2 or q^{-2} and use the appropriate commutator relation (11) (appropriate means that we should get rid of the q coefficients) then we find that the condition above is equivalent to condition (12) for the values $(n-1, m_1, m_2+1)$ and $(n-1, m_1+1, m_2)$.

Equivalently, this means that if $m_1 < m_2$ then (12) for (n, m_1, m_2) is implied by $(n+1, m_1-1, m_2)$ and (n, m_1-1, m_2+1). Thus repeating the argument we can reduce to the case when $m_1 = m_2 = m$. In this case (12) becomes

$$(E_{j,n})(E_{i,m})^2 + (E_{i,m})^2(E_{j,n}) = [2](E_{i,m})(E_{j,n})(E_{i,m}).$$

If $m = n = 0$ then this relation follows formally from the other relations (see for instance [N2]). But in the more general case, $E'_i := (E_{i,m})$ and $E'_j := (E_{j,n})$ also generate an $\mathfrak{sl}_3$ action (when combined with $F'_i := (F_{i,-m})$ and $F'_j := (F_{j,-n})$) so the argument used in the case $m = n = 0$ still applies.

4. Minimal realization

The presentation from Section 2.6 can probably be stripped down even further. For generators, consider the algebra generated by $E_{i,r}1_\lambda$ and $F_{i,r}1_\lambda$, where 1_λ are idempotents as before and where $i \in I$ and $r \in \mathbb{Z}$. The relations are as follows.

(1) $\{1_\lambda : \lambda \in \widehat{X}\}$ are mutually orthogonal idempotents with

$$\begin{aligned} E_{i,r}1_\lambda = 1_\mu E_{i,r} 1_\lambda = 1_\mu E_{i,r} \\ F_{i,r}1_\mu = 1_\lambda E_{i,r} 1_\mu = 1_\lambda E_{i,r} \end{aligned}$$

where $\mu = \lambda + \alpha_i + rc\delta$

(2) For $a, b, a', b' \in \mathbb{Z}$ we have

$$\begin{aligned} q^{-ac}[E_{i,a}, F_{i,b}]1_\lambda &= q^{-a'c}[E_{i,a'}, F_{i,b'}]1_\lambda \text{ if } a+b = a'+b' > 0 \\ q^{-bc}[E_{i,a}, F_{i,b}]1_\lambda &= q^{-b'c}[E_{i,a'}, F_{i,b'}]1_\lambda \text{ if } a+b = a'+b' < 0 \\ [E_{i,a}, F_{i,b}]1_\lambda &= [\langle \lambda, i \rangle + ac]1_\lambda \text{ if } a+b = 0. \end{aligned}$$

If $i \neq j$ then $[E_{i,a}, F_{j,b}]1_\lambda = 0$.

(3) For any $n \in \mathbb{Z}$ we have $E_{i,n}E_{i,n-1} = q^2 E_{i,n-1}E_{i,n}$ and $F_{i,n-1}F_{i,n} = q^2 F_{i,n}F_{i,n-1}$.
(4) For any $i \neq j \in I$ and $n \in \mathbb{Z}$ we have

$$\begin{aligned} E_{i,1}E_j 1_\lambda + E_{j,1}E_i 1_\lambda &= q^{-1}\left(E_j E_{i,1} 1_\lambda + E_i E_{j,1} 1_\lambda\right) \quad \text{if} \quad \langle i,j\rangle = -1 \\ E_i E_{j,n} 1_\lambda &= E_{j,n} E_i 1_\lambda \quad \text{if} \quad \langle i,j\rangle = 0 \end{aligned}$$

and similarly

$$\begin{aligned} F_{i,-1}F_j 1_\lambda + F_{j,-1}F_i 1_\lambda &= q^{-1}\left(F_j F_{i,-1} 1_\lambda + F_i F_{j,-1} 1_\lambda\right) \quad \text{if} \quad \langle i,j\rangle = -1 \\ F_i F_{j,n} 1_\lambda &= F_{j,n} F_i 1_\lambda \quad \text{if} \quad \langle i,j\rangle = 0. \end{aligned}$$

Conjecture 1. *In an integrable representation, all the Drinfeld relations are a consequence of the relations above.*

The most glaring omission above are the Ps and Qs. These can be defined using the Es and Fs as follows

$$P_i^{[1^{-a-b}]} 1_\lambda := q^{-ac} q^{-\langle \lambda, i\rangle} [E_{i,a}, F_{i,b}] 1_\lambda \text{ and } Q_i^{[1^{a+b}]} 1_\lambda := q^{-bc} q^{\langle \lambda, i\rangle} [E_{i,a}, F_{i,b}] 1_\lambda.$$

Lemma 1. *Let P_i and Q_i be defined by $-q^{-1}P_i = P_i^{[1]}$ and $-qQ_i = Q_i^{[1]}$. The relations between $E_{i,a}$ and $F_{i,b}$ imply that $[Q_i, P_i]1_\lambda = [2][c]1_\lambda$, $[Q_i, P_j]1_\lambda = -[c]1_\lambda$ if $\langle i,j\rangle = -1$, and $[Q_i, P_i]1_\lambda = 0$ otherwise.*

Proof. First we compute $Q_i P_i 1_\lambda$. To do this we use

$$P_i 1_\lambda = q^c q^{-\langle \lambda, i\rangle} (E_{i,-1} F_i 1_\lambda - F_i, E_{i,-1}) \text{ and } 1_\lambda Q_i = q^{-c} q^{\langle \lambda, i\rangle} (E_i F_{i,1} 1_\lambda - F_{i,1} E_i 1_\lambda).$$

Thus

$$Q_i P_i 1_\lambda = E_i F_{i,1} E_{i,-1} F_i 1_\lambda - E_i F_{i,1} F_i E_{i,-1} 1_\lambda - F_{i,1} E_i E_{i,-1} F_i 1_\lambda + F_{i,1} E_i F_i E_{i,-1} 1_\lambda.$$

Consider the first term. We have

$$\begin{aligned} & E_i F_{i,1} E_{i,-1} F_i 1_\lambda \\ = \quad & E_i E_{i,-1} F_{i,1} F_i 1_\lambda - [\langle \lambda - i, i\rangle - c] E_i F_i 1_\lambda \\ = \quad & E_{i,-1} E_i F_i F_{i,1} 1_\lambda - [\langle \lambda, i\rangle - 2 - c] E_i F_i 1_\lambda \\ = \quad & E_{i,-1} F_i E_i F_{i,1} 1_\lambda + [\langle \lambda - i, i\rangle] E_{i,-1} F_{i,1} 1_\lambda - [\langle \lambda, i\rangle - 2 - c] F_i E_i 1_\lambda - [\langle \lambda, i\rangle - 2 - c][\langle \lambda, i\rangle] 1_\lambda. \end{aligned}$$

Likewise, the other three terms give

$$\begin{aligned} & -q^{-2}\left(F_i E_i E_{i,-1} F_{i,1} 1_\lambda - F_i E_i 1_\lambda [\langle \lambda, i\rangle - c] + E_{i,-1} F_{i,1} 1_\lambda [\langle \lambda, i\rangle] - [\langle \lambda, i\rangle][\langle \lambda, i\rangle - c] 1_\lambda\right) \\ & -q^{2}\left(E_{i,-1} F_{i,1} F_i E_i 1_\lambda + E_{i-1} F_{i,1} 1_\lambda [\langle \lambda, i\rangle] - [\langle \lambda, i\rangle - c] F_i E_i 1_\lambda - [\langle \lambda, i\rangle - c][\langle \lambda, i\rangle] 1_\lambda\right) \\ & F_i E_{i,-1} F_{i,1} E_i 1_\lambda - F_i E_i 1_\lambda [\langle \lambda, i\rangle + 2 - c] + [\langle \lambda, i\rangle + 2] E_{i,-1} F_{i,1} 1_\lambda - [\langle \lambda, i\rangle + 2][\langle \lambda, i\rangle - c] 1_\lambda \end{aligned}$$

Now consider the coefficient of $F_i E_i 1_\lambda$ after summing up these four expressions. It equals

$$-[\langle \lambda, i\rangle - 2 - c] + q^{-2}[\langle \lambda, i\rangle - c] + q^2[\langle \lambda, i\rangle - c] - [\langle \lambda, i\rangle + 2 - c] = 0.$$

Likewise the coefficient of $E_{i,-1}F_{i,1}$ is zero. The coefficient of 1_λ is

$$-[\langle \lambda, i\rangle - 2 - c][\langle \lambda, i\rangle] + q^{-2}[\langle \lambda, i\rangle][\langle \lambda, i\rangle - c] + q^2[\langle \lambda, i\rangle][\langle \lambda, i\rangle - c] - [\langle \lambda, i\rangle + 2][\langle \lambda, i\rangle - c]$$

which simplifies to give $[2][c]$. The remaining terms are grouped together to give $P_i Q_i 1_\lambda$. Thus we end up with $Q_i P_i 1_\lambda = P_i Q_i 1_\lambda + [2][c] 1_\lambda$.

An analogous but slightly longer calculation, which we omit, can be used to show that $Q_i P_j 1_\lambda = P_j Q_i 1_\lambda - [c] 1_\lambda$ when $\langle i,j\rangle = -1$. The fact that $Q_i P_j 1_\lambda = P_j Q_i 1_\lambda$ when $\langle i,j\rangle = 0$ is immediate. □

In general, the commutator relations beween $P_i^{[1^n]}$ and $Q_j^{[1^n]}$ can be read off from the Drinfeld realization. They are defined recursively as follows.

- If $i = j$ and $a, b \geq 0$ then
$$\begin{aligned} & P_i^{[1^a]} Q_j^{[1^b]} - (qq^{-c} + q^{-1}q^c) P_i^{[1^{a+1}]} Q_j^{[1^{b+1}]} + P_i^{[1^{a+2}]} Q_j^{[1^{b+2}]} \\ = \quad & Q_j^{[1^b]} P_i^{[1^a]} - (q^{-1}q^{-c} + qq^c) Q_j^{[1^{b+1}]} P_i^{[1^{a+1}]} + Q_j^{[1^{b+2}]} P_i^{[1^{a+2}]} \end{aligned}$$
while $[Q_j^{[1^{b+1}]}, P_i^{[1]}] = -(q - q^{-1})[c] Q_j^{[1^b]}$ and $[Q_j^{[1]}, P_i^{[1^{b+1}]}] = (q - q^{-1})[c] P_i^{[1^b]}$ unless $b = 0$ in which case $[Q_j^{[1]}, P_i^{[1]}] = -[c]$.
- If $\langle i, j\rangle = -1$ and $a, b \geq 0$ then
$$\begin{aligned} & P_i^{[1^a]} Q_j^{[1^b]} - (qq^{-c} + q^{-1}q^c) P_i^{[1^{a+1}]} Q_j^{[1^{b+1}]} + P_i^{[1^{a+2}]} Q_j^{[1^{b+2}]} \\ = \quad & Q_j^{[1^b]} P_i^{[1^a]} - (q^{-1}q^{-c} + qq^c) Q_j^{[1^{b+1}]} P_i^{[1^{a+1}]} + Q_j^{[1^{b+2}]} P_i^{[1^{a+2}]} \end{aligned}$$
while $[Q_j^{[1^{b+1}]}, P_i^{[1]}] = -(q - q^{-1})[c] Q_j^{[1^b]}$ and $[Q_j^{[1]}, P_i^{[1^{b+1}]}] = (q - q^{-1})[c] P_i^{[1^b]}$ unless $b = 0$ in which case $[Q_j^{[1]}, P_i^{[1]}] = -[c]$.
- If $\langle i, j\rangle = 0$ then $P_i^{[1^a]} Q_j^{[1^b]} = Q_j^{[1^b]} P_i^{[1^a]}$.

One would expect that the proof in Lemma 1 extends to deduce these relations as a formal consequence of the commutator relations between Es and Fs. However, such a computation is much more complex than the one in Lemma 1 and we did not perform it.

5. Remarks

5.1. **Break of symmetry.** The algebra generated by Ps and Qs is known as the quantum Heisenberg algebra. It has a natural involution ψ defined by
$$P_i^{(1^n)} \mapsto (-1)^n P_i^{(n)} \text{ and } Q_i^{(1^n)} \mapsto (-1)^n Q_i^{(n)}.$$
In terms of the generators $h_{i,m}$ this involution corresponds to $h_{i,m} \mapsto -h_{i,m}$. In section 2.4 we used these Heisenberg generators to define two sets of algebra elements, $\{P_i^{[1^n]}, Q_i^{[1^n]}\}$ and $\{P_i^{[n]}, Q_i^{[n]}\}$. However, in subsequent presentations we used only $\{P_i^{[1^n]}, Q_i^{[1^n]}\}$ and never used $\{P_i^{[n]}, Q_i^{[n]}\}$. This apparent break in symmetry is explained by the definition
$$\psi_i^{\pm}(z) = \sum_{n \geq 0} \psi_i^{\pm}(\pm n) z^{\mp n} = q^{\pm h_i} \exp\big(\pm (q - q^{-1}) \sum_{n>0} h_{i,\pm n} z^{\mp n}\big)$$
from section 2.3. Changing $h_{i,m}$ to $-h_{i,m}$ in this definition would have the effect of exchanging $P_i^{[1^n]} \leftrightarrow P_i^{[n]}$ and $Q_i^{[1^n]} \leftrightarrow Q_i^{[n]}$, because $\psi(P_i^{[1^n]}) = (-1)^n P_i^{[n]}$ and $\psi(Q_i^{[1^n]}) = (-1)^n Q_i^{[n]}$.

5.2. **Divided powers.** To define the quantum affine algebra over $\mathbb{Z}[q, q^{-1}]$ one needs to include divided powers, as observed by Lusztig. These divided powers are defined as follows
$$E_{i,m}^{(r)} := \frac{E_{i,m}^r}{[r]!} \text{ and } F_{i,n}^{(r)} := \frac{F_{i,n}^r}{[r]!} \tag{13}$$
where $r \geq 0$ (by convention, $r = 0$ gives the identity). Ideally one would like to understand the relations involving these divided powers; these relations should be defined over $\mathbb{Z}[q, q^{-1}]$, and they should arise in geometric and categorical constructions. The lemma below gives some examples of relations involving divided powers and Ps and Qs.

Lemma 2. *The following identities hold:*
$$\begin{aligned} [Q_i, E_{i,b}^{(r)}] &= q^{r-1}[2] E_{i,b}^{(r-1)} E_{i,b+1} & \qquad [Q_i, F_{i,b}^{(r)}] &= -q^c q^{r-1}[2] F_{i,b+1} F_{i,b}^{(r-1)} \\ [P_i, E_{i,b+1}^{(r)}] &= q^{-c} q^{r-1}[2] E_{i,b} E_{i,b+1}^{(r-1)} & \qquad [P_i, F_{i,b+1}^{(r)}] &= -q^{r-1}[2] F_{i,b+1}^{(r-1)} F_{i,b} \end{aligned}$$

$$[Q_j, E^{(r)}_{i,b}] = -q^{r-1} E^{(r-1)}_{i,b} E_{i,b+1} \qquad [Q_j, F^{(r)}_{i,b}] = q^c q^{r-1} F_{i,b+1} F^{(r-1)}_{i,b}$$
$$[P_j, E^{(r)}_{i,b+1}] = -q^{-c} q^{r-1} E_{i,b} E^{(r-1)}_{i,b+1} \qquad [P_i, F^{(r)}_{i,b+1}] = q^{r-1} F^{(r-1)}_{i,b+1} F_{i,b}$$

Proof. We prove the first relation by induction on r (the others are proved in exactly the same way). The base case $r = 1$ is part of the definition. To prove the induction step we have

$$\begin{aligned} Q_i E^{(r)}_{i,b} E_{i,b} &= E^{(r)}_{i,b} Q_i E_{i,b} + q^{r-1}[2] E^{(r-1)}_{i,b} E_{i,b+1} E_{i,b} \\ &= E^{(r)}_{i,b} E_{i,b} Q_i + [2] E^{(r)}_{i,b} E_{i,b+1} + q^{r-1}[2] q^2 E^{(r-1)}_{i,b} E_{i,b} E_{i,b+1} \\ &= E^{(r)}_{i,b} E_{i,b} Q_i + E^{(r)}_{i,b} E_{i,b+1} [2] \left(1 + q^{r-1} q^2 [r]\right) \\ &= E^{(r)}_{i,b} E_{i,b} Q_i + E^{(r)}_{i,b} E_{i,b+1} [2][r+1] q^r. \end{aligned}$$

This means that $[r+1][Q_i, E^{(r+1)}_{i,b}] = q^r [2][r+1] E^{(r)}_{i,b} E_{i,b+1}$ which implies the induction step. $\square$

Let $U^{\mathbb{Z}}_q(\widehat{\mathfrak{g}})$ be the subalgebra of $U_q(\widehat{\mathfrak{g}})$ generated over $\mathbb{Z}[q, q^{-1}]$ by the divided powers $E^{(r)}_{i,m}$ and $F^{(r)}_{i,n}$ together with $Q^{[1^a]}$ and $P^{[1^a]}$. One expects that the algebra $U^{\mathbb{Z}}_q(\widehat{\mathfrak{g}})$ is free over $\mathbb{Z}[q, q^{-1}]$ and that the natural map $U^{\mathbb{Z}}_q(\widehat{\mathfrak{g}}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{k}(q) \to U_q(\widehat{\mathfrak{g}})$ should be an isomorphism. This observation appears in [CP] and also in the remark in section 1 of [N1].

5.3. **Renormalization.** In [CL2] we construct a categorical action of the quantum affine algebra using the Heisenberg algebra from [CL1]. However, the resulting decategorified relations do not match up identically with those in the above presentations. More precisely, they are off by a sign or some power of q in a few instances. We now renormalize the generators of the idempotent vertex realization $\dot{U}_q(\widehat{\mathfrak{g}})$ of the quantum affine algebra so that the resulting presentation matches up with that in [CL2]. So the point of this renormalized realization of $\dot{U}_q(\widehat{\mathfrak{g}})$ is that it occurs naturally in categorification.

To define the renormalization we need an asymmetric pairing $(\cdot,\cdot)$ on $\widehat{X}$. To define this pairing, fix an orientation of the original Dynkin diagram of $\mathfrak{g}$, so that an edge between i and j in the Dynkin diagram is now oriented $i \to j$ or $j \to i$, but not both. Then

$$(i,j) := \begin{cases} 1 \text{ if } i = j \\ -1 \text{ if } i \to j \\ 0 \text{ otherwise} \end{cases}$$

while $(i,\delta) = 0 = (\delta, i)$ and $(\delta,\delta) = 0$. Moreover, $(\Lambda_i, j) = \delta_{i,j}$ and $(j, \Lambda_i) = 0$ while $(\Lambda_i, \delta) = 1$ and $(\delta, \Lambda_i) = 0$. In particular, this means that if λ is a weight appearing in the representation of level c then $(\lambda,\delta) = c$ and $(\delta,\lambda) = 0$. It is easy to check using this definition that $\langle \lambda_1, \lambda_2 \rangle = (\lambda_1,\lambda_2) + (\lambda_2,\lambda_1)$ for any $\lambda_1 \in \widehat{X}$ and $\lambda_2 \in \widehat{Y}$.

We define the renormalization as follows. It takes $q \mapsto -q$ and

$$E_{i,m} 1_\lambda \mapsto (-1)^{(\lambda,i)} q^{mc} E_{i,m} 1_\lambda \quad 1_\lambda F_{i,m} \mapsto -(-1)^{(i,\lambda)} (-q)^{mc} 1_\lambda F_{i,m}$$
$$P^{(n)}_i 1_\lambda \mapsto -P^{(n)}_i 1_\lambda \;\; P^{(1^n)}_i 1_\lambda \mapsto -P^{(1^n)}_i 1_\lambda \;\; 1_\lambda Q^{(n)}_i \mapsto -(-1)^{cn} 1_\lambda Q^{(n)}_i \;\; 1_\lambda Q^{(1^n)}_i \mapsto -(-1)^{cn} 1_\lambda Q^{(1^n)}_i$$

where $c = (\lambda,\delta)$ is the level.

For example, the relation $q^c [Q_i, E_{i,b}] 1_\lambda = [2] E_{i,b+1} 1_\lambda$ becomes

$$(-q)^c \left((-1)^{c+1} Q_i (-1)^{(\lambda,i)} q^{bc} E_{i,b} - (-1)^{(\lambda+\delta,i)} q^{bc} E_{i,b} (-1)^{c+1} Q_i \right) 1_\lambda = -[2](-1)^{(\lambda,i)} q^{(b+1)c} E_{i,b+1} 1_\lambda$$

which simplifies to give $[Q_i, E_{i,b}] 1_\lambda = [2] E_{i,b+1} 1_\lambda$. The resulting relations turn out to eb independent of the choice of orientation of the Dynkin diagram. We summarize them below.

The renormalized (idempotent) vertex realization has generators

$$E_{i,r} 1_\lambda, F_{i,r} 1_\lambda, Q^{(n)}_i 1_\lambda, P^{(n)}_i 1_\lambda, Q^{(1^n)}_i 1_\lambda, P^{(1^n)}_i 1_\lambda, \text{ where } i \in I \text{ and } r, k \in \mathbb{Z}.$$

The $P_i^{[1^n]}, Q_i^{[1^n]}, P_i^{[n]}, Q_i^{[n]}$ are defined as before. The relations in this renormalized realization are then modified as follows.

(1) This condition is redundant.
(2) This condition is unchanged.
(3) This condition is unchanged.
(4) We have

$$Q_j^{(n)} P_i^{(m)} 1_\lambda = \begin{cases} \sum_{k\geq 0} \mathrm{Sym}^k([2][c]) P_i^{(m-k)} Q_i^{(n-k)} 1_\lambda \text{ if } i=j \\ \sum_{k\geq 0} \mathrm{Sym}^k([c]) P_i^{(m-k)} Q_j^{(n-k)} 1_\lambda \text{ if } \langle i,j\rangle = -1 \\ P_i^{(m)} Q_j^{(n)} 1_\lambda \text{ if } \langle i,j\rangle = 0 \end{cases} \tag{14}$$

$$Q_j^{(1^n)} P_i^{(m)} 1_\lambda = \begin{cases} \sum_{k\geq 0} \Lambda^k([2][c]) P_i^{(m-k)} Q_i^{(1^{n-k})} 1_\lambda \text{ if } i=j \\ \sum_{k\geq 0} \Lambda^k([c]) P_i^{(m-k)} Q_j^{(1^{n-k})} 1_\lambda \text{ if } \langle i,j\rangle = -1 \\ P_i^{(m)} Q_j^{(1^n)} 1_\lambda \text{ if } \langle i,j\rangle = 0 \end{cases} \tag{15}$$

and likewise if you exchange (a) with (1^a) everywhere.

(5) We have

$$P_i^{(n)} 1_\lambda = 1_\mu P_i^{(n)} 1_\lambda = 1_\mu P_i^{(n)} \quad \text{and} \quad P_i^{(1^n)} 1_\lambda = 1_\mu P_i^{(1^n)} 1_\lambda = 1_\mu P_i^{(1^n)}$$
$$Q_i^{(n)} 1_\mu = 1_\lambda Q_i^{(n)} 1_\mu = 1_\lambda Q_i^{(n)} \quad \text{and} \quad Q_i^{(1^n)} 1_\mu = 1_\lambda Q_i^{(1^n)} 1_\mu = 1_\lambda Q_i^{(1^n)}$$

where $\mu = \lambda + nc\delta$.

(6) We have

$$[Q_i^{[1^{a+1}]}, E_{i,b}] 1_\lambda = \begin{cases} q^2 Q_i^{[1^a]} E_{i,b+1} 1_\lambda - q^{-2} E_{i,b+1} Q_i^{[1^a]} 1_\lambda \text{ if } a>0 \\ [2] E_{i,b+1} 1_\lambda \text{ if } a=0. \end{cases}$$

$$q^{-c}[Q_i^{[1^{a+1}]}, F_{i,b}] 1_\lambda = \begin{cases} q^{-2} Q_i^{[1^a]} F_{i,b+1} 1_\lambda - q^{2} F_{i,b+1} Q_i^{[1^a]} 1_\lambda \text{ if } a>0 \\ -[2] F_{i,b+1} 1_\lambda \text{ if } a=0. \end{cases}$$

$$q^{c}[P_i^{[1^{a+1}]}, E_{i,b+1}] 1_\lambda = \begin{cases} q^2 E_{i,b} P_i^{[1^a]} 1_\lambda - q^{-2} P_i^{[1^a]} E_{i,b} 1_\lambda \text{ if } a>0 \\ [2] E_{i,b} 1_\lambda \text{ if } a=0 \end{cases}$$

$$[P_i^{[1^{a+1}]}, F_{i,b+1}] 1_\lambda = \begin{cases} q^{-2} F_{i,b} P_i^{[1^a]} 1_\lambda - q^{2} P_i^{[1^a]} F_{i,b} 1_\lambda \text{ if } a>0 \\ -[2] F_{i,b} 1_\lambda \text{ if } a=0. \end{cases}$$

while if $\langle i,j\rangle = -1$ we have

$$[Q_j^{[1^{a+1}]}, E_{i,b}] 1_\lambda = \begin{cases} q E_{i,b+1} Q_j^{[1^a]} 1_\lambda - q^{-1} Q_j^{[1^a]} E_{i,b+1} 1_\lambda \text{ if } a>0 \\ E_{i,b+1} 1_\lambda \text{ if } a=0. \end{cases}$$

$$q^{-c}[Q_j^{[1^{a+1}]}, F_{i,b}] 1_\lambda = \begin{cases} q^{-1} F_{i,b+1} Q_j^{[1^a]} 1_\lambda - q Q_j^{[1^a]} F_{i,b+1} 1_\lambda \text{ if } a>0 \\ -F_{i,b+1} 1_\lambda \text{ if } a=0 \end{cases}$$

$$q^{c}[P_j^{[1^{a+1}]}, E_{i,b+1}] 1_\lambda = \begin{cases} q^{-1} E_{i,b} P_j^{[1^a]} 1_\lambda - q P_j^{[1^a]} E_{i,b} 1_\lambda \text{ if } a>0 \\ E_{i,b} 1_\lambda \text{ if } a=0 \end{cases}$$

$$[P_j^{[1^{a+1}]}, F_{i,b+1}] 1_\lambda = \begin{cases} q F_{i,b} P_j^{[1^a]} 1_\lambda - q^{-1} P_j^{[1^a]} F_{i,b} 1_\lambda \text{ if } a>0 \\ -F_{i,b} 1_\lambda \text{ if } a=0. \end{cases}$$

(7) We have
$$[E_{i,a}, F_{i,b}]1_\lambda = \begin{cases} q^{-bc} q^{\langle \lambda, i\rangle} Q_i^{[1^{a+b}]} 1_\lambda \text{ if } a+b>0 \\ q^{-ac} q^{-\langle \lambda, i\rangle} P_i^{[1^{-a-b}]} 1_\lambda \text{ if } a+b<0 \\ [\langle \lambda, i\rangle + ac] 1_\lambda \text{ if } a+b=0. \end{cases}$$
while if $i \neq j$ then $[E_{i,a}, F_{j,b}]1_\lambda = 0$.

(8) This condition is unchanged.

(9) For any $m, n \in \mathbb{Z}$, if $\langle i, j\rangle = -1$ we have
$$\begin{aligned} E_{i,m}E_{j,n+1}1_\lambda - qE_{j,n+1}E_{i,m}1_\lambda &= E_{j,n}E_{i,m+1}1_\lambda - qE_{i,m+1}E_{j,n}1_\lambda \\ F_{i,m+1}F_{j,n}1_\lambda - qF_{j,n}F_{i,m+1}1_\lambda &= F_{j,n+1}F_{i,m}1_\lambda - qF_{i,m}F_{j,n+1}1_\lambda \end{aligned}$$
while if $\langle i, j\rangle = 0$ then
$$E_{i,m}E_{j,n}1_\lambda = E_{j,n}E_{i,m}1_\lambda \text{ and } F_{i,m}F_{j,n}1_\lambda = F_{j,n}F_{i,m}1_\lambda.$$

(10) This condition is unchanged.

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