Whittaker Functions and Demazure Operators

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Abstract

We consider a natural basis of the Iwahori fixed vectors in the Whittaker model of an unramified principal series representation of a split semisimple p-adic group, indexed by the Weyl group. We show that the elements of this basis may be computed from one another by applying Demazure-Lusztig operators. The precise identities involve correction terms, which may be calculated by a combinatorial algorithm that is identical to the computation of the fibers of the Bott-Samelson resolution of a Schubert variety. The Demazure-Lusztig operators satisfy the braid and quadratic relations satisfied by the ordinary Hecke operators, and this leads to an action of the affine Hecke algebra on functions on the maximal torus of the L-group. This action was previously described by Lusztig using equivariant K-theory of the flag variety, leading to the proof of the Deligne-Langlands conjecture by Kazhdan and Lusztig. In the present paper, the action is applied to give a simple formula for the basis vectors of the Iwahori Whittaker functions.

It is well known that there are relations between the representation theory of p-adic groups and the topology of flag varieties. More precisely, let G be a split semisimple group over a nonarchimedean local field F, and let $\hat{G}(\mathbb{C})$ be the (connected) Langlands dual group. Then the representation theory of $G(F)$ is closely related to the topology of the complex flag variety X of $\hat{G}(\mathbb{C})$. For example, the same affine Hecke algebra appears as both contexts. On the one hand, Iwahori and Matsumoto [13], in introducing this important ring, showed that it is a convolution ring of functions on $G(F)$ acting on the Iwahori fixed vectors of any representation. On the other hand, Lusztig [18] interpreted it as a ring of endomorphisms of the equivariant K-theory of the flag variety of $\hat{G}(F)$.

With this in mind, we study Whittaker functions for unramified principal series representations of $G(F)$. Our investigations were motivated by earlier work of

Reeder [20] and we begin with a brief summary of his results. Further discussion of this may be found in Section 1.

Let τ be an unramified character of the split maximal torus $T(F)$ where $B = TN$ is the standard Borel subgroup. Then we may form the principal series representation $M(\tau) = \mathrm{Ind}_B^G(\tau)$. A Whittaker model for $M(\tau)$ is an intertwining map $\mathcal{W}_\tau : M(\tau) \to \mathrm{Ind}_N^G(\psi)$ where ψ is in an open T-orbit of the space of characters of N; such an intertwiner is unique up to constant. An important problem is the characterization of functions $\mathcal{W}_\tau(\phi) : G \to \mathbb{C}$ for distinguished vectors $\phi \in M(\tau)$.

Let J be the Iwahori subgroup of $G(F)$. The space $M(\tau)^J$ of J-fixed vectors has dimension equal to the order of the Weyl group W of G. We will be primarily concerned with two particular bases of $M(\tau)^J$. First, we have the "standard basis" $\{\Phi_w^\tau \mid w \in W\}$ whose elements are defined by

$$\Phi_w^\tau(buk) = \begin{cases} \delta^{1/2}\tau(b) & \text{if } u = w, \\ 0 & \text{otherwise,} \end{cases} \qquad (b \in B(F), u \in W \text{and } k \in J) \tag{1}$$

where δ is the modular character of $B(F)$. These functions are well-defined according to the decomposition $G = \coprod_{w \in W} BwJ$. Our formulas will be simpler if we use the basis obtained by summing Φ_w's according to the Bruhat order:

$$\tilde{\Phi}_w^\tau = \sum_{u \geqslant w} \Phi_u^\tau, \tag{2}$$

so that if $u, w \in W$

$$\tilde{\Phi}_w^\tau(buk) = \begin{cases} \delta^{1/2}\tau(b) & \text{if } u \geqslant w, \\ 0 & \text{otherwise}. \end{cases}$$

In particular $\tilde{\Phi}_1^\tau$ is the standard spherical vector:

$$\tilde{\Phi}_1^\tau(bk) = \delta^{1/2}\tau(b) \qquad b \in B, k \in K.$$

In [20] Reeder investigated the functions $\mathcal{W}_\tau(\tilde{\Phi}_w^\tau)$ evaluated on the maximal torus, using methods of Casselman and Shalika. The key idea is that one can compute the effect of intertwining operators on Iwahori fixed vectors before or after applying the Whittaker functional.

Let Λ be the weight lattice of $\hat{G}$, which is the group $X^*(\hat{T})$ of rational characters of the maximal torus $\hat{T}$ of $\hat{G}$ that is dual to T. Thus Λ may be identified with the group $X_*(T)$ of rational one parameter subgroups of T, isomorphic to $T(F)/T(\mathfrak{o})$, where $\mathfrak{o}$ is the ring of integers in F. If λ is a dominant weight of $\hat{T}$, let $a_\lambda \in T(F)$ be a representative of the corresponding coset in $T(F)/T(\mathfrak{o})$. Casselman [7] describes, in addition to the basis Φ_w of Iwahori fixed vectors, a more subtly defined basis which we refer to as the *Casselman basis* $\{f_w^\tau\}$. Reeder gives a simple formula for $\mathcal{W}_\tau(f_w^\tau)(a_\lambda)$, but a similar closed formula for $\mathcal{W}_\tau(\tilde{\Phi}_w^\tau)(a_\lambda)$ is more difficult. However he found a recursive algorithm for the change of basis between the Casselman basis and the basis Φ_w. This algorithm, which we implemented in `Sage`, allowed us to compute the Whittaker functions evaluated at any fixed a_λ and these calculations were an important tool in our investigation.

Furthermore, Reeder saw that the $\mathcal{W}_\tau(\tilde{\Phi}_w^\tau)(a_\lambda)$ should be related to the coherent cohomology of line bundles over the flag varieties and their Schubert varieties, and gave such interpretations for particular Weyl group elements—those corresponding to the long element of a Levi subgroup of W.

We will exhibit connections between Iwahori Whittaker functions and the geometry of Schubert varieties that hold for all Weyl group elements. Our starting point is a recursive relation for the Whittaker function of $\tilde{\Phi}_w$ in terms of Bruhat order (Theorem 17). It is obtained by considering an operator on $M(\tau)$ which acts as an idempotent in the isomorphism with the finite Hecke algebra. From this we prove that the Iwahori Whittaker functions become Demazure characters when q^{-1} is specialized to 0, where q is the cardinality of the residue field. See Theorem 18 for a precise statement.

Instead of the Whittaker model, one may consider the Iwahori fixed vectors in the spherical model of the representation. This has been investigated by Ion [13], who also finds that Demazure operators play a role, and (generalizing the Macdonald formula for the spherical function) the functions on the p-adic group may be expressed in terms of the nonsymmetric Macdonald polynomials.

We briefly recall the definition of Demazure characters and their relation to cohomology of Schubert varieties. Given $w \in W$, let X_w be the corresponding Schubert cell in $X = \hat{G}(\mathbb{C})/\hat{B}(\mathbb{C})$, where $\hat{B}$ is the standard Borel subgroup. Thus X_w is the closure of the open Schubert cell Y_w, which is the image in X of $\hat{B}(\mathbb{C})w\hat{B}(\mathbb{C})$. If λ is a dominant weight of $\hat{G}$, then λ determines a line bundle $\mathcal{L}_\lambda$ on X. The space $H^0(X_w, \mathcal{L}_\lambda)$ of sections is a module for the standard maximal torus $\hat{T}(\mathbb{C})$ of $\hat{G}(\mathbb{C})$, and the Demazure character formula computes its character.

To describe the Demazure character formula, let $\alpha_1, \cdots, \alpha_r$ be the simple roots of $\hat{G}$ and let $s_1, \cdots, s_r$ be the corresponding simple reflections. The Demazure operators are defined on the ring $\mathcal{O}(\hat{T})$ of rational functions on $\hat{T}$ by

$$\partial_{\alpha_i} f(\boldsymbol{z}) = \partial_i f(\boldsymbol{z}) = \frac{f(\boldsymbol{z}) - \boldsymbol{z}^{-\alpha_i} f(s_i \boldsymbol{z})}{1 - \boldsymbol{z}^{-\alpha_i}}, \tag{3}$$

for $\boldsymbol{z} \in \hat{T}(\mathbb{C})$. The operators ∂_i are idempotent and satisfy $s_i \circ \partial_i = \partial_i$. They also satisfy the braid relations for the Weyl group. This implies that if $\mathfrak{w} = (s_{h_1}, \cdots, s_{h_d})$ is a reduced word for w, so that $w = s_{h_1} \cdots s_{h_d}$ is a reduced decomposition of w into a product of simple reflections then we may define $\partial_w := \partial_{h_1} \cdots \partial_{h_d}$ and this is well-defined. In particular, if w_0 is the long element and λ is a dominant weight, then $\partial_{w_0} \boldsymbol{z}^\lambda$ is the character of the irreducible representation with highest weight λ. The Demazure character formula asserts that for an arbitrary Weyl group element w the trace of $\boldsymbol{z} \in \hat{T}(\mathbb{C})$ on $H^0(X_w, \mathcal{L}_\lambda)$ is $\partial_w \boldsymbol{z}^\lambda$.

A key ingredient of the proof of the Demazure character formula is the Bott-Samelson resolution of the (possibly singular) Schubert variety X_w. Depending on the reduced word $\mathfrak{w}$ we may construct the nonsingular *Bott-Samelson variety* $Z_{\mathfrak{w}}$. See Bott-Samelson [4], Demazure [11], Andersen [1] and Kumar [17]. There is then a birational morphism $Z_{\mathfrak{w}} \longrightarrow X_w$. Since the map is birational, the fiber over a generic point is just is a single point. However even if X_w is nonsingular, this map may have nontrivial fibers over some points. Pulling the line bundle back to $Z_{\mathfrak{w}}$ does not change its space of sections, and over the Bott-Samelson variety, the cohomology of $\mathcal{L}_\lambda$ may be computed inductively using the Leray spectral sequence, leading to the Demazure character formula.

On the other hand, returning to Whittaker functions over a nonarchimedean local field F, $\boldsymbol{z} \in \hat{T}(\mathbb{C})$ parametrizes an unramified character $\tau = \tau_{\boldsymbol{z}}$ of $T(F)$, which may be parabolically induced to $G(F)$. The set $\{\mathcal{W}_\tau \tilde{\Phi}_w^\tau\}$ indexed by elements w of the Weyl group gives a natural basis of the space of Iwahori fixed vectors in the Whittaker model. If λ is a dominant weight of $\hat{T}$, then λ parametrizes a coset in $T(F)/T(\mathfrak{o})$, where $\mathfrak{o}$ is the ring of integers in F. Let $a_\lambda \in T(F)$ be a representative. Then

$$\mathcal{W}_\tau \tilde{\Phi}_w^\tau(a_\lambda) = \delta^{1/2}(a_\lambda) P_{\lambda,w}(\boldsymbol{z}, q^{-1})$$

where δ is the modular character of $B(F)$, and $P_{\lambda,w}$ is a rational function in $\boldsymbol{z}$ – that is, a finite linear combination of weights – whose coefficients are polynomials in q^{-1}. The factor $\delta^{1/2}(a_\lambda)$ is a constant, independent of $\boldsymbol{z}$, and removing it is harmless. For example, multiplying by $\delta^{1/2}(a_\lambda)$ commutes with the Demazure operators.

We will use Bott-Samelson varieties to exhibit a further connection between representations and geometry—a certain dictionary between Whittaker functions and Schubert varieties. Our philosophy is that in the geometric picture, the various varieties that appear, including Schubert and Bott-Samelson varieties, can be associated with polynomials depending on the spectral parameter $\boldsymbol{z}$, a dominant weight λ and a parameter v. Thus given a variety V with an action of the Borel subgroup of $\hat{G}$, together with an equivariant map $V \to X$, one may hope to construct an invariant $\boldsymbol{V}(\boldsymbol{z}, \lambda, v)$ which is a polynomial in v. We will not give a systematic theory of such invariants, but we will make such an association for the particular varieties that come up in our study of Whittaker functions.

In order to make this dictionary more apparent, let us use the notation

$$\boldsymbol{X}_w(\boldsymbol{z},\lambda,v) := P_{-w_0\lambda,ww_0}(\boldsymbol{z}^{-1},v), \tag{4}$$

where v is an indeterminate and w_0 is the long Weyl group element. Note that if λ is a dominant weight, then so is $-w_0\lambda$. The construction of $\boldsymbol{X}_w$ was by way of Whittaker functions, but we are now thinking of it as an invariant associated to the Schubert variety X_w. We will also use the notation $\boldsymbol{Y}_w$ where we set

$$\boldsymbol{X}_w = \sum_{u\leqslant w} \boldsymbol{Y}_u, \qquad \text{and so} \qquad \boldsymbol{Y}_w = \sum_{u\leqslant w} (-1)^{l(w)-l(u)} \boldsymbol{X}_u.$$

Thus $\boldsymbol{Y}_w$ may be obtained from Whittaker functions using $\mathcal{W}_\tau \Phi_w^\tau$ instead of $\mathcal{W}_\tau \tilde{\Phi}_w^\tau$.

We will see that if we replace v by 0 then $\boldsymbol{X}_w$ becomes a Demazure character; that is,

$$\boldsymbol{X}_w(\boldsymbol{z},\lambda,0) = \partial_w \boldsymbol{z}^\lambda. \tag{5}$$

Since the Demazure character is the coherent cohomology of a line bundle on X_w, this is the first evidence of a connection between Whittaker functions and the geometry of the Schubert varieties.

In order to describe a deeper connection, it is necessary to understand the complete polynomial $P_{\lambda,w}$, not just its constant term, in relation to Schubert varieties. Corresponding to the Bott-Samelson variety $Z_{\mathfrak{w}}$, define

$$\boldsymbol{Z}_{\mathfrak{w}}(\boldsymbol{z},\lambda,v) = \mathfrak{D}_{h_1} \cdots \mathfrak{D}_{h_d} \boldsymbol{z}^\lambda \tag{6}$$

where

$$\mathfrak{D}_i = (1 - v\boldsymbol{z}^{-\alpha_i})\partial_i. \tag{7}$$

The Bott-Samelson variety may be built up from a point by successive fiberings by $\mathbb{P}^1$, and we think of the the application of the operators $\mathfrak{D}_i$ as an algebraic analog of this process.

Our thesis is that the relationship between $\boldsymbol{X}_w$, which is essentially the Whittaker function, and $\boldsymbol{Z}_{\mathfrak{w}}$ is identical to the relationship between the Schubert varieties X_w and the Bott-Samelson varieties $Z_{\mathfrak{w}}$. The $\boldsymbol{Y}_w$ then correspond to the open Schubert varieties Y_w.

To explain further, we may write

$$\boldsymbol{X}_w = \boldsymbol{Z}_{\mathfrak{w}} - v \times \text{“correction terms”}.$$

The correction terms are linear combinations of $\boldsymbol{Z}_{\mathfrak{u}}$ for a few particular reduced words $\mathfrak{u}$ of $u \leqslant w$ in the Bruhat order. The determination of the correction terms is a combinatorial matter, and the point is that the combinatorics are identical between the Whittaker ($\boldsymbol{X}_w$ and $\boldsymbol{Z}_{\mathfrak{w}}$) and the Schubert ($X_w$ and $Z_{\mathfrak{w}}$) relationships.

In order to explain the relationship most simply, it is useful to introduce *partial Bott-Samelson varieties* $Z_{s,w}$, where $s = s_\alpha$ is a simple reflection and w is a Weyl group element such that $sw > w$. This is a fiber bundle over $\mathbb{P}^1$ in which the fibers are X_w. Furthermore, it is equipped with a birational morphism $Z_{s,w} \to X_{sw}$. In keeping with the algebraic analogy described above, we define

$$\boldsymbol{Z}_{s,w} = \mathfrak{D}_i \boldsymbol{X}_w, \quad \text{where} \quad \alpha = \alpha_i.$$

Just as in (6), the operator $\mathfrak{D}_i$ should be thought of as an algebraic analog of the $\mathbb{P}^1$.

We find that

$$\boldsymbol{X}_{sw} = \boldsymbol{Z}_{s,w} - v \times \text{“correction terms”}.$$

Here the correction terms have the following combinatorial description. Let $H(w,s)$ be the set of $u \in W$ such that both $u, su \leq w$ in the Bruhat order. If $u \in H(w,s)$ and $t \leqslant u$ then $t \in H(w,s)$. The set $H(w,s)$ may be empty or it may have a few maximal elements $u_1, \cdots, u_k$. If it is empty, then $\boldsymbol{X}_{sw} = \boldsymbol{Z}_{s,w}$, while if $H(w,s)$ has a unique maximal element u_1, then

$$\boldsymbol{X}_{sw} = \boldsymbol{Z}_{s,w} - v\boldsymbol{X}_{u_1}.$$

If $H(w,s)$ has two maximal elements u_1 and u_2, and $\{x \leqslant u_1, u_2\}$ has a unique maximal element u_3, then

$$\boldsymbol{X}_{sw} = \boldsymbol{Z}_{s,w} - v\boldsymbol{X}_{u_1} - v\boldsymbol{X}_{u_2} + v\boldsymbol{X}_{u_3}. \tag{8}$$

This latter case occurs, for example, in Type A_3 when $s = s_1$ and $w = s_2 s_1 s_3 s_2$, with $u_1 = s_1 s_2 s_1$, $u_2 = s_1 s_3 s_2$ and $u_3 = s_1 s_2$. This illustrates the combinatorial description of the Whittaker functions in terms of the operators $\mathfrak{D}_i$.

Now let us consider an analogous geometric problem leading to the *same combinatorial decomposition* as in (8). We recall that there is a birational morphism $\mu : Z_{s,w} \longrightarrow X_{sw}$. The fiber over every point $x \in X_{sw}$ is either a point or a $\mathbb{P}^1$. Let us determine the subvariety E of X_{sw} where the fiber is $\mathbb{P}^1$. The combinatorics of this question are again controlled by the same set $H(w,s)$. Indeed, E is just the union of the X_u as u runs through the maximal elements of $H(w,s)$. For computational purposes, however, it may be useful to describe E by recording the overlapping of its irreducible components. Thus in the case where $H(w,s)$ has two maximal elements

u_1 and u_2, and $\{x \leqslant u_1, u_2\}$ has a unique maximal element u_3, E has two irreducible components X_{u_1} and X_{u_2}, whose intersection is X_{u_3}. So we write formally

$$E = X_{u_1} + X_{u_2} - X_{u_3}. \tag{9}$$

The three Weyl group elements that appear with nonzero coefficient are the same in (8) and (9), and we will prove that this is always true.

The notation (9) is intended to be suggestive, but we can make it precise as follows. If S is a subset of X we will write symbolically

$$S = \sum_{w \in W} c(w) X_w, \qquad c(w) \in \mathbb{Z}, \tag{10}$$

to mean that if $y \in X$ then

$$\sum_{\substack{w \in W \\ y \in X_w}} c(w) = \begin{cases} 1 & \text{if } y \in S \\ 0 & \text{otherwise.} \end{cases}$$

We may now state a theorem connecting the correction terms in the expansions of the Whittaker functions and the fibers of the maps $Z_{s,w} \longrightarrow X_{sw}$.

Theorem 1. *Let s be a simple reflection, and let $w \in W$ such that $sw > w$. Then there exist integer coefficients $c_{s,w}(u) = c(u)$ defined for $u \in W$ such that*

$$\boldsymbol{X}_{sw} = \boldsymbol{Z}_{s,w} - v \sum_u c(u) \boldsymbol{X}_u.$$

Moreover, there is a corresponding geometric identity

$$S = \sum_{u \in W} c(u) X_u$$

where S is the subset of X_{sw} such that the fiber of the map $Z_{s,w} \longrightarrow X_{sw}$ over $y \in X_{sw}$ is a point if $y \notin S$, or $\mathbb{P}^1$ if $y \in S$. The coefficient $c(u)$ is zero unless both $u, su < w$.

In the special case $v = 0$, this immediately implies the specialization to Demazure characters in (5). When $v = 1$ they also have a simple specialization. Let ρ be the Weyl vector for $\hat{G}(\mathbb{C})$, that is, half the sum of the positive roots.

Theorem 2. *Given a dominant weight λ, for any $w \in W$:*

$$\boldsymbol{X}_w(\boldsymbol{z}, \lambda, 1) = \sum_{u \leqslant w} (-1)^{l(u)} \boldsymbol{z}^{u(\rho+\lambda)-\rho}. \tag{11}$$

As alluded to earlier, the operators $\mathfrak{D}_i$ have an interpretation in terms of Hecke algebras. The Hecke algebra $\mathcal{H}_v$ associated with the Weyl group W has generators T_i, one for each Weyl group element s_i. They satisfy the same braid relations as the s_i, together with the quadratic relations $T_i^2 = (v-1)T_i + v$. We will show that the operators $\mathfrak{D}_i - 1$ satisfy these relations. Therefore we obtain a representation of $\mathcal{H}_v$ on $\mathbb{C}[v, v^{-1}] \otimes \mathcal{O}(\hat{T})$ in which T_i acts as the operator $\mathfrak{D}_i - 1$. The operators $\mathfrak{D}_i - 1$ are essentially the same as the *Demazure-Lusztig operators* which first appeared in Lusztig [18], equation (8.1). See equation (36) below.

Once this representation of $\mathcal{H}_v$ is constructed, the functions $\boldsymbol{X}_w$ or $\boldsymbol{Y}_w$ have the following explicit description. Denote the effect of $\phi \in \mathcal{H}_v$ on $f \in \mathbb{C}[v, v^{-1}] \otimes \mathcal{O}(\hat{T})$ by $\phi \cdot f$.

Theorem 3. *Let λ be a dominant weight. Then*

$$\boldsymbol{Y}_w(\lambda) = T_w \cdot \boldsymbol{z}^\lambda, \qquad \boldsymbol{X}_w(\lambda) = \sum_{u \leqslant w} T_u \cdot \boldsymbol{z}^\lambda.$$

This shows that the action of the Hecke algebra by Demazure-Lusztig operators gives a tidy formula for a basis of the Iwahori fixed Whittaker functions.

An equivalent formulation more directly in terms of Whittaker functions is given in Theorem 14. The relation between $\mathcal{W}_\tau$ and T_w appearing in the statement of Theorem 3 is more elaborate and we supply a second proof based on machinery developed in Section 5. This machinery is also used to obtain the extension of the action to the affine Hecke algebra, a point we discuss next.

The affine Iwahori Hecke algebra, denoted $\tilde{\mathcal{H}}_v$, is generated by $\mathcal{H}_v$ together an abelian subgroup ζ^Λ isomorphic with the weight lattice Λ of $\hat{T}$. We will show in Theorem 28 that the above representation of $\mathcal{H}_v$ may be extended to a representation of $\tilde{\mathcal{H}}_v$ in which ζ^Λ acts by translation: if $\lambda \in \Lambda$ then $\zeta^\lambda \in \zeta^\Lambda$ is an element that acts on functions of $\mathbb{C}[v, v^{-1}] \otimes \mathcal{O}(\hat{T})$ by multiplication by $\boldsymbol{z}^{-\lambda}$. The resulting representation of $\tilde{\mathcal{H}}_v$ was previously considered by Lusztig [18]. Indeed $\mathbb{C}[v, v^{-1}] \otimes \mathcal{O}(\hat{T})$ is isomorphic to the complexified equivariant K-group $\mathbb{C} \otimes_\mathbb{Z} K_M(X)$ of the flag variety X, where $M = \hat{G} \times GL_1$, and Lusztig constructions an action of the Hecke algebra (over $\mathbb{Z}$) on $K_M(X)$. The same representation also appeared in Cherednik [9] in the context of double affine Hecke algebras. In our context, it appears naturally in the theory of Whittaker functions. This representation is irreducible, and may be described as follows. The element $\boldsymbol{z}^{-\rho}$ of $\mathbb{C}[v, v^{-1}] \otimes \mathcal{O}(\hat{T})$, where ρ is the Weyl vector (half the sum of the positive roots) for $\hat{G}$, is annihilated by the $\mathfrak{D}_i$, so $\mathcal{H}_v$ acts on the one-dimensional span of this vector by the sign character that sends T_i to -1. The representation of $\tilde{\mathcal{H}}_v$ on $\mathcal{O}(\hat{T})$ is the irreducible representation induced by this one-dimensional representation of $\mathcal{H}_v$.

Let $\widetilde{\mathcal{H}}$ be the specialization of $\widetilde{\mathcal{H}}_v$ to a complex algebra obtained by putting $v = q^{-1}$. It was shown by Iwahori and Matsumoto [13] that $\tilde{\mathcal{H}}$ is the convolution ring of compactly supported J-biinvariant functions on $G(F)$. It therefore acts on the finite-dimensional vector space space $M(\tau)^J$ of Iwahori invariants by convolution on the right. This action depends on τ, and the isomorphism class of $M(\tau)$ is determined by the isomorphism class of this representation of $\tilde{\mathcal{H}}$. If $\boldsymbol{z} \in \hat{T}(\mathbb{C})$ is in general position then $M(\tau_{\boldsymbol{z}})$ is irreducible, and in this case, the isomorphism class of $M(\tau_{\boldsymbol{z}})$ and $M(\tau_{\boldsymbol{z}'})$ are the same if and only if $\boldsymbol{z}, \boldsymbol{z}' \in \hat{T}(\mathbb{C})$ are in the same W-orbit.

On the other hand, we have seen that there is a representation of $\tilde{\mathcal{H}}$ on $\mathbb{C}[q, q^{-1}] \otimes \mathcal{O}(\hat{T})$ which is constructed in this paper using the theory of Whittaker functions, and which was constructed earlier by Lusztig in connection with equivariant K-theory. This representation was applied by Lusztig in [18] and in Kazhdan and Lusztig [15, 16] to construct all representations having an Iwahori fixed vector. To see that one may construct at least the principal series representations, fix $\boldsymbol{z}_0 \in \hat{T}(\mathbb{C})$. Consider the ideal $\mathcal{J}_{\boldsymbol{z}_0}$ of functions in $\mathcal{O}(\hat{T})$ that vanish on the W-orbit of $\boldsymbol{z}_0$. It is clear from the definitions that this ideal is closed under the operators $\mathfrak{D}_i$, as well as ζ^Λ, and so there is an induced action of $\tilde{\mathcal{H}}$ on $\mathcal{O}(\hat{T})/\mathcal{J}_{\boldsymbol{z}_0}$. By Lemma 7 of Bernstein and Rumelhart [2] every irreducible constitutent of this representation is the space of Iwahori fixed vectors in an irreducible representation of $G(F)$.

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1 Preliminaries

Let G be a split semisimple Chevalley group. By this we mean an affine algebraic group scheme over $\mathbb{Z}$, whose Lie algebra $\mathfrak{g}_\mathbb{Z}$ has a fixed Chevalley basis defined over $\mathbb{Z}$ corresponding to a root system Δ. The elements of the Chevalley basis are the nilpotent elements X_α where α runs through all roots and the coroots $H_\alpha = [X_\alpha, X_{-\alpha}]$ where α runs through the simple positive roots. The structure constants of the Chevalley basis are all integers (cf. [10]). If α is a root, let $x_\alpha : \mathbb{G}_\mathrm{a} \longrightarrow G$ be the one parameter subgroup tangent to X_α. Let T be the split maximal torus whose Lie algebra is spanned by the H_α, and let N (resp. N_-) be the unipotent subgroup whose Lie algebra is spanned by the X_α (resp. $X_{-\alpha}$) as α runs through the positive roots. Then $B = TN$ is the standard Borel subgroup of G.

Let F be a nonarchimedean local field and $\mathfrak{o}$ its ring of integers, $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$, and let $q = |\mathfrak{o}/\mathfrak{p}|$. We will denote the residue field $\mathbb{F}_q$. The group $G(F)$ has $K = G(\mathfrak{o})$ as a maximal compact subgroup. The group $X_*(T)$ of rational cocharacters is isomorphic to $T(F)/T(\mathfrak{o})$, where the one-parameter subgroup $\varphi \in X_*(T)$ corresponds to the coset $\varphi(\varpi)T(\mathfrak{o})$ with ϖ a prime element.

If w is an element of the Weyl group W, we will choose a fixed representative of w in $G(\mathfrak{o})$, and by abuse of notation we will denote this element also as w. Nothing will depend on this choice in any essential way. We will denote by w_0 the long Weyl group element.

A character τ of $T(F)$ is called *unramified* if it is trivial on $T(\mathfrak{o})$. We will let W act on the right on unramified characters τ, so that

$$(\tau w)(t) = \tau(wtw^{-1}), \qquad w \in W, \qquad t \in T(F).$$

Since τ is unramified, this does not depend on the choice of representative w of the Weyl group element.

Let $\hat{G}$ be the connected Langlands dual group. It is an algebraic group over $\mathbb{C}$ whose root data are dual to G. Thus if Δ is the root system of G with respect to T, we will denote by $\alpha \longrightarrow \alpha^\vee$ the bijection of Δ with system $\Delta^\vee$ of coroots, and $\Delta^\vee$ may be regarded as the root system of $\hat{G}$. If $\hat{T}$ is a maximal torus of $\hat{G}$ then the group $X_*(T)$ of rational one-parameter subgroups of T is identified with the group $X^*(\hat{T})$ of rational characters. Thus we have a homomorphism

$$T(F) \longrightarrow T(F)/T(\mathfrak{o}) \cong X_*(T) \cong X^*(\hat{T}). \tag{12}$$

If $\boldsymbol{z} \in \hat{T}(\mathbb{C})$ and $\lambda \in X^*(\hat{T})$ we will denote by $\boldsymbol{z}^\lambda$ the application of the character λ to $\boldsymbol{z}$. Also if $t \in T(F)$ we may apply the homomorphism (12) to t and apply the resulting rational character of $\hat{T}$ to $\boldsymbol{z}$; we will denote the result by $\tau_{\boldsymbol{z}}(t)$. Thus $\tau_{\boldsymbol{z}}$ is an unramified character of T and $\boldsymbol{z} \longmapsto \tau_{\boldsymbol{z}}$ is an isomorphism of $\hat{T}(\mathbb{C})$ with the group of unramified characters of $T(F)$.

If $\lambda \in X^*(\hat{T})$, we will denote by a_λ a representative of the coset in $T(F)/T(\mathfrak{o})$ corresponding to λ by the isomorphism in (12).

Let $\tau = \tau_{\boldsymbol{z}}$ be such an unramified character. Let $M(\tau)$ be the space of the representation of $G(F)$ induced from τ. This is the space of locally constant functions $f : G(F) \longrightarrow \mathbb{C}$ that satisfy

$$f(bg) = (\delta^{1/2}\tau)(b)f(g), \qquad b \in B(F),$$

where $\delta : B(F) \longrightarrow \mathbb{R}^\times$ is the modular character. The standard intertwining integral $\mathcal{A}_w : M(\tau) \longrightarrow M(\tau w)$ is

$$\mathcal{A}_w^\tau \Phi(g) = \int_{N(F)\cap w^{-1}N_-(F)w} \Phi(wng)\, dn.$$

The integral is convergent for $\tau = \tau_{\boldsymbol{z}}$ with $|\alpha^\vee(\boldsymbol{z})| < 1$ when $\alpha \in \Delta^+$. It makes sense for other $\boldsymbol{z}$ by meromorphic continuation in a suitable sense.

In order to convert between this notation (which is the same as Reeder [20]) and that of Casselman [7], bear in mind that $\mathcal{A}_w^\tau$ is Casselman's $T_{w^{-1}}$. Due to this difference, the Weyl group action on characters is a right action: $w : \tau \longrightarrow \tau w$.

Let J be the Iwahori subgroup of $G(F)$. It is the inverse image of $B(\mathbb{F}_q)$ under the natural map $K \longrightarrow G(\mathbb{F}_q)$. The dimension of the space $M(\tau)^J$ of J-fixed vectors is $|W|$. Let $\{\Phi_w^\tau\}$ and $\{\widetilde{\Phi}_w^\tau\}$ be the bases of $M(\tau)^J$ defined by (1) and (2). The basis element Φ_w^τ is denoted ϕ_w by Casselman [7].

Lemma 4. *If $n \in N(F)$ and $w_0 n \in B w_0 J$ then $n \in N(\mathfrak{o})$.*

Proof. Using the Iwahori factorization $J = N_-(\mathfrak{p}) T(\mathfrak{o}) N(\mathfrak{o})$ we may write $w_0 n = b w_0 \gamma n_1$ with $b \in B(F)$, $\gamma \in N_-(\mathfrak{p}) T(\mathfrak{o})$ and $n_1 \in N(\mathfrak{o})$. Since $w_0 \gamma w_0^{-1} \in B(F)$, this implies that $n n_1^{-1} \in N(F) \cap w_0^{-1} B(F) w_0$, so $n = n_1 \in N(\mathfrak{o})$. $\square$

Lemma 5. *Let $\lambda \in X^*(\hat{T})$. Then $\mathcal{W}_\tau \Phi_w^\tau(a_\lambda) = 0$ if λ is not dominant.*

Proof. See Casselman and Shalika [8] Lemma 5.1. (The proof there obviously applies to all Iwahori-fixed Whittaker functions since J contains $N(\mathfrak{o})$.) $\square$

Proposition 6. *Given an unramified character τ of $T(F)$ we have $\Phi_{w_0}^\tau = \tilde{\Phi}_{w_0}^\tau$ and*

$$\mathcal{W}_\tau \Phi_{w_0}^\tau(a_\lambda) = \begin{cases} \delta^{1/2}(a_\lambda) \boldsymbol{z}^{w_0 \lambda} & \text{if } \lambda \text{ is dominant,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The fact that $\tilde{\Phi}_{w_0} = \Phi_{w_0}$ is clear since w_0 is maximal in W. For the last assertion by Lemma 5 we may assume that λ is dominant. Denoting $a = a_\lambda$,

$$\mathcal{W}_\tau \Phi_{w_0}^\tau(a) = \int_{N(F)} \Phi_{w_0}^\tau(w_0 n a) \psi(n)\, dn.$$

We make the variable change $n \longrightarrow a n a^{-1}$ whose Jacobian is $\delta(a)$, and the integral becomes

$$\delta(a) \int_{N(F)} \Phi_{w_0}^\tau(w_0 a n) \psi(a n a^{-1})\, dn \quad =$$

$$\delta(a) \cdot \delta^{1/2} \tau(w_0 a w_0^{-1}) \int_{N(F)} \Phi_{w_0}^\tau(w_0 n) \psi(a n a^{-1})\, dn.$$

Since $\delta^{1/2}(w_0 a w_0^{-1}) = \delta^{-1/2}(a)$,

$$\delta(a) \cdot \delta^{1/2} \tau(w_0 a w_0^{-1}) = \delta^{1/2}(a) (\tau w_0)(a) = \delta^{1/2}(a) \boldsymbol{z}^{w_0 \lambda}.$$

By Lemma 4, $\Phi_{w_0}^\tau(w_0 n) = 1$ if $n \in N(\mathfrak{o})$ and 0 otherwise. Since λ is dominant, $\psi(a n a^{-1}) = 1$ when $n \in N(\mathfrak{o})$, and the statement follows. $\square$

Let $\tau = \tau_{\boldsymbol{z}}$ and define

$$C_\alpha(\tau) = \frac{1 - q^{-1}\boldsymbol{z}^\alpha}{1 - \boldsymbol{z}^\alpha}. \tag{13}$$

Proposition 7. *We have*

$$\mathcal{W}_{\tau w}\mathcal{A}_w^\tau = \prod_{\substack{\alpha \in \Delta^+ \\ w^{-1}\alpha \in \Delta^-}} \frac{1 - q^{-1}\boldsymbol{z}^{-\alpha}}{1 - \boldsymbol{z}^\alpha}\mathcal{W}_\tau. \tag{14}$$

Proof. This is Casselman and Shalika [8], Proposition 4.3. $\square$

In (3) we defined the Demazure operator $\partial_i = \partial_{\alpha_i}$ corresponding to a simple reflection α_i. We will also make use of alternative version defined by

$$\partial'_{\alpha_i} f(\boldsymbol{z}) = \partial'_i f(\boldsymbol{z}) = \frac{f(\boldsymbol{z}) - \boldsymbol{z}^{\alpha_i} f(s_i \boldsymbol{z})}{1 - \boldsymbol{z}^{\alpha_i}} = \frac{f(s_i \boldsymbol{z}) - \boldsymbol{z}^{-\alpha_i} f(\boldsymbol{z})}{1 - \boldsymbol{z}^{-\alpha_i}}.$$

Again we may define $\partial'_w = \partial'_{i_1} \cdots \partial'_{i_k}$ if $w = s_{i_1} \cdots s_{i_k}$ is a reduced decomposition. If $w = w_0$ and μ is dominant, then $\partial'_{w_0}(\boldsymbol{z}^{w_0\mu})$ is the character of the irreducible representation of highest weight μ.

2 Whittaker functions

In this section we will prove that the Iwahori Whittaker functions $W(\Phi_w)$ are polynomials in q^{-1} whose constant terms are Demazure characters.

Proposition 8. *Let $s = s_\alpha$ with α a simple root. Then*

$$\mathcal{A}_s \Phi_w^{\tau s} + C_\alpha(\tau)\Phi_w^\tau = \begin{cases} \Phi_w^\tau + \Phi_{sw}^\tau & \text{if } sw < w, \\ q^{-1}(\Phi_w^\tau + \Phi_{sw}^\tau) & \text{if } sw > w. \end{cases} \tag{15}$$

Proof. The identities

$$\mathcal{A}_s \Phi_w^\tau = \begin{cases} (C_\alpha(\tau) - q^{-1})\Phi_w^{\tau s} + \Phi_{sw}^{\tau s} & \text{if } sw < w, \\ (C_\alpha(\tau) - 1)\Phi_w^{\tau s} + \Phi_{sw}^{\tau s} & \text{if } sw > w, \end{cases}$$

are Casselman [7], Theorem 3.4. These imply that

$$\mathcal{A}_s \Phi_w^\tau - (C_\alpha(\tau) - q^{-1} - 1)\Phi_w^{\tau s} = \begin{cases} \Phi_w^{\tau s} + \Phi_{sw}^{\tau s} & \text{if } sw < w, \\ q^{-1}(\Phi_w^{\tau s} + \Phi_{sw}^{\tau s}) & \text{if } sw > w. \end{cases} \tag{16}$$

Replacing τ by τs, noting that $C_\alpha(\tau s) = C_{-\alpha}(\tau)$ and using

$$C_\alpha(\tau) + C_{-\alpha}(\tau) = 1 + q^{-1}.$$

we obtain (15). $\square$

Proposition 9. *Let α be a simple root, and $s = s_\alpha$ the corresponding reflection. Then*

$$\mathcal{W}_\tau(\mathcal{A}_s \Phi_w^{\tau s} + C_\alpha(\tau)\Phi_w^\tau) = (1 - q^{-1}\boldsymbol{z}^\alpha)\partial'_\alpha \mathcal{W}_\tau \Phi_w^\tau. \tag{17}$$

Proof. By (14), we have

$$\mathcal{W}_{\tau s}\mathcal{A}_s^\tau = \frac{1 - q^{-1}\boldsymbol{z}^{-\alpha}}{1 - \boldsymbol{z}^\alpha}\mathcal{W}_\tau,$$

which we rewrite as

$$\mathcal{W}_\tau \mathcal{A}_s^{\tau s} = \frac{1 - q^{-1}\boldsymbol{z}^\alpha}{1 - \boldsymbol{z}^{-\alpha}}\mathcal{W}_{\tau s}.$$

Now the left-hand side in (17) equals

$$\frac{1 - q^{-1}\boldsymbol{z}^\alpha}{1 - \boldsymbol{z}^{-\alpha}}\mathcal{W}_{\tau s}\Phi_w^{\tau s} + \frac{1 - q^{-1}\boldsymbol{z}^\alpha}{1 - z^\alpha}\mathcal{W}_\tau \Phi_w^\tau = (1 - q^{-1}\boldsymbol{z}^\alpha)\frac{1}{1 - \boldsymbol{z}^\alpha}(\mathcal{W}_\tau \Phi_w^\tau - \boldsymbol{z}^\alpha \mathcal{W}_{\tau s}\Phi^{\tau s}).$$

By the definition of the Demazure operator we have

$$\frac{1}{1 - \boldsymbol{z}^\alpha}(\mathcal{W}_\tau \Phi_w^\tau - \boldsymbol{z}^\alpha \mathcal{W}_{\tau s}\Phi^{\tau s}) = \partial'_\alpha \mathcal{W}_\tau \Phi_w^\tau,$$

and we are done. □

Define operators on the ring $\mathcal{O}(\hat{T})$ of functions on $\hat{T}(\mathbb{C})$ as follows. Let s_i be a simple reflection corresponding to the simple root $\alpha = \alpha_i$. Define

$$\mathfrak{D}'_i = \mathfrak{D}'_\alpha = (1 - q^{-1}\boldsymbol{z}^\alpha)\partial'_\alpha, \qquad \mathfrak{T}'_i = \mathfrak{T}'_\alpha = \mathfrak{D}'_i - 1. \tag{18}$$

Proposition 10. *Suppose that $s = s_\alpha$ is a simple reflection and $sw < w$. Then*

$$\mathcal{W}_\tau \Phi_{sw}^\tau = \mathfrak{T}'_\alpha \mathcal{W}_\tau \Phi_w^\tau.$$

Proof. This follows immediately from Propositions 9 and 8. □

The operator $\mathfrak{T}'_i$ is closely related to the Demazure-Lusztig operator. We will return to this point in Section 5. At the moment, we make use of the Whittaker function to give a short proof that they satisfy the braid relation. Let $\mathcal{D}$ be the ring of expressions of the form $\sum_{w \in W} f_w \cdot w$ where $f_w \in \mathcal{O}(\hat{T})$, and the multiplication is defined by $(f_1 \cdot w_1)(f_2 \cdot w_2) = f_1{}^{w_1} f_2 \cdot w_1 w_2$. The $\mathfrak{D}_i$ are naturally elements of this ring. The ring $\mathcal{D}$ acts on $\mathcal{O}(\hat{T})$ in the obvious way.

Lemma 11. *Suppose that $D \in \mathcal{D}$ and that D annihilates $\boldsymbol{z}^{w_0\lambda}$ for every dominant weight λ. Then $D = 0$.*

Proof. Define the *support* of $f \in \mathcal{O}(\hat{T})$ to be the finite set of weights with nonzero coefficients in f. Let $D = \sum_{w \in W} f_w \cdot w$. We may choose the dominant weight λ so that the functions $f_w \boldsymbol{z}^{ww_0\lambda}$ have disjoint support. Then $D\boldsymbol{z}^{w_0\lambda} = 0$ implies that each $f_w = 0$ and so $D = 0$. $\square$

Proposition 12. *Let $s = s_i$ and s_j be simple reflections. Then the operators $\mathfrak{T}'_i$ and $\mathfrak{T}'_j$ satisfy the same braid relations as s_i and s_j. That is, if k is the order of $s_i s_j$ then*

$$\mathfrak{T}'_i \mathfrak{T}'_j \mathfrak{T}'_i \cdots = \mathfrak{T}'_j \mathfrak{T}'_i \mathfrak{T}'_j \cdots , \tag{19}$$

where k is the number of factors on both sides of this equation.

Proof. By Lemma 11 it is enough to show that these both have the same effect on $\boldsymbol{z}^{w_0\lambda}$ where λ is a dominant weight. By Proposition 6, $\boldsymbol{z}^{w_0\lambda} = c\mathcal{W}_\tau \Phi_{w_0}(a_\lambda)$ where the constant $c = (\delta^{1/2}\tau)(a_\lambda)$ is independent of $\boldsymbol{z}$ and hence commutes with operators in $\mathcal{D}$. Applying either side of (19) to $\mathcal{W}_\tau \Phi_{w_0}(a_\lambda)$ gives $\mathcal{W}_\tau \Phi_{ww_0}(a_\lambda)$ where w is the long element of the rank two Weyl group $\langle s_i, s_j \rangle$. Indeed, Proposition 10 is applicable taking $(s, w) = (s_i, w_0)$, $(s, w) = (s_j, s_i w_0)$, etc. until we reach ww_0 since at each stage s is a descent of w. $\square$

Proposition 13. *The operators satisfy the quadratic relations*

$$(\mathfrak{D}'_i)^2 = (1 + q^{-1})\mathfrak{D}'_i, \qquad (\mathfrak{T}'_i)^2 = (q^{-1} - 1)\mathfrak{T}'_i + q^{-1}. \tag{20}$$

Proof. The two relations are equivalent. We prove the first. Writing $\alpha = \alpha_i$,

$$\mathfrak{D}_i^2 = (1 - q^{-1}\boldsymbol{z}^\alpha)\partial'_i(1 - q^{-1}\boldsymbol{z}^\alpha)\partial'_i = (1 - q^{-1}\boldsymbol{z}^\alpha)(\partial'_i)^2 - (1 - q^{-1}\boldsymbol{z}^\alpha)q^{-1}\partial'_i \boldsymbol{z}^\alpha \partial'_i$$

and the quadratic relation in the form $\mathfrak{D}_i^2 = (v + 1)\mathfrak{D}_i$ follows from the properties $(\partial'_i)^2 = \partial'_i$ and $\partial'_i \boldsymbol{z}^\alpha \partial'_i = -\partial'_i$ of the usual Demazure operator. $\square$

If v is either an element of $\mathbb{C}$ or a field containing it, let $\mathcal{H}_v$ be the complex algebra generated by T_i subject to the quadratic relations $T_i^2 = (v - 1)T_i + v$ together with the braid relations. We see from Propositions 13 and 12 that there is a representation of $\mathcal{H}_{q^{-1}}$ on the Whittaker model $\mathcal{W}_\tau M(\tau)^J$ in which T_i acts by $\mathfrak{T}'_i$.

Given any $w \in W$ we may construct an element $\mathfrak{T}'_w$ of $\mathcal{D}$ as follows. Let $w = s_{i_1} \cdots s_{i_k}$ be any reduced decomposition of w into a product of simple reflections. Then $\mathfrak{T}'_w = \mathfrak{T}'_{i_1} \cdots \mathfrak{T}'_{i_k}$. This is well-defined by Proposition 12.

Iwahori and Matsumoto [13] showed that the convolution ring of J-bi-invariant functions supported in $G(\mathfrak{o})$ is isomorphic to $\mathcal{H}_q$ and that algebra acts by right convolution on $\mathcal{W}M(\tau)^J$. The rings $\mathcal{H}_q$ and $\mathcal{H}_{q^{-1}}$ are isomorphic, and one might wonder whether the action of $\mathcal{H}_{q^{-1}}$ we have just defined is the same. It is not, since the isomorphism class of the convolution action depends on τ, while the isomorphism class of the representation that we have just defined does not.

Theorem 14. *Let $w \in W$ and let λ be a dominant weight. Then*

$$\mathcal{W}_\tau \Phi^\tau_{ww_0}(a_\lambda) = \delta^{1/2}(a_\lambda) \mathfrak{T}'_w \boldsymbol{z}^{w_0\lambda}. \tag{21}$$

Proof. If $w = 1$, this follows from Proposition 6. The general case follows by repeated applications of Proposition 10. □

In passing from the functions $\mathcal{W}_\tau \Phi^\tau_w$ to $\mathcal{W}_\tau \widetilde{\Phi}^\tau_w$, the combinatorics of the Bruhat order begins to play a role.

Proposition 15. *Let s be a simple reflection and $w_1, w_2 \in W$.*

(i) Assume that $sw_1 < w_1$ and $sw_2 < w_2$. Then $w_1 \leqslant w_2$ if and only if $sw_1 \leqslant w_2$ if and only if $sw_1 \leqslant sw_2$.

(ii) Assume that $sw_1 > w_1$ and $sw_2 > w_2$. Then $w_1 \geqslant w_2$ if and only if $sw_1 \geqslant w_2$ if and only if $sw_1 \geqslant sw_2$.

Proof. Part (i) is a well-known property of Coxeter groups, called property $Z(s, w_1, w_2)$ by Deodhar [12]. Note that $w \longmapsto ww_0$ is an order reversing bijection of W. Applying this gives (ii). □

Suppose that $s = s_\alpha$ is a descent of $w \in W$: $sw < w$. Then we will define

$$H'(w, s) = \{u \in W | u, su \geqslant w\}.$$

Proposition 16. *The set $H'(w, s)$ is cofinal in W in the sense that if $u \in H'(w, s)$ and $t \geqslant u$ then $t \in H'(w, s)$.*

Proof. We have $t \geqslant u$ with both $u, su \geqslant w$. We wish to show that $t \in H'(w, s)$. We may assume without loss of generality that $su > u$. For if not, then $su < u$ so $t \geqslant su$. Thus interchanging u and su if necessary, we may assume that $su > u$. Also without loss of generality, $t > st$ since otherwise both t, st are $\geqslant u \geqslant w$ as required. Now taking $w_1 = su$ and $w_2 = t$ in Proposition 15 (i), we see that both $t, st \geqslant u$ and so $t \in H'(w, s)$. □

Define an integer-valued function $c'_{w,s}$ on W by

$$c'_{w,s}(u) = \sum_{\substack{t \in H'(w,s) \\ t \leqslant u}} (-1)^{l(t) - l(u)}.$$

Theorem 17. *Let α be a simple root, and let $s = s_\alpha$ denote the corresponding reflection. Assume that $sw < w$. Then*

$$\mathcal{W}_\tau \tilde{\Phi}^\tau_{sw} = (1 - q^{-1}\boldsymbol{z}^\alpha)\partial'_\alpha \mathcal{W}_\tau \tilde{\Phi}^\tau_w - q^{-1} \sum_{u \in H'(w,s)} \mathcal{W}_\tau \Phi^\tau_u. \tag{22}$$

Equivalently,

$$\mathcal{W}_\tau \tilde{\Phi}^\tau_{sw} = (1 - q^{-1}\boldsymbol{z}^\alpha)\partial'_\alpha \mathcal{W}_\tau \tilde{\Phi}^\tau_w - q^{-1} \sum_{u \in H'(w,s)} c'_{w,s}(u) \mathcal{W}_\tau \tilde{\Phi}^\tau_u. \tag{23}$$

Proof. By Proposition 9

$$(1 - q^{-1}\boldsymbol{z}^\alpha)\partial'_\alpha \mathcal{W}_\tau \tilde{\Phi}^\tau_w = \sum_{x \geqslant w} \mathcal{W}_\tau(\mathcal{A}_s \Phi^{\tau s}_x + C_\alpha(\tau)\Phi^\tau_x).$$

We split the sum into two parts according as $sx < x$ or $sx > x$ and use Proposition 8. We have

$$\sum_{\substack{x \geqslant w \\ sx < x}} \mathcal{W}_\tau(\mathcal{A}_s \Phi^{\tau s}_x + C_\alpha(\tau)\Phi^\tau_x) = \sum_{\substack{x \geqslant w \\ sx < x}} (\mathcal{W}_\tau \Phi^\tau_x + \mathcal{W}_\tau \Phi^\tau_{sx}).$$

By Proposition 15 (i) with $w_1 = w$ and $w_2 = x$, we see that

$$\bigcup_{\substack{x \geqslant w \\ sx < x}} \{x, sx\} = \{u \in W | u \geqslant sw\}.$$

Therefore this contribution equals $\mathcal{W}_\tau \tilde{\Phi}^\tau_{sw}$. On the other hand, let us consider the contributions from $sx > x$. By Proposition 8 these contribute

$$q^{-1} \sum_{\substack{x \geqslant w \\ sx > x}} (\mathcal{W}_\tau \Phi^\tau_x + \mathcal{W}_\tau \Phi^\tau_{sx}) = q^{-1} \sum_{u \in H'(w,s)} \mathcal{W}_\tau \Phi^\tau_u.$$

This proves (22). By Möbius inversion (Verma [22], Deodhar [12]) we may write

$$\Phi^\tau_u = \sum_{t \leqslant u} (-1)^{l(t)-l(u)} \tilde{\Phi}^\tau_t,$$

and substituting this gives (23). ∎

The function $c'_{w,s}$ has a tendency to take on only a few nonzero values. It vanishes off $H'(w,s)$. This sparseness means there are usually only a few terms on the right-hand side in (23). For example, in A_3, if we consider the pairs w,s where s is a left descent of w, we find sixteen such pairs where $c'_{w,s}$ is always zero. Thus for these pairs the identity (23) takes the form

$$\mathcal{W}_{\tau s}(\tilde{\Phi}^{\tau s}_{sw}) = (1 - q^{-1}\boldsymbol{z}^{-\alpha})\partial'_\alpha \mathcal{W}_\tau(\tilde{\Phi}^{\tau s}_w).$$

There are seventeen pairs (w,s) such that $c'_{w,s}(u) \neq 0$ for only one particular u. Then

$$\mathcal{W}_{\tau s}(\tilde{\Phi}^{\tau s}_{sw}) = (1 - q^{-1}\boldsymbol{z}^{-\alpha})\partial'_\alpha \mathcal{W}_\tau(\tilde{\Phi}^{\tau s}_w) - q^{-1}\mathcal{W}_{\tau s}(\tilde{\Phi}^{\tau s}_u).$$

Finally, there are three cases where

$$\mathcal{W}_{\tau s}(\tilde{\Phi}^{\tau s}_w) = (1 - q^{-1}\boldsymbol{z}^{-\alpha})\partial'_\alpha \mathcal{W}_\tau(\tilde{\Phi}^{\tau s}_w) - q^{-1}\mathcal{W}_{\tau s}(\tilde{\Phi}^{\tau s}_u) - q^{-1}\mathcal{W}_{\tau s}(\tilde{\Phi}^{\tau s}_t) + q^{-1}\mathcal{W}_{\tau s}(\tilde{\Phi}^{\tau s}_x).$$

These are:

w	s	u	t	x
s_2	s_2	s_1s_2	s_3s_2	$s_1s_3s_2$
s_3s_1	s_1	$s_2s_3s_1$	$s_3s_2s_1$	$s_2s_3s_2s_1$
s_3s_1	s_3	$s_2s_3s_1$	$s_1s_2s_3$	$s_2s_1s_2s_3$

The ring $\mathcal{O}(\hat{T})$ of regular functions on $\hat{T}$ is the complex group algebra over the weight lattice Λ of $\hat{T}$. Let v be an indeterminate, and let $\mathcal{R} = \mathbb{C}[v] \otimes \mathcal{O}(\hat{T})$. If $P \in \mathcal{R}$ we will denote by $P(\boldsymbol{z}, q^{-1})$ the image of P under the specialization map $\mathcal{R} \longrightarrow \mathcal{O}(\hat{T})$ that sends v to q^{-1}.

Theorem 18. *Let λ be dominant. There exists an element $P_{\lambda,w}$ of $\mathcal{R}$ such that*

$$\mathcal{W}\tilde{\Phi}^\tau_w(a_\lambda) = \delta^{1/2}(a_\lambda) P_{\lambda,w}(\boldsymbol{z}, q^{-1}).$$

We have

$$P_{\lambda,w}(\boldsymbol{z}, 0) = \partial'_w \boldsymbol{z}^{w_0\lambda}.$$

It should be emphasized that $P_{\lambda,w}$ is independent of q and of the field F.

Proof. This may be proved by induction on $l(ww_0)$. If $w = w_0$ then by Proposition 6 we have $P_{\lambda,w_0}(\boldsymbol{z}, q^{-1}) = \boldsymbol{z}^{w_0\lambda}$, so the assertion is true. In general, we may apply (23) to deduce both statements for sw when it is known for Weyl group elements $\geqslant w$. Indeed, by induction every term on the right-hand side is in $\mathcal{R}$, so $P_{\lambda,sw} \in \mathcal{R}$, and specializing $q^{-1} \longrightarrow 0$ produces simply $\partial'_i P_{\lambda,w} = \partial'_i \partial'_w \boldsymbol{z}^{w_0\lambda} = \partial'_{sw} \boldsymbol{z}^{w_0\lambda}$. □

3 Fibers of Partial Bott-Samelson Varieties

Let G be a complex reductive group, and let B be a Borel subgroup. In the application to Whittaker functions, G will be the group formerly denoted $\hat{G}(\mathbb{C})$, but we suppress the hat in this section.

Let $X = G/B$ be the flag variety. If w is an element of the Weyl group W, let X_w° be the image of BwB in X. The closure

$$X_w = \bigcup_{u \leqslant w} X_u^\circ$$

is the closed Schubert cell.

Choose a reduced decomposition $\mathfrak{w} = (s_{h_1}, \cdots, s_{h_k})$ of $w = s_{h_1} \cdots s_{h_k}$ into a product of simple reflections. To define the Bott-Samelson variety $Z_\mathfrak{w}$, let P_i be the parabolic subgroup generated by B and s_i. The group B^k acts on $P_{h_1} \times \cdots \times P_{h_k}$ on the right by

$$(p_1, \cdots, p_k)(b_1, \cdots, b_k) = (p_1 b_1, b_1^{-1} p_2 b_2, \cdots, b_{r-1} p_r b_r). \tag{24}$$

Then $Z_\mathfrak{w}$ is the quotient variety. The multiplication map $P_{h_1} \times \cdots \times P_{h_k} \longrightarrow G$ induces a rational map $Z_\mathfrak{w} \longrightarrow X_w$ that is a birational equivalence. Although X_w may be singular $Z_\mathfrak{w}$ is always smooth, so this gives a resolution of the singularities of X_w.

If s is an ascent of w with respect to the Bruhat order then we have a partial Schubert variety $Z_{s,w}$ which is the quotient $(P \times X_w)/B$ where if $s = s_i$ then $P = P_i$ and B acts on the right by $(p, x) \cdot b = (pb^{-1}, bx)$. It is birationally equivalent to X_{sw}.

We are interested in the map $\mu : Z_{s,w} \longrightarrow X_{sw}$ from the partial Bott-Samelson variety to the Schubert variety. The fiber over an open Schubert cell Y_u (with $u \leqslant sw$) is either a single point or a $\mathbb{P}^1$, and we will find a combinatorial criterion for these cases.

Let T be a maximal torus of G contained in B. Since the fibers of μ are constant on Schubert cells $Y_t \subset X_{sw}$ with $t \in W$, it suffices to study the fiber $\mu^{-1}(y_t)$, where $y_t \in Y_t^T$ is the unique T-fixed point in the Schubert cell Y_t. Since the fiber $\mu^{-1}(y_t)$ is either a single point or has dimension 1, it is determined by its Euler characteristic $\chi(\mu^{-1}(y_t))$, and this is what we will compute.

Lemma 19. *Let Y be a projective complex algebraic variety with a T action whose fixed point set Y^T consists of isolated points. Then $\chi(Y) = \#Y^T$.*

Proof. Choosing a regular element λ of $\mathrm{Hom}(\mathbb{C}^\times, T)$, it follows that Y has a $\mathbb{C}^\times$ action with the same fixed point set, that is, $Y^{\mathbb{C}^\times} = Y^T$. This $\mathbb{C}^\times$ action defines a Bialynicki-Birula cellular decomposition of Y, with cells $\{U_x\}_{x \in Y^{\mathbb{C}^\times}}$ defined by

$$U_x = \{z \in Y \mid \lim_{\varepsilon \to 0} \varepsilon \cdot z = x\},$$

one cell for each x. See Bialynicki-Birula, Carrell and McGovern [3]. Since the cells are all of even real dimension, the Euler characteristic of Y is simply the number of cells - that is, the number of fixed points. □

Proposition 20. *The fiber of μ over Y_u is $\mathbb{P}^1$ if and only if both $u, su \leqslant w$, and is a point otherwise.*

Proof. In view of Lemma 19, in order to compute $\chi(\mu^{-1}(y_t))$, we need to compute the number of fixed points in the set $\mu^{-1}(y_t)^T$. This is straightforward since the fixed point set $Z_{s,w}^T$ equals $\{(u,t) \mid u \in \langle s \rangle,\ t \leq w\}$ and the map $\mu^T : Z_{s,w}^T \longrightarrow X_{sw}^T$ is multiplication $(u,t) \mapsto ut$. Discussions of these facts may be found in many places, usually for “standard” Bott-Samelson varieties rather than these partial ones. See, for example Brion [5].

Now, from these two facts, we compute $(\mu^T)^{-1}(y_t)$ for $t \leqslant w$. In general, we have

$$(\mu^T)^{-1}(y_t) = \{(u,x) \mid u \in \langle s \rangle, \, x \leqslant w \text{ and } ux = t\}.$$

Since $\langle s \rangle$ has order two, there are at most two points in $(\mu^T)^{-1}(y_t)$. One of them is the point $y_{(1,t)}$, which is the image of the affine point $(1,t)$ in $\mathbb{P}^1$. The other possibility is the $y_{(s,st)}$; but this point is only a point of $Z_{s,w}^T$ if $st \leqslant w$. Thus we conclude that $(\mu^T)^{-1}(y_t)$ is in bijection with the elements z such that t and st are less than or equal to w.

The map μ is an isomorphism over the big cell Y_{sw}. Thus it remains to study the fibers over the cells Y_u with $u \leqslant w$.

It now suffices to to show that the fiber over u is a $\mathbb{P}^1$ if and only if both $u, su \leqslant w$, and is a point otherwise. Thus we must show that the Euler characteristic of $\mu^{-1}(y_u)$ is equal to 2 if and only if both $u, su \leqslant w$, and is equal to one otherwise. By Lemma 19, we must show that the cardinality of $(\mu^T)^{-1}(y_u)$ is equal to 2 if and only if both $u, su \leqslant w$, and is equal to 1 otherwise. But, as explained above, $(\mu^T)^{-1}(y_u)$ is the one element set $y_{(1,u)}$ unless both $u, su \leqslant w$, in which case $(\mu^T)^{-1}(y_u)$ is the 2 element set $\{y_{(1,u)}, y_{(s,su)}\}$. □

Let us reformulate this proposition using the notation (10) from the introduction. Assuming that $sw > w$, define

$$H(w,s) = \{u \in W | u, su \leqslant w\}.$$

As in Proposition 16 the set $H(w,s)$ has the property that if $u \in H(w,s)$ and $t \leqslant u$ then $t \in H(w,s)$. Define

$$c_{w,s}(u) = \sum_{\substack{t \in H(w,s) \\ t \geqslant u}} (-1)^{l(t)-l(u)}.$$

Proposition 21. *Let s be a left ascent of w. Then*

$$\{y \in X_{sw} \mid \mu^{-1}(y) \text{ is nontrivial}\} = \sum_{u \in H(w,s)} c_{w,s}(u) X_u.$$

Proof. Let $y \in X$. Let $t \in W$ such that $y \in Y_t$. Then

$$\sum_{\substack{u \in H(w,s) \\ y \in X_u}} c_{w,s}(u) = \sum_{\substack{u \in H(w,s) \\ t \leqslant u}} c_{w,s}(u).$$

It follows from Möbius inversion (Verma [22] or Deodhar [12]) that given $y \in X$ that this is 1 if $t \in H(w,s)$ and 0 otherwise. Thus the statement follows from Proposition 20. $\square$

4 Proof of Theorems 1 and 2

To prove Theorem 1 we may specialize $v = q^{-1}$ with q the cardinality of the residue field. Let $\theta : \hat{T}(\mathbb{C}) \longrightarrow \hat{T}(\mathbb{C})$ be the map that sends $\boldsymbol{z} \longmapsto \boldsymbol{z}^{-1}$. Then

$$\partial_i = \theta \partial_i' \theta \tag{25}$$

Let λ be a dominant weight of $\hat{G}(\mathbb{C})$. Then $-w_0\lambda$ is also a dominant weight, and (4) may be written

$$\boldsymbol{X}_w(\lambda) = \delta(a_{-w_0\lambda})^{-1/2} \theta \mathcal{W}_\tau \tilde{\Phi}^\tau_{ww_0}(a_{-w_0\lambda}), \tag{26}$$

where $\tau = \tau_{\boldsymbol{z}}$. Similarly

$$\boldsymbol{Y}_w(\lambda) = \delta(a_{-w_0\lambda})^{-1/2} \theta \mathcal{W}_\tau \Phi^\tau_{ww_0}(a_{-w_0\lambda}). \tag{27}$$

We have $u \in H(s,w)$ if and only if $uw_0 \in H'(s, ww_0)$. Note that $sww_0 < ww_0$ so that $H'(s, ww_0)$ is defined, and it is also easy to see that $c_{w,s}(u) = c'_{ww_0,s}(uw_0)$.

Proposition 22. *Let λ be a dominant weight. Then*

$$\boldsymbol{X}_1(\lambda) = \boldsymbol{Y}_1(\lambda) = \boldsymbol{z}^\lambda. \tag{28}$$

Assume that $s = s_\alpha$ is a simple reflection and that $sw > w$. Then

$$\boldsymbol{X}_{sw}(\lambda) = (1 - q^{-1}\boldsymbol{z}^{-\alpha})\partial_\alpha \boldsymbol{X}_w(\lambda) - q^{-1} \sum_{u \,\in\, H(w,\,s)} c_{w,s}(u)\boldsymbol{X}_u(\lambda). \tag{29}$$

Proof. Equation (28) follows from Proposition 6. To prove (29) we begin with (23), with w replaced by ww_0. Applying θ and using (25) we obtain

$$\theta\mathcal{W}_\tau\tilde{\Phi}^\tau_{sww_0} = (1 - q^{-1}\boldsymbol{z}^{-\alpha})\partial_\alpha\theta\mathcal{W}_\tau\tilde{\Phi}^\tau_{ww_0} - q^{-1} \sum_{uw_0 \in H'(ww_0,s)} c'_{ww_0,s}(uw_0)\theta\mathcal{W}_\tau\tilde{\Phi}^\tau_u.$$

Now evaluating this at $a_{-w_0\lambda}$ and multiplying by the constant $\delta(a_{-w_0\lambda})^{-1/2}$ (which depends on λ but not $\boldsymbol{z}$) we obtain (29). $\square$

Combining Propositions 21 and 22 gives Theorem 1.

To prove Theorem 2, we will take $q = 1$. Then the operators $\mathfrak{D}_i$ simplify to

$$\mathfrak{D}_i f(\boldsymbol{z}) = f(z) - \boldsymbol{z}^{-\alpha_i} f(s_i\boldsymbol{z}).$$

These operators do not satisfy the braid relation. We show that if we define $\widehat{\boldsymbol{X}}_w(\boldsymbol{z})$ to be the right-hand side of (11), and if $s = s_i$ with $l(sw) > l(w)$, then

$$\widehat{\boldsymbol{X}}_{sw} = \mathfrak{D}_i\widehat{\boldsymbol{X}}_w - \sum_{u \in H(w,s)} c(u)\widehat{\boldsymbol{X}}_u, \tag{30}$$

where $c(u)$ is as in Theorem 1. The dominant weight λ will be fixed and we suppress it from the notation. By comparison with Theorem 1, $\boldsymbol{X}_w$ (with $q = 1$) and $\widehat{\boldsymbol{X}}_w$ both satisfy the same recursion, and agree when $w = 1$; therefore by induction on the Bruhat order, they are equal.

Let $\widehat{\boldsymbol{Y}}_w = \boldsymbol{z}^{w(\rho+\lambda)-\rho}$. The identity (30) is equivalent by Möbius inversion to

$$\mathfrak{D}_i\widehat{\boldsymbol{X}}_w - \sum_{u \in H(w,s)} \widehat{\boldsymbol{Y}}_u, \tag{31}$$

Since $\boldsymbol{z}^{-\alpha}\boldsymbol{z}^{-s\rho} = \boldsymbol{z}^{-s\rho}$, we have

$$\mathfrak{D}_i\widehat{\boldsymbol{Y}}_u = \widehat{\boldsymbol{Y}}_u - \widehat{\boldsymbol{Y}}_{su}.$$

Therefore

$$\mathfrak{D}_i \widehat{\boldsymbol{X}}_w = \sum_{u \leqslant w} (-1)^{l(u)} \widehat{\boldsymbol{Y}}_u + \sum_{su \leqslant w} (-1)^{l(u)} \widehat{\boldsymbol{Y}}_u, \tag{32}$$

where in the second term we have replaced u by su.

By Proposition 15 (ii), we have $u \leqslant sw$ if and only $\min(u, su) \leqslant w$ where the notation $\min(u, su)$ makes sense since u and su are always comparable in the Bruhat order. So

$$\{u | u \leqslant sw\} = \{u | u \leqslant w\} \cup \{u | su \leqslant w\}$$

and we may write

$$\widehat{\boldsymbol{X}}_{sw} = \sum_{u \leqslant w} (-1)^{l(u)} \widehat{\boldsymbol{Y}}_u + \sum_{su \leqslant w} (-1)^{l(u)} \widehat{\boldsymbol{Y}}_u - \sum_{u, su \leqslant w} (-1)^{l(u)} \widehat{\boldsymbol{Y}}_u$$

where we have subtracted the terms that are double counted in the first two sums. Now using (32) we obtain (31).

5 Hecke algebras

The space of Iwahori fixed vectors $M(\tau)^J$ for a fixed unramified character τ is isomorphic to the finite-dimensional Hecke algebra $\mathcal{H}$ of compactly supported J-bi-invariant functions on $G(F)$ which have support inside of $G(\mathfrak{o})$. We recall the isomorphism, following Bump and Nakasuji [6], who in turn follow Rogawski [21].

In this isomorphism, $f \in M(\tau)^J$ corresponds to the element $\varrho_\tau(f) \in \mathcal{H}$ where

$$\varrho_\tau(f)(g) = \begin{cases} f(g^{-1}) & \text{if } g \in G(\mathfrak{o}), \\ 0 & \text{otherwise.} \end{cases}$$

Both $M(\tau)^J$ and $\mathcal{H}$ are left $\mathcal{H}$-modules, where the action of $\mathcal{H}$ on $M(\tau)$ is

$$\phi \cdot f(g) = \int_{G(\mathfrak{o})} \phi(x) f(gx)\, dx.$$

Then ϱ_τ is a homomorphism of left $\mathcal{H}$-modules.

The Hecke algebra $\mathcal{H}$ has the following description by generators and relations, due to Iwahori and Matsumoto [13]. If t_w is the characteristic function of the double coset JwJ then $\mathcal{H}$ is generated by the $t_i = t_{s_i}$ where s_i is a simple reflection. The generators satisfy the quadratic relation

$$t_i^2 = (q-1) t_i + q,$$

together with the braid relations. Thus $t_i = t_{s_i}$. The braid relations and the quadratic relations give a presentation of $\mathcal{H}$ which is valid even if q is not a prime.

Let v be an indeterminate. Let $\mathcal{H}_v$ be the abstract algebra over $\mathbb{C}[v, v^{-1}]$ generated by $T_1, \cdots, T_r$ subject to the braid relations together with the condition that

$$T_i^2 = (v-1)T_i + v.$$

There is an antihomomorphism $\sigma : \mathcal{H}_v \to \mathcal{H}$ defined by $\sigma(v) = q^{-1}$ and $\sigma(T_w) = t_w^{-1}$. In particular

$$\sigma(T_i + 1) = q^{-1}(t_i + 1). \tag{33}$$

The operators $\mathfrak{T}_i = \mathfrak{D}_i - 1$ are closely related to the *Demazure-Lusztig operators* that were introduced in Lusztig [18] equation (8.1). These are the operators defined by

$$\mathcal{L}_i(\boldsymbol{z}^\lambda) = \mathcal{L}_{i,v}(\boldsymbol{z}^\lambda) = \frac{\boldsymbol{z}^\lambda - \boldsymbol{z}^{s_i\lambda}}{\boldsymbol{z}^{\alpha_i} - 1} - v\frac{\boldsymbol{z}^\lambda - \boldsymbol{z}^{\alpha + s_i\lambda}}{\boldsymbol{z}^{\alpha_i} - 1}.$$

Lusztig shows that these satisfy the same relations as the T_i in the (finite) Hecke algebra, and we will also prove this in the equivalent form of Proposition 23 below. The precise relationship between our operators and Lusztig's is as follows. Lusztig's operators $\mathcal{L}_i$ satisfy the quadratic relation

$$\mathcal{L}_i^2 = (v-1)\mathcal{L}_i + v. \tag{34}$$

Replacing v by v^{-1} and multiplying by $-v$ let $\mathcal{L}'_i = -v\mathcal{L}_{i,v^{-1}}$. We have

$$\mathcal{L}'_i \boldsymbol{z}^\lambda = \frac{-v\boldsymbol{z}^\lambda - \boldsymbol{z}^{\alpha_i + s_i\lambda} + v\boldsymbol{z}^{s_i\lambda} + \boldsymbol{z}^\lambda}{\boldsymbol{z}^{\alpha_i} - 1}. \tag{35}$$

It follows from (34) that the $\mathcal{L}'_i$ satisfy the same braid and quadratic relations as the $\mathcal{L}_i$ and hence there is a representation of $\mathcal{H}_v$ in which $T_i \longrightarrow \mathcal{L}'_i$. We recall that if λ is a weight, then ζ^λ is the operation on $\mathcal{O}(\hat{T})$ defined by multiplication by $\boldsymbol{z}^{-\lambda}$. It follows easily from (35) that

$$\zeta^{-\rho}(\mathfrak{D}_i - 1)\zeta^\rho = \mathcal{L}'_i. \tag{36}$$

Thus the following proposition is equivalent to the result of [18]. Our proof is different, since we use Whittaker functions to prove the braid relations.

Proposition 23. (Lusztig [18]) *The operators $\mathfrak{T}_i = \mathfrak{D}_i - 1$ satisfy the braid relations and the quadratic relations of $\mathcal{H}_v$. Thus $T_i \longmapsto \mathfrak{T}_i$ is a representation of $\mathcal{H}_v$.*

Proof. The corresponding facts for $\mathfrak{T}'_i$, which is defined by (18), are Proposition 13 for the quadratic relation and Proposition 12 for the braid relation. Since $\mathfrak{T}_i = \theta\mathfrak{T}'_i\theta$, the result follows. $\square$

We have now defined an action of $\mathcal{H}_v$ on $\mathbb{C}[v, v^{-1}] \otimes \mathcal{O}(\hat{T})$, so Theorem 3 now has meaning. In order to prove it we will have define certain maps $\xi_\lambda : \mathcal{H}_v \to \mathbb{C}[v, v^{-1}] \otimes \mathcal{O}(\hat{T})$ and that is our next goal. These maps ξ_λ also turn out to be the key to extend the action to the affine Hecke algebra $\widetilde{\mathcal{H}}_v$.

Lemma 24. *Let $s = s_\alpha$ be a simple reflection, and let $A_s^{\tau s} : \mathcal{H} \longrightarrow \mathcal{H}$ be the homomorphism of left $\mathcal{H}$-modules that makes the following diagram commutative:*

$$\begin{array}{ccc} M(\tau s) & \xrightarrow{\mathcal{A}_s^{\tau s}} & M(\tau) \\ \Big\downarrow \varrho_{\tau s} & & \Big\downarrow \varrho_\tau \\ \mathcal{H} & \xrightarrow{A_s^{\tau s}} & \mathcal{H} \end{array}$$

Then $A_s^{\tau s}$ is right multiplication by

$$\frac{1}{q} t_i + \left(1 - \frac{1}{q}\right) \frac{\boldsymbol{z}^{-\alpha}}{1 - \boldsymbol{z}^{-\alpha}}, \tag{37}$$

where $\tau = \tau_{\boldsymbol{z}}$.

Proof. We recall that ϱ_τ is a homomorphism of left $\mathcal{H}$-modules. The composition $\varrho_\tau \circ \mathcal{A}_s^{\tau s} \circ \varrho_{\tau s}^{-1}$ is thus a homomorphism of left $\mathcal{H}$-modules $\mathcal{H} \longrightarrow \mathcal{H}$ and since $\mathcal{H}$ is a ring it must consist of right multiplication by some element. The scalar may be evaluated by applying $\varrho_\tau \circ \mathcal{A}_s^{\tau s} \circ \varrho_{\tau s}$ to the unit element of $\mathcal{H}$, which corresponds under ϱ^{-1} to $\Phi_1^{\tau s}$. By Casselman [7], Theorem 3.4 we have

$$\mathcal{A}_s^\tau \Phi_1^\tau = \frac{1}{q} \Phi_s^{\tau s} + \left(1 - \frac{1}{q}\right) \frac{\boldsymbol{z}^{\alpha}}{1 - \boldsymbol{z}^{\alpha}} \Phi_1^{\tau s}$$

so

$$\mathcal{A}_s^{\tau s} \varrho_{\tau s}^{-1}(1) = \mathcal{A}_s^\tau \Phi_1^{\tau s} = \frac{1}{q} \Phi_s^\tau + \left(1 - \frac{1}{q}\right) \frac{\boldsymbol{z}^{-\alpha}}{1 - \boldsymbol{z}^{-\alpha}} \Phi_1^\tau.$$

Applying ρ_τ gives

$$\frac{1}{q} t_i + \left(1 - \frac{1}{q}\right) \frac{\boldsymbol{z}^{-\alpha}}{1 - \boldsymbol{z}^{-\alpha}}.$$

$\square$

Let us fix a dominant weight λ for $\hat{T}$. Then we will define a map $\xi'_\lambda : \mathcal{H} \longrightarrow \mathcal{O}(\hat{T})$, where $\mathcal{O}(\hat{T})$ is the ring of rational functions on $\hat{T}$, which is the group algebra of the weight lattice Λ of $\hat{T}$. We regard a rational function on $\hat{T}$ as a function on the complex points, and to specify $\xi_\lambda(\phi)$ for $\phi \in \mathcal{H}$, it is enough to specify its value $\xi_\lambda(\phi)(\boldsymbol{z})$ for $\boldsymbol{z} \in \hat{T}(\mathbb{C})$. Let $\tau = \tau_{\boldsymbol{z}}$ and $\Phi = \varrho_\tau^{-1}(\phi) \in M(\tau)$. Define $\xi'_\lambda(\phi)$ by

$$\xi'_\lambda(\phi)(\boldsymbol{z}) = \delta^{-1/2}(a_\lambda)\, \mathcal{W}_\tau \Phi(a_\lambda).$$

Proposition 25. *The following diagram is commutative for any dominant weight λ of $\hat{T}$. Let $s = s_i$ be a simple reflection, and let $\alpha = \alpha_i$ be the corresponding simple root.*

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\xi'_\lambda} & \mathcal{O}(\hat{T}) \\ \Big\downarrow {\scriptstyle \times q^{-1}(1+t_i)} & & \Big\downarrow {\scriptstyle (1-q^{-1}\boldsymbol{z}^\alpha)\partial'_\alpha} \\ \mathcal{H} & \xrightarrow{\xi'_\lambda} & \mathcal{O}(\hat{T}) \end{array} \tag{38}$$

where the left vertical arrow is right multiplication by $q^{-1}(1+t_i)$. Moreover

$$\xi'_\lambda(t_{w^{-1}}) = \delta^{-1/2}(a_\lambda)\, \mathcal{W}_\tau \Phi^\tau_w(a_\lambda). \tag{39}$$

Proof. Let us consider both ways of traversing the commutative diagram (38) as applied to $t_{w^{-1}} \in \mathcal{H}$. Then with $\tau = \tau_{\boldsymbol{z}}$ we have $\varrho_\tau^{-1} t_{w^{-1}} = \Phi^\tau_w$ and so we obtain (39). Furthermore by Proposition 9

$$\delta^{1/2}(a_\lambda)(1 - q^{-1}\boldsymbol{z}^\alpha)\partial'_\alpha \xi'_\lambda(t_{w^{-1}}) = \mathcal{W}_\tau(\mathcal{A}^{\tau s}_s \Phi^{\tau s}_w + C_\alpha(\tau)\Phi^\tau_w)(a_\lambda).$$

By Lemma 24,

$$\mathcal{A}^{\tau s}_s \Phi^{\tau s}_w = \mathcal{A}^{\tau s}_s \varrho_{\tau s}^{-1} t_{w^{-1}} = \varrho_\tau^{-1}\left(t_{w^{-1}}\left(\frac{1}{q}t_i + (1-q^{-1})\frac{\boldsymbol{z}^{-\alpha}}{1-\boldsymbol{z}^{-\alpha}}\right)\right).$$

We have

$$\left(1 - \frac{1}{q}\right)\frac{\boldsymbol{z}^{-\alpha}}{1-\boldsymbol{z}^{-\alpha}} + C_\alpha(\tau) = \frac{1}{q}.$$

Therefore

$$\mathcal{A}^{\tau s}_s \Phi^{\tau s}_w + C_\alpha(\tau)\Phi^\tau_w = \varrho_\tau^{-1}\left(t_{w^{-1}} q^{-1}(t_i + 1)\right).$$

Applying $\mathcal{W}_\tau$ and evaluating at a_λ, we see that

$$(1 - q^{-1}\boldsymbol{z}^\alpha)\partial'_\alpha \xi'_\lambda(t_{w^{-1}}) = \xi'_\lambda(t_{w^{-1}} q^{-1}(t_i + 1)),$$

which is the required commutativity. $\square$

If λ is dominant, we will define ξ_λ as follows. By Theorem 2, if $\phi \in \mathcal{H}_v$ then $\theta\xi'_{-w_0\lambda}(\sigma\phi)$ is a polynomial in q^{-1}, so there exists an element $\xi_\lambda(\phi)$ of $\mathbb{C}[v,v^{-1}]\otimes\mathcal{O}(\hat{T})$ that corresponds to it. Thus $\xi_\lambda : \mathcal{H}_v \longrightarrow \mathbb{C}[v,v^{-1}] \otimes \mathcal{O}(\hat{T})$ is the map such that

$$\mathrm{ev} \circ \xi_\lambda = \theta \circ \xi'_{-w_0\lambda} \circ \sigma \ .$$

where $\mathrm{ev} : \mathbb{C}[v,v^{-1}] \otimes \mathcal{O}(\hat{T}) \to \mathcal{O}(\hat{T})$ is the evaluation map that sends v to q^{-1}.

Proposition 26. *Let λ be a weight, and let $\alpha = \alpha_i$ be a simple root The following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{H}_v & \xrightarrow{\xi_\lambda} & \mathbb{C}[v,v^{-1}] \otimes \mathcal{O}(\hat{T}) \\ \Big\downarrow{\scriptstyle (1+T_i)\times\cdot} & & \Big\downarrow{\scriptstyle \mathfrak{D}_i} \\ \mathcal{H}_v & \xrightarrow{\xi_\lambda} & \mathbb{C}[v,v^{-1}] \otimes \mathcal{O}(\hat{T}) \end{array} \tag{40}$$

where the left vertical arrow is left multiplication by $1 + T_i$. Moreover

$$\xi_\lambda(T^{-1}_{(ww_0)^{-1}}) = \boldsymbol{Y}_w(\lambda). \tag{41}$$

Proof. We observe that (41) follows from the definitions of ξ_λ and $\boldsymbol{Y}_w(\lambda)$ and (39). As for (40), consider the diagram

$$\begin{array}{ccccccc} \mathcal{H}_v & \xrightarrow{\quad\sigma\quad} & \mathcal{H} & \xrightarrow{\xi'_{-w_0\lambda}} & \mathcal{O}(\hat{T}) & \xrightarrow{\quad\theta\quad} & \mathcal{O}(\hat{T}) \\ {\scriptstyle (1+T_i)\times}\Big\downarrow & & {\scriptstyle q^{-1}(1+t_i)}\Big\downarrow & & \Big\downarrow{\scriptstyle (1-q^{-1}\boldsymbol{z}^{\alpha})\partial'_\alpha} & & \Big\downarrow{\scriptstyle (1-q^{-1}\boldsymbol{z}^{-\alpha})\partial_\alpha} \\ \mathcal{H}_v & \xrightarrow{\quad\sigma\quad} & \mathcal{H} & \xrightarrow{\xi'_{-w_0\lambda}} & \mathcal{O}(\hat{T}) & \xrightarrow{\quad\theta\quad} & \mathcal{O}(\hat{T}) \end{array}$$

The commutativity of the left square follows from (33). The commutativity of the middle square is (38). The commutativity of the right square is the fact that $\theta\partial_\alpha\theta = \partial'_\alpha$. The composition on the top is $\mathrm{ev} \circ \xi_\lambda$ and the statement follows. $\square$

The (extended) affine algebra $\tilde{\mathcal{H}}_v$ is generated by $\mathcal{H}_v$ and another commutative subalgebra ζ^Λ isomorphic to the weight lattice Λ. If $\lambda \in \Lambda$ let ζ^λ be the corresponding element of ζ^Λ. To complete the presentation of $\tilde{\mathcal{H}}$ we have the relation

$$T_i\zeta^\lambda - \zeta^{s_i\lambda}T_i = \zeta^\lambda T_i - T_i\zeta^{s_i\lambda} = \left(\frac{v-1}{1-\zeta^{-\alpha_i}}\right)(\zeta^\lambda - \zeta^{s_i\lambda}). \tag{42}$$

Lemma 27. *Let λ run through the dominant weights, and for each λ let w run through a set of coset representatives for W/W_λ where W_λ is the stabilizer of λ. Then $\boldsymbol{Y}_w(\lambda)$ runs through a basis of the $\mathbb{C}(v)$-vector space $\mathbb{C}(v) \otimes \mathcal{O}(\hat{T})$. Moreover*

$$\mathbb{C}[v, v^{-1}] \otimes \mathcal{O}(\hat{T}) = \bigoplus_{\lambda \text{ dominant}} \xi_\lambda(\mathcal{H}_v). \tag{43}$$

Proof. Every weight may be written as $w\lambda$ with λ dominant. Here λ is uniquely determined, and if there is more than one choice for w we take the one of smallest length. We make a partial order on the weights by $w_1\lambda_1 \preccurlyeq w_2\lambda_2$ with either $\lambda_1 < \lambda_2$ in the usual partial order on weights, or $\lambda_1 = \lambda_2$ and $w_1 \leqslant w_2$ in the Bruhat order. Then $\boldsymbol{Y}_w(\lambda) = \boldsymbol{z}^{w\lambda}$ plus terms that are lower in the partial order, so these are a basis. The direct sum decomposition follows from (41). $\square$

Theorem 28. *There is a right action of $\tilde{\mathcal{H}}_v$ on $\mathbb{C}[v, v^{-1}] \otimes \mathcal{O}(\hat{T})$ such that ξ_λ is a homomorphism of right $\mathcal{H}_v$-modules. In this action*

$$\zeta^\lambda \cdot \boldsymbol{z}^\mu = \boldsymbol{z}^{\mu-\lambda}, \qquad (1 + T_i) \cdot \boldsymbol{z}^\mu = \mathfrak{D}_i(\boldsymbol{z}^\mu) = \left(\frac{1 - v\boldsymbol{z}^{-\alpha_i}}{1 - \boldsymbol{z}^{-\alpha_i}} \right) (\boldsymbol{z}^\mu - \boldsymbol{z}^{s_i\mu-\alpha_i}). \tag{44}$$

This action of $\tilde{\mathcal{H}}_v$ appeared previously in Lusztig [18]. The interpretation here in terms of Whittaker functions appears to be new.

Proof. We make use of (43). Each summand may be given a right $\mathcal{H}_v$-module structure such that ξ_λ is a right $\mathcal{H}_v$-module homomorphism. Thus $\mathbb{C}[v, v^{-1}] \otimes \mathcal{O}(\hat{T})$ becomes an $\mathcal{H}_v$-module. By the commutativity of (40)

$$(1 + T_i) \cdot \xi_\lambda(\phi) = \xi_\lambda((1 + T_i)\phi) = \mathfrak{D}_i \xi_\lambda(\phi),$$

which gives the second identity in (44). As for the first, we make this a definition and then must check compatibility with the relation (42) in the equivalent form

$$(T_i + 1)\zeta^\lambda - \zeta^{s_i\lambda}(T_i + 1) = \left(\frac{v - \zeta^{-\alpha_i}}{1 - \zeta^{-\alpha_i}} \right) (\zeta^\lambda - \zeta^{s_i\lambda}). \tag{45}$$

We will write $s = s_i$ and $\alpha = \alpha_i$. Applying the left-hand side of (45) to $\boldsymbol{z}^\mu$ gives

$$\left(\frac{1 - v\boldsymbol{z}^{-\alpha}}{1 - \boldsymbol{z}^{-\alpha}} \right) (\boldsymbol{z}^{\mu-\lambda} - \boldsymbol{z}^{-\alpha+s\mu-s\lambda}) - \left(\frac{1 - v\boldsymbol{z}^{-\alpha}}{1 - \boldsymbol{z}^{-\alpha}} \right) (\boldsymbol{z}^{\mu-s\lambda} - \boldsymbol{z}^{-\alpha+s\mu-s\lambda})$$

This equals

$$\left(\frac{1 - v\boldsymbol{z}^{-\alpha}}{1 - \boldsymbol{z}^{-\alpha}} \right) (\boldsymbol{z}^{\mu-\lambda} - \boldsymbol{z}^{\mu-s\lambda}) = \boldsymbol{z}^\mu \left(\frac{v - \boldsymbol{z}^{\alpha}}{1 - \boldsymbol{z}^{\alpha}} \right) (\boldsymbol{z}^{-\lambda} - \boldsymbol{z}^{-s\lambda}).$$

which equals the right-hand side of (45) using the action $\zeta^\lambda \cdot \boldsymbol{z}^\mu = \boldsymbol{z}^{\mu-\lambda}$, as desired. $\square$

The maps ξ_λ have two applications in this paper. The first is that they allow one to extend the action of $\mathcal{H}_v$ to all of $\widetilde{\mathcal{H}}_v$ as in the previous theorem. The second application is Theorem 3.

Proof of Theorem 3. Since $l(w_0) = l(w) + l(w_0 w^{-1})$ we have $T_{(ww_0)^{-1}} T_w = T_{w_0^{-1}}$ or $T^{-1}_{(ww_0)^{-1}} = T_w T^{-1}_{w_0^{-1}}$. Using (41) and the fact that ξ_λ is $\mathcal{H}_v$-equivariant,

$$\boldsymbol{Y}_w(\lambda) = \xi_\lambda(T^{-1}_{(ww_0)^{-1}}) = T_w \cdot \xi_\lambda(T^{-1}_{w_0^{-1}}).$$

But using Proposition 22 and (41) again, $\xi_\lambda(T^{-1}_{w_0}) = \boldsymbol{Y}_1(\lambda) = \boldsymbol{z}^\lambda$. □

6 Complements

It is easy to see that $\partial_i \boldsymbol{z}^{-\rho} = 0$ for each Demazure operator ∂_i. As a consequence, $\mathfrak{D}_i \boldsymbol{z}^{-\rho} = 0$ and so in the action of Theorem 28 we have

$$T_i \cdot \boldsymbol{z}^{-\rho} = -\boldsymbol{z}^{-\rho}.$$

Hence this vector spans a one-dimensional $\mathcal{H}_v$-invariant subspace affording the sign representation of $\mathcal{H}_v$.

Theorem 29. *The $\tilde{\mathcal{H}}_v$-module $\mathbb{C}[v, v^{-1}] \otimes \mathcal{O}(\hat{T})$ is isomorphic to the representation of $\tilde{\mathcal{H}}_v$ induced from the sign representation of $\mathcal{H}_v$.*

Proof. The module $\mathbb{C}[v, v^{-1}] \otimes \mathcal{O}(\hat{T})$ is generated by this vector $\boldsymbol{z}^{-\rho}$ since each coset of $\mathcal{H}_v$ has as a coset representation ζ^λ for a unique weight λ, which maps $\boldsymbol{z}^{-\rho}$ to a basis vector $\boldsymbol{z}^{-\lambda-\rho}$. The statement is thus clear. □

Finally, let us define certain elements $\boldsymbol{C}_w(\lambda)$ that resemble the $\boldsymbol{X}_w(\lambda)$ but may in some sense be more natural. We begin by recalling the Kazhdan-Lusztig basis C'_w for the (finite) Hecke algebra $\mathcal{H}_v$. It is uniquely characterized by the following properties. First, $\overline{C_w}' = C'_w$, where $x \longmapsto \bar{x}$ is the Kazhdan-Lusztig involution that sends $q \to q^{-1}$ and $T_w \to T^{-1}_{w^{-1}}$; and second, $v^{l(w)/2} C'_w = \sum_{u \leqslant w} P_{u,w}(v) T_u$, where $P_{u,w}(v) \in \mathbb{Z}[v]$ of degree $\leqslant \frac{1}{2}(l(w) - l(u) - 1)$ for $y < w$ and $P_{w,w} = 1$. The $P_{u,w}$ are the Kazhdan-Lusztig polynomials. (Note that P is used differently in other parts of our paper.) Let $\boldsymbol{C}_w(\lambda) = v^{l(w)/2} C'_w \cdot \boldsymbol{z}^\lambda$, where the action is that of Theorem 28.

Proposition 30. *Let λ be a dominant weight and $w \in W$. We have $\boldsymbol{X}_w(\lambda) = \boldsymbol{C}_w(\lambda)$ if and only if the Schubert variety X_w is rationally smooth in the sense of Kazhdan and Lusztig [14].*

Proof. Each Kazhdan-Lusztig $P_{u,w}$ with $u \leqslant w$ is a polynomial in v with constant term equal to 1, so $\boldsymbol{X}_w(\lambda) = \boldsymbol{C}_w(\lambda)$ if and only if $P_{u,w} = 1$ for all $u \leqslant w$. By Theorem A2 of [14], this is true if and only if X_w is rationally smooth. $\square$

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