ODD KHOVANOV HOMOLOGY FOR HYPERPLANE ARRANGEMENTS

ZSUZSANNA DANCSO AND ANTHONY LICATA

Abstract. We define several homology theories for central hyperplane arrangements, categorifying well-known polynomial invariants including the characteristic polynomial, Poincare polynomial, and Tutte polynomial. We consider basic algebraic properties of such chain complexes, including long-exact sequences associated to deletion-restriction triples and dg-algebra structures. We also consider signed hyperplane arrangements, and generalize the Odd Khovanov homology of Ozsvath-Rassmussen-Szabo from link projections to signed arrangements. We define hyperplane Reidemeister moves which generalize the usual Reidemeister moves from link projections to signed arrangements, and prove that the chain homotopy type associated to a signed arrangement is invariant under hyperplane Reidemeister moves.

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1. Introduction

Let $\Bbbk$ be a field, and let V be a vector space over $\Bbbk$ endowed with an inner product $\langle -, - \rangle : V \times V \to \Bbbk$. A *vector arrangement* is a collection of vectors $\nu_1, ..., \nu_n$ in V. A vector arrangement determines an arrangement of hyperplanes $\mathcal{H} = \{V; H_1, ..., H_n\}$ in V, with $H_i = \nu_i^\perp = \{v \in V : \langle v, \nu_i \rangle = 0\}$. We allow the degenerate case $\nu_i = 0$, in which case $H_i = V$. For $S \subset [n] = \{1, \ldots, n\}$, set $H_S = \cap_{s \in S} H_s$. Important features of the hyperplane arrangement $\mathcal{H}$ are captured by polynomial invariants associated to the arrangement; here "polynomial invariant" means a polynomial which depends only on the associated matroid, that is, only on the linear dependencies between hyperplanes. An example is the *characteristic polynomial* $\chi(\mathcal{H})$ of $\mathcal{H}$,

$$\chi(\mathcal{H}) = \sum_{S \subseteq [n]} (-1)^{|S|} t^{\dim H_S},$$

which generalizes the chromatic polynomial of a graph, [TO]. An important feature of the characteristic polynomial is that it satisfies a deletion-restriction relation,

$$\chi(\mathcal{H}) = \chi(\mathcal{H} - H_i) - \chi(\mathcal{H}^{H_i}), \tag{1}$$

where $\mathcal{H}^{H_i}$ is the restriction of $\mathcal{H}$ to a hyperplane H_i and $\mathcal{H} - H_i$ is the arrangement with the hyperplane H_i deleted. (We refer to Section 2 for complete definitions of deletion and restriction). Similar deletion-restriction relations hold for other polynomials associated to hyperplane arrangements, including the Poincare polynomial and the two-variable Tutte polynomial.

In this paper we categorify these invariants, upgrading them from polynomials to homology theories. Our constructions are modeled on the Odd Khovanov homology of Ozsvath-Rassmussen-Szabo [ORSz] which categorifies the Jones polynomial of links in the three-sphere. In particular, the relation (1) becomes a long exact sequence in homology, as expected by analogy with the Skein relation for the Jones polynomial and the resulting long-exact sequence in (odd) Khovanov homology.

The first part of the paper considers homology theories for unsigned hyperplane arrangements. These constructions are free from the restrictions imposed by isotopy invariance in the theory of link homologies, and as a result there is a lot of freedom in the definition of boundary maps between chain groups. For example, in considering categorifications of the characteristic polynomial, we see that essentially the same chain groups can be made into a chain complex using two very different choices of boundary maps. One choice of boundary maps gives a chain complex $\mathcal{C}_d(\mathcal{H})$ which essentially generalizes to hyperplane arrangements earlier work [HGR] on categorification of the chromatic polynomial of a graph. A second choice of boundary map, however, gives a completely different complex $\mathcal{C}_\partial(\mathcal{H})$, which turns out to be compatible with a multiplication defined at the chain level; thus this second choice assigns a differential graded algebra to each central hyperplane arrangement. Similar categorifications of the Poincare and Tutte polynomials are defined in the body of the paper. A deletion-restriction triple $(\mathcal{H}, \mathcal{H} - H_i, \mathcal{H}^{H_i})$ gives rise to a short exact sequence of chain complexes in all of our homology theories, though the resulting long exact sequences of homology behave quite differently for the two different choices of boundary map.

The second part of the paper considers signed hyperplane arrangements, that is, arrangements with a sign assigned to each hyperplane. A planar projection of a link in the three-sphere defines a signed hyperplane arrangement, and in Section 7 we generalize the Odd Khovanov homology of [ORSz] from link projections to arbitrary signed central arrangements. In low dimensional topology one considers link projections up to the equivalence relations defined by Reidemeister moves. These moves, too, admit a generalization to the combinatorial world of signed arrangements, and the main point of the second part of the paper is to explain how much of the basic structure of Odd Khovanov homology, including the Skein long-exact sequence and Reidemeister invariance, generalizes.

All of the chain complexes we use are essentially straightforward generalizations to hyperplane arrangements of existing constructions for links [Kh, ORSz]. Nevertheless, these generalizations seem both sufficiently natural and sufficiently interesting as to warrant further investigation. For example, the differential graded algebra structure on chain groups associated to unsigned arrangements does not (as far as we are aware) occur at the chain level in other polynomial categorifications. Moreover, hyperplane arrangements and their signed analogs admit a duality, known as Gale duality, which greatly generalizes duality of planar graphs. The dual of a non-planar graph thus makes sense as a hyperplane arrangement. The considerations in Section 7 for signed hyperplane arrangements are perhaps the most interesting in the paper, though an unfortunate feature of moving from arrangements to signed arrangements is that the dg-algebra structure on chain groups is lost; in contrast, the Gale duality statement for signed arrangements becomes cleaner, as there is an obvious isomorphism between the chain complexes assigned to a signed arrangement and its Gale dual.

Finally, we point out that it has been a question for some time to find, given an arrangement $\mathcal{H}$, a natural bigraded vector space whose graded dimension is the Tutte poylnomial of $\mathcal{H}$. In light of the categorifications of low dimensional topology, it is also reasonable to modify this question and search instead for a chain complex whose graded Euler characteristic is the Tutte polynomial. For graphs, one such complex was defined in [JHR]. Two of the homology theories of the current paper give a solution for hyperplane arrangements. A homology theory which categorifies the Tutte polynomial of a matroid has also recently been investigated independently by A. Lowrance and M. Cohen.

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2. Linear algebra of vector arrangements

Let $\mathcal{V} = \{V; \nu_1, ..., \nu_n\}$ be a vector arrangement in a vector space V over $\mathbb{k}$, and define $\mathcal{H} = \{V; H_1, ..., H_n\}$ to be the associated arrangement of hyperplanes in V. The arrangement $\mathcal{H}$ is central, meaning that the intersection $\cap_i H_i \neq \emptyset$. Let

$$W = \{(w_1, \dots, w_n) \in \mathbb{k}^n \mid \sum_i w_i \nu_i = 0\} \tag{2}$$

denote the space of linear dependencies in $\mathcal{V}$, that is, the orthogonal complement of V in $\mathbb{k}^n$. The inner product on $\mathbb{k}^n$ induces an inner product $\langle -, - \rangle$ on W, thus identifying W with W^*.

To a subset $S \subset [n]$ there are three naturally associated vector spaces:

- $H_S = \cap_{i \in S} H_S$,
- $V_S = span\{\nu_i\}_{i \in S}$, and
- $W_S = \{w = (w_1, \dots, w_n) \in W \mid w_r = 0 \text{ for } r \notin S\}$.

For $s \in S$ and $r \notin S$, there are natural inclusions

$$H_S \hookrightarrow H_{S-s}, \quad V_S \hookrightarrow V_{S \cup r}, \quad W_S \hookrightarrow W_{S \cup r}, \tag{3}$$

and orthogonal projection maps

$$H_S \to H_{S \cup r}, \quad V_S \to V_{S-s}, \quad W_S \to W_{S-s}. \tag{4}$$

Remark 1. The spaces H_S and V_S are related by standard linear duality, in that V_S is the space of vectors orthogonal to H_S (eqvivalently, the space of linear functionals which vanish on it). The relationship between H_S and W_S is more subtle: the space of linear dependencies W comes equipped with n linear functionals, namely, the coordinate projections $\nu_i^\vee : w = (w_1, \dots, w_n) \mapsto w_i$. Via the identification $W \cong W^*$, $\nu_i^\vee$ can be thought of as the orthogonal projection of the standard basis vector $(0, ..., 1, ..., 0)$ onto W (where 1 appears in the i-th coordinate). Thus $\mathcal{V}^\vee = \{W; \nu_1^\vee, ..., \nu_n^\vee\}$ is another vector arrangement, known as the *Gale dual* of $\mathcal{V}$. Let $\mathcal{H}^\vee$ be the hyperplane arrangement associated to the vector arrangement $\mathcal{V}^\vee$; the defining hyperplanes of $\mathcal{H}^\vee$ are $H_i^\vee = \ker(\nu_i^\vee)$. Then the space W_S above is given by $W_S = H_{S^c}^\vee = \cap_{i \notin S} H_i^\vee$. Thus the spaces $\{H_S\}_{S \subset [n]}$ and $\{W_S\}_{S \subset [n]}$ are exchanged by Gale duality. Note that if the vectors ν_i generate V, then the canonical inner products on V and W are related by $\langle \nu_i, \nu_j \rangle = -\langle \nu_i^\vee, \nu_j^\vee \rangle$. This follows from the fact that $\nu_i + \nu_i^\vee = x_i$ for all i, where the x_i are the standard orthonormal basis vectors of $\mathbb{k}^n$.

The inclusions and projections (3) and (4) induce maps of exterior algebras. We will denote all the maps which increase the size of the subset S by d's, and all those which decrease subset size by b's ("b" is a backwards "d"). Thus we have

$$\wedge^\bullet(H_S) \xrightarrow{d_{S,r}} \wedge^\bullet(H_{S \cup r}), \quad \wedge^\bullet(V_S) \overset{d_{S,r}}{\hookrightarrow} \wedge^\bullet(V_{S \cup r}), \quad \wedge^\bullet(W_S) \overset{d_{S,r}}{\hookrightarrow} \wedge^\bullet(W_{S \cup r}), \tag{5}$$

and

$$\wedge^\bullet(H_S) \overset{b_{S,s}}{\hookrightarrow} \wedge^\bullet(H_{S-s}), \quad \wedge^\bullet(V_S) \xrightarrow{b_{S,s}} \wedge^\bullet(V_{S-s}), \quad \wedge^\bullet(W_S) \xrightarrow{b_{S,s}} \wedge^\bullet(W_{S-s}). \tag{6}$$

However, the vectors from the original vector arrangement $\mathcal{V}$ can also be used to define maps between exterior algebras by wedging and contracting. For $s \in S$, $r \notin S$, we define:

$$\wedge^\bullet H_S \xrightarrow{w_{S,s}} \wedge^{\bullet+1} H_{S-s}, \quad \wedge^\bullet V_S \xrightarrow{w_{S,r}} \wedge^{\bullet+1} V_{S \cup r}, \quad \wedge^\bullet W_S \xrightarrow{w_{S,r}} \wedge^{\bullet+1} W_{S \cup r}. \tag{7}$$

$$(8) \qquad \wedge^\bullet H_S \xrightarrow{c_{S,r}} \wedge^{\bullet-1} H_{S\cup r}, \quad \wedge^\bullet V_S \xrightarrow{c_{S,s}} \wedge^{\bullet-1} V_S, \quad \wedge^\bullet W_S \xrightarrow{c_{S,s}} \wedge^{\bullet-1} W_{S-s}.$$

The maps in (7) act by wedging on the left by ν_s, ν_r, and $\nu_r^\vee$, respectively. In the first of these, note that ν_s is considered as an element of H_{S-s} by orthogonal projection from V. Similarly, for the third map we consider $\nu_r^\vee$ as an element of $W_{S\cup r}$.

The maps in (8) act by contraction with the vectors ν_r, ν_s and $\nu_s^\vee$ respectively. To define the first map, think of ν_r as an element of H_S by projecting it there, and note that for any element $h \in \wedge^\bullet H_S$ the image $(\nu_r \perp h)$ lies in $\wedge^\bullet H_{S\cup r}$, since ν_r is orthogonal to $H_{S\cup r}$. Similarly, contraction with $\nu_s^\vee$ on $\wedge^\bullet W_S$ has image in $\wedge^{\bullet-1} W_{S-s}$ since $\nu_s^\vee$ is orthogonal to W_{S-s}. However, in the case of V_S, ν_s is not necessarily orthogonal to V_{S-s}, so the image does not lie in $\wedge^{\bullet-1} V_{S-s}$. Note also that in case of H_S and W_S, the wedge and contraction maps are linear duals to each other (via the identifications $H_S \cong H_S^*$ and $W_S \cong W_S^*$ induced by the inner product). This is not true for V_S: the linear dual of $w_{S,r}$ is $c_{S\cup r,r}$ composed with an orthogonal projection onto V_{S-s}. This difference will account for the extra dg-algebra structure that can be introduced on chain complexes which are constructed from the spaces H_S and W_S and the maps w and c, but that cannot (as far as we know) be introduced on chain complexes constructed from the spaces V_S.

Remark 2. All of the algebras we consider in this paper are naturally $\mathbb{Z}$-graded, and they will be considered as superalgebras for the $\mathbb{Z}_2$ grading induced from the $\mathbb{Z}$ grading. Tensor products are always taken in the category of superalgebras. Thus $(a\otimes 1)(1\otimes b) = (-1)^{\deg a \cdot \deg b}(1\otimes b)(a \otimes 1)$, and in this way there is an algebra isomorphism $\wedge^\bullet(V \oplus W) \cong \wedge^\bullet V \otimes \wedge^\bullet W$.

Remark 3. We have taken $\Bbbk$ to be a field for convenience, but indeed almost all constructions in this paper may be carried out over the integers, or over an arbitrary commutative ring. The one exception is in one part of Section 7, where we must work over a field (see Remark 6).

2.1. **Deletion and Restriction.** On the level of vector arrangements, a *subarrangment* of $\mathcal{V}$ is an arrangement in the same ambient space consisting of a subset of the vectors in $\mathcal{V}$. For $\nu_i \in \mathcal{V}$, the *deletion* of ν_i is the operation which results in the subarrangement with ν_i removed, denoted $\mathcal{V} - \nu_i$. Given $\nu_i \in \mathcal{V}$, the *restriction* $\mathcal{V}^{\nu_i}$ is an arrangement in the orthogonal complement $\nu_i^\perp \subseteq V$, consisting of the vectors $\{P(\nu_j) : j \neq i\}$, where P is the orthogonal projection to $\nu_i^\perp$.

The corresponding notions for hyperplane arrangements follow from the above. A *subarrangement* of $\mathcal{H}$ is an arrangement consisting of a subset of hyperplanes in $\mathcal{H}$, in the same ambient vectorspace. The arrangement obtained by deleting the hyperplane H_i from $\mathcal{H}$ is denoted $\mathcal{H} - H_i$.

Given a hyperplane $H_i \in \mathcal{H}$, the *restriction* of $\mathcal{H}$ to H_i is the arrangement $\mathcal{H}^{H_i} = \{H_i; H_1 \cap H_i, ..., H_{i-1} \cap H_i, H_{i+1} \cap H_i, ..., H_k \cap H_i\}$.

Deletion and restriction are Gale dual notions: $(\mathcal{H} - H_i)^\vee = (\mathcal{H}^\vee)^{H_i^\vee}$ and $(\mathcal{H}^{H_i})^\vee = \mathcal{H}^\vee - H_i^\vee$, and similarly for vector arrangements.

3. Polynomials associated to Hyperplane arrangements

3.1. **The Characteristic Polynomial.** As before, for $S \subseteq [n]$, let $H_S = \bigcap_{s\in S} H_s$. The characteristic polynomial (see [TO]) of the central hyperplane arrangement $\mathcal{H}$ is defined as

$$\chi(\mathcal{H}, q) = \sum_{S\subseteq[n]} (-1)^{|S|}(1+q)^{\dim H_S}. \tag{9}$$

This is a slightly non-standard normalization, and the definition in [TO] would be given by the substitution $t = 1 + q$.

There is a closely related, though distinct, polynomial that also occurs as an Euler characteristic in categorification:

$$\bar{\chi}(\mathcal{H}, q) = \sum_{S\subseteq[n]} q^{|S|}(1-q)^{\dim H_S} \tag{10}$$

Informally we will refer to both of the above polynomials as characteristic polynomials.

3.2. **The Poincare Polynomial.** The Poincare polynomial of a hyperplane arrangement $\pi(\mathcal{H}, t)$ contains the same information as the characteristic polynomial, as they are related by a change of variables [TO]. A convenient state sum definition of the Poincare polynomial (in a slightly unusual normalization) is

$$\pi(\mathcal{H}, q) = \sum_{S\subseteq[n]} (-1)^{|S|}(1+q)^{\dim V_S} \tag{11}$$

3.3. **The Tutte Polynomial.** The Tutte polynomial of $\mathcal{H}$ is usually defined as

$$T(\mathcal{H}; x, y) = \sum_{S\subseteq[n]} (x-1)^{\dim H_S - \dim H_{[n]}}(y-1)^{\dim W_S}.$$

The version we will categorify is the analogue of that used for graphs in [HGR], given by the state sum formula

$$\hat{T}(\mathcal{H}; x, y) = \sum_{S\subseteq[n]} (-1)^{|S|}(1+x)^{\dim H_S}(1+y)^{\dim W_S}. \tag{12}$$

The relationship between these two polynomials is $\hat{T}(\mathcal{H}; x, y) = (-1)^k(-x-1)^{dim H_{[n]}}T(-x,-y)$.

3.4. **Relations from deletion-restriction.** All of the polynomials defined above satisfy *deletion-restriction formulas*. If $H_l \in \mathcal{H}$ is a given hyperplane in the arrangement $\mathcal{H}$, $\mathcal{H} - H_l$ is the subarrangement produced by deleting H_l from $\mathcal{H}$, and $\mathcal{H}^{H_l}$ is the restriction to H_l, then

$$\chi(\mathcal{H}, q) = \chi(\mathcal{H} - H_l, q) - \chi(\mathcal{H}^{H_l}, q), \quad \text{and} \quad \bar{\chi}(\mathcal{H}, q) = \chi(\mathcal{H} - H_l, q) + q\chi(\mathcal{H}^{H_l}, q). \tag{13}$$

Similar relations hold for the Poincare and Tutte polynomials if H_l is non-degenerate, i.e., if $\nu_l \neq 0$:

$$\pi(\mathcal{H}, q) = \pi(\mathcal{H} - H_l) - (1+q)\pi(\mathcal{H}^{H_l}), \tag{14}$$

and

$$\hat{T}(\mathcal{H}; x, y) = \hat{T}(\mathcal{H} - H_l; x, y) - \hat{T}(\mathcal{H}^{H_l}; x, y). \tag{15}$$

4. Categorifications for unsigned arrangements

In this section we will describe several homology theories which categorify the characteristic, Poincare, and Tutte polynomials. The different constructions arise from the freedom to choose between the H_S, V_S and W_S spaces (and their tensor products) to build chain groups, and between the natural inclusion/projection maps versus the wedge/contraction maps to construct differentials. We develop categorifications of the two characteristic polynomials, using the spaces H_S, in this section. The homology theories categorifying the Poincare and Tutte polynomials are very similar, and we will only state the results and highlight where the proofs differ from those for the characteristic polynomials.

4.1. **Hypercubes associated to a hyperplane arrangement.** In the vein of [Kh] and [HGR], we use the state sum formula (9) to construct a chain complex, the graded Euler characteristic of which is the characteristic polynomial by design. The first step is to arrange the terms of the formula on the vertices of a cube, in this case the vertices correspond to subsets $S \subseteq [n]$. The space $\wedge^\bullet H_S$ is placed at the vertex corresponding to S.

We illustrate the cube on the example of the *braid arrangement* in $\mathbb{R}^3$. This arrangement consists of three hyperplanes defined by the equations vector arrangement $\nu_1 = (1, -1, 0)$, $\nu_2 = (0, 1, -1)$, and $\nu_3 = (-1, 0, 1)$. By placing the spaces $\wedge^\bullet H_S$ at vertices, and connecting them by an edge if the subsets S differ only by one element, we obtain the following 3-dimensional cube:

$$\begin{tikzcd}
 & \wedge^\bullet H_1 \arrow[r] \arrow[dr] & \wedge^\bullet(H_1\cap H_2) \arrow[dr] & \\
\wedge^\bullet \mathbb{R}^3 \arrow[ur] \arrow[r] \arrow[dr] & \wedge^\bullet H_2 \arrow[ur] \arrow[dr] & \wedge^\bullet(H_3\cap H_1) \arrow[r] & \wedge^\bullet(H_1\cap H_2\cap H_3) \\
 & \wedge^\bullet H_3 \arrow[ur] \arrow[r] & \wedge^\bullet(H_2\cap H_3) \arrow[ur] &
\end{tikzcd}$$

Note that the vertices of the cube are organized into columns according to the size of the subsets S.

In the next section we will discuss the maps associated to the cube edges which make up the differentials; here we only define the chain groups, which are obtained by "flattening" the cube along the "columns". Thus we get chain groups

$$C^i = \bigoplus_{S\subseteq[n],\,|S|=i} \wedge^\bullet H_S.$$

Thus the chain complex $\mathcal{C}$ is given by

$$\mathcal{C} = \bigoplus_{S\subseteq[n]} C_S, \text{ where } C_S = \wedge^\bullet H_S. \tag{16}$$

This vector space has a natural bi-grading given by $\deg \wedge^j H_S = (|S|, j)$. Note that the Euler characteristic with respect to the first grading component is

$$\chi_q(\mathcal{C}) = \sum_{S\subseteq[n]} (-1)^{|S|}(1+q)^{\dim H_S} = \chi(\mathcal{H}, q).$$

Thus, if we impose differentials for which the homological degree of C_S is $|S|$, as we will do in our first construction, the graded Euler characteristic of the complex will be the characteristic polynomial. In the second construction, discussed in Section 4.2.2, we will need to shift the grading, and the resulting graded Euler characteristic will yield the second characteristic polynomial $\bar{\chi}(\mathcal{H}, q)$.

There are also natural cubes involving the spaces V_S,

$$\mathcal{C}^P = \bigoplus_{S \subseteq [n]} \wedge^\bullet V_S,$$

leading to a categorification of the Poincare polynomial.

To categorify the Tutte polynomial, we will use the tensor product of the spaces H_S and W_S.

$$\mathcal{C}^T = \bigoplus_{S \subseteq [n]} \wedge^\bullet H_S \otimes \wedge^\bullet W_S.$$

This vector space is triply-graded, with $\deg \wedge^i H_S \otimes \wedge^j W_S = (|S|, i, j)$. As with the chain groups used to categorify the characteristic polynomial, one choice of differential will be homogeneous for this grading convention, while another choice will require us to make minor adjustments to the definition of the gradings.

4.2. **Boundary maps.** The first class of categorifications uses differentials which arise from the natural inclusion and orthogonal projection maps explained in Section 2. In fact these inclusion and projection maps can be used to define chain complexes in even more general settings; for example, instead of exterior algebras, we could use instead symmetric algebras, or in principle any other exact functor from vector spaces to graded vector spaces. Moreover, these boundary maps are rather straightforward odd generalization from graphs to hyperplane arrangements of the differentials defined in [HGR, JHR].

In contrast, the boundary maps defined in Section 4.2.2 use the wedge maps as differentials. These maps use in a fundamental way the structure in the exterior algebra, and the resulting chain complexes have a dg-algebra structure and simpler theorems for deletion/restriction.

4.2.1. *Boundary maps arising from natural inclusion or orthogonal projection.* We describe the construction for the characteristic polynomial in detail. Recall that for each edge of the cube corresponding to a subset $S \subseteq [n]$ and an element $r \notin S$ there are maps induced by the orthogonal projections

$$d_{S,r} : \wedge^\bullet H_S \to \wedge^\bullet H_{S \cup r}.$$

These are of degree $(1, 0)$ with respect to the natural bi-grading defined in Section 4.1.

To define differentials, we want to take the direct sums of the maps. However, as in Khovanov homology, we first need to introduce signs to make the (a priori commutative) cube anti-commutative, this is needed for the square of the differential to be zero. To achieve this, we set

$$\varepsilon_{S,r} = \begin{cases} -1 & \text{if } |\{s \in S, s < r\}| = \text{odd} \\ 1 & \text{otherwise.} \end{cases}$$

The differentials are the sums with appropriate signs of the edge maps $d_{S,r}$ going from the algebras in column i to the ones in column $(i+1)$:

$$d^i = \bigoplus_{|S|=i,\ r \notin S} \varepsilon_{S,r} d_{S,r}.$$

Let us illustrate this on the braid arrangement example:

$$
\begin{tikzcd}
 & \wedge^\bullet H_1 \arrow[r, "-d"] \arrow[dr, "-d"'] & \wedge^\bullet(H_1\cap H_2) \arrow[dr, "d"] & \\
\wedge^\bullet\mathbb{R}^3 \arrow[ur, "d_{\emptyset,1}"] \arrow[r, "d"] \arrow[dr, "d"'] \arrow[ddd, dashed] & \wedge^\bullet H_2 \arrow[ur, "d"] \arrow[dr, "d"'] & \wedge^\bullet(H_3\cap H_1) \arrow[r, "-d"] & \wedge^\bullet(H_1\cap H_2\cap H_3) \arrow[ddd, dashed] \\
 & \wedge^\bullet H_3 \arrow[ur, "-d"] \arrow[r, "d"'] \arrow[dd, dashed] & \wedge^\bullet(H_2\cap H_3) \arrow[ur, "d"'] \arrow[dd, dashed] & \\
 & & & \\
\wedge^\bullet\mathbb{R}^3 \arrow[r, "\oplus d_{\dots}"] & \bigoplus\wedge^\bullet H_i \arrow[r, "\oplus\pm d_{\dots}"] & \bigoplus\wedge^\bullet(H_i\cap H_j) \arrow[r, "\oplus\pm d_{\dots}"] & \wedge^\bullet(H_1\cap H_2\cap H_3)
\end{tikzcd}
$$

Definition 1. We denote the resulting chain complex by $C_d^\bullet(\mathcal{H},\Bbbk)$, and the homology by $H_d^\bullet(\mathcal{H},\Bbbk)$, and call it (odd) *characteristic homology.*

Proposition 4.1. *The homology $H_d^\bullet(\mathcal{H},\Bbbk)$ has graded Euler characteristic equal to the characteristic polynomial:*

$$\chi_q(H_d^\bullet(\mathcal{H},\Bbbk)) = \chi(\mathcal{H},q).$$

Proof. The graded Euler characteristic of the chain complex is the characteristic polynomial by design, as noted in Section 4.1. As the chain groups are finite dimensional and the differential is degree zero with respect to the second grading, the graded Euler characteristic of the homology is the same. $\square$

Note that the sign assignment for the differentials made use of the ordering of the hyperplanes. The following lemma states that the end result is, up to isomorphism, order-independent.

Lemma 4.2. *For any permutation $\sigma \in S_n$ and arrangement $\mathcal{H} = \{V; H_1, ..., H_n\}$, let $\mathcal{H}_\sigma := \{V; H_{\sigma(1)}, ..., H_{\sigma(n)}\}$ denote the permuted arrangement. Then*

$$H_d^\bullet(\mathcal{H}_\sigma) \cong H_d^\bullet(\mathcal{H}).$$

Proof. Since S_n is generated by transpositions, it is enough to prove the theorem for $\sigma = (i, i+1)$, $i \in \{1, ..., n-1\}$.

We prove that the chain complexes $\mathcal{C}(\mathcal{H})$ and $\mathcal{C}(\mathcal{H}_\sigma)$ are isomorphic. Acting by σ does not change the chain groups, it just permutes the direct summands of a fixed chain group. However, some of the signs for the differentials differ in $\mathcal{C}(\mathcal{H})$ and $\mathcal{C}(\mathcal{H}_\sigma)$. Specifically, $\epsilon_{S,r}$ changes if and only if $r = i$ and $(i+1) \in S$ or $r = i+1$ and $i \in S$.

An isomorphism of the chain complexes $\Phi : \mathcal{C}(\mathcal{H}) \to \mathcal{C}(\mathcal{H}_\sigma)$ is given by letting Φ be multiplication by (-1) on the components $\wedge^\bullet H_S$ where $\{i, i+1\} \subseteq S$, and letting Φ act by the identity on all other summands. It is simple combinatorics to check that this map commutes with the differentials, hence it gives rise to a chain isomorphism. $\square$

Example 1. One could in principle construct a chain complex from the spaces H_S directly, rather than first taking the exterior algebra. The resulting complex is less interesting, however, as this simple example will illustrate. Consider the hyperplane arrangement in $\mathbb{R}^3$ consisting of two planes defined by vectors $\nu_1 = (1,-1,0)$ and $\nu_2 = (0,1,-1)$. The cube in this case is a square. The proposed "simple" complex,

$$
\begin{tikzcd}
 & H_1 \arrow[dr, "-d_3"] & \\
\mathbb{R}^3 \arrow[ur, "d_1"'] \arrow[dr, "d_2"] & & H_1\cap H_2 \\
 & H_2 \arrow[ur, "d_4"'] &
\end{tikzcd}
$$

is acyclic. However, by taking exterior algebras before flattening the cube, we get non-zero homology in homological degree 0, linearly spanned by the two elements $\{x_1x_2x_3, x_1x_3 - x_1x_2 - x_2x_3\}$.

Note that the graded Euler characteristic of this is in fact $q^3 + q^2$, in agreement with the characteristic polynomial. □

We can assign differentials to the V_S spaces the same way: the complex $\mathcal{C}_d^P = \bigoplus_{S\subseteq[n]} \wedge^\bullet V_S$ with the natural bi-grading and differentials

$$d = \bigoplus_{S\subseteq[n];r\notin S} \varepsilon_{S,r} d_{S,r}$$

gives rise to homology groups $H_d^{P\bullet}(\mathcal{H})$.

Similarly, consider $\mathcal{C}_d^T := \bigoplus_{S\subseteq[n]} \wedge^\bullet H_S \otimes \wedge^\bullet W_S$ with the triple grading $\deg \wedge^i H_S \otimes \wedge^j W_S = (|S|, i, j)$ and differentials

$$d = \bigoplus_{S\subseteq[n];r\notin S} \varepsilon_{S,r} d_{S,r} \otimes d_{S,r},$$

and call the resulting homology $H_d^{T\bullet}(\mathcal{H})$.

The proof of the theorems regarding the characteristic polynomial can be repeated word by word to prove the following:

Proposition 4.3. *The cohomologies $H_d^{P\bullet}(\mathcal{H})$ and $H_d^{T\bullet}(\mathcal{H})$ categorify the Poincare polynomial and the Tutte polynomial, respectively:*

$$\chi_q(H_d^{P\bullet}(\mathcal{H})) = \pi(\mathcal{H}, q), \text{ and } \chi_q(H_d^{T\bullet}(\mathcal{H})) = \hat{T}(\mathcal{H}, x, y).$$

Furthermore, permuting the vectors in the vector arrangement induces isomorphisms of the chain groups.

4.2.2. *Wedge maps.* We now define a second type of differentials using the same underlying chain groups but shifting the grading. This construction has several advantages over the previous one, most notably these chain complexes admit a differential graded algebra structure, so the cohomologies are themselves algebras. Moreover, the long exact sequences for deletion and restriction (described in Section 5.1) are simpler than those for the homology theories defined in the previous section.

The differentials of this section only work for the characteristic and Tutte complexes – that is, the complexes which use the spaces H_S and W_S – not for the Poincare complex (which uses V_S).

For the characteristic homology, the differential $\partial : \mathcal{C} \to \mathcal{C}$ is defined using the wedge maps $w_{S,s} : \wedge^\bullet H_S \to \wedge^{\bullet+1} H_{S-s}$ explained in Section 2, Equation (7). We set

$$\partial := \bigoplus_{S\subseteq[n],\, s\in S} w_{S,s}.$$

From now on we will denote the maps $w_{S,s}$ by the name $\partial_{S,s}$ as well. To distinguish between the chain complexes with different differentials when needed, we will write $\mathcal{C}_d$ and $\mathcal{C}_\partial$.

Note that the hypercube with vertices $\wedge^\bullet H_S$ and edge maps $\partial_{S,s}$ is anticommutative by definition, hence $\partial^2 = 0$.

Observe that $\deg \partial = (-1, 1)$ with respect to the natural bigrading of the previous section. Thus in this section we will redefine the bigrading by setting $\deg \wedge^i H_S = (|S| + i, i)$. With respect to this grading $\deg \partial = (0, 1)$.

Let $H^\bullet_\partial$ denote the resulting homology. (Note that the two gradings switched roles: now the second degree is the homological degree.) In this construction we never used that the hyperplanes were ordered, so $H^\bullet_\partial$ is order-independent for free. Note also that there were no sign choices required in the definition of the differential on the complex.

Lemma 4.4. *The graded Euler characteristic of $H^\bullet_\partial$ is*

$$\chi_q(H^\bullet_\partial)(\mathcal{H}) = \bar{\chi}(\mathcal{H}, q),$$

where $\bar{\chi}$ is the version of the characteristic polynomial defined in (10).

Proof. Immediate from the definition. □

As for the Tutte polynomial, there are several possible differentials to chose from. One condition which is convenient to impose is that the cube should be anti-commutative naturally, without the order-dependent sign assignments. One such differential is as follows.

We define the differentials on $\mathcal{C}^T$ by

$$\partial^T = \bigoplus_{S \subseteq [n], s \in S} \partial^T_{S,s}, \text{ where } \partial^T_{S,s} = w_{S,s} \otimes b_{S,s}. \tag{17}$$

With respect to the natural grading, $\deg \partial^T = (-1, 1, 0)$. We set the new grading convention to be $\deg \wedge^i H_S \otimes \wedge^j W_S = (|S| + i, i, j)$. In this grading $\deg \partial^T = (0, 1, 0)$. We denote the homology by $H^{T\bullet}_\partial$.

Proposition 4.5. *The graded Euler characteristic of $H^{T\bullet}_\partial$ is*

$$\chi_q(H^{T\bullet}_\partial(\mathcal{H})) = (-x)^k \hat{T}(\mathcal{H}; -\frac{1}{x}, -2 - xy).$$

□

4.3. **Differential graded algebra structure.** One advantage of the wedge differentials of the previous subsection is that the resulting homology groups admit a compatible multiplication.

For the characteristic homology, this multiplication is defined at the chain level as a map

$$m : C_S \otimes C_T \to C_{S \cup T}.$$

We set m to be 0 when $S \cap T \neq \emptyset$; for $S \cap T = \emptyset$ and $h \in C_S$, $h' \in C_T$, we set

$$m(h \otimes h') = h \wedge h'.$$

Here the wedging takes place inside $\wedge^\bullet(H_{S \cup T})$, which is well-defined after first using orthogonal projection to send both h and h' into $\wedge^\bullet(H_{S \cup T})$. If h and h' are homogeneous elements of respective bi-degrees $(|S| + i, i)$ and $(|T| + j, j)$, then

$$\deg m(h \otimes h') = \deg(h \wedge h') = (|S| + |T| + i + j, i + j),$$

so the multiplication respects both gradings.

Proposition 4.6. *$(\mathcal{C}, \partial, m)$ is a differential graded algebra, and hence $H^\bullet_\partial(\mathcal{H})$ is a graded algebra.*

Proof. We need to show that the multiplication is compatible with the differential:

$$\partial(m(h \otimes h') = m((\partial h) \otimes h') + (-1)^j m(h \otimes (\partial h')),$$

where $j = \deg_2(h)$ is the second degree of h (i.e., its exterior algebra degree). A short computation shows that both sides are equal to $\sum_{r \in S \cup T} \nu_r \wedge h \wedge h' \in \bigoplus_{r \in S \cup T} C_{S \cup T - r}$. $\square$

For the Tutte chain groups, we define the multiplication on $\mathcal{C}^T_\partial(\mathcal{H})$ in a similar way:

$$m^T : (\wedge^{i_1} H_S \otimes \wedge^{j_1} W_S) \otimes (\wedge^{i_2} H_T \otimes \wedge^{j_2} W_T) \to (\wedge^{i_1+i_2} H_{S\cup T} \otimes \wedge^{j_1+j_2} W_{S \cup T}),$$

$$m^T((h_1 \otimes w_1) \otimes (h_2 \otimes w_2)) = (h_1 \wedge h_2) \otimes (w_1 \wedge w_2) \quad \text{if} \quad S \cap T = \emptyset, \tag{18}$$

and set the multiplication to be zero when $S \cap T \neq \emptyset$.

Proposition 4.7. *The multiplication m^T is compatible with all three gradings and makes $\mathcal{C}^T$ into a triply-graded dg-algebra. As a result, $H^{T\bullet}_\partial$ is a triply-graded algebra.*

Proof. What needs to be verified is that for

$$h_1 \otimes w_1 \in \wedge^{i_1} H_S \otimes \wedge^{j_1} W_S \quad \text{and} \quad h_2 \otimes w_2 \in \wedge^{i_2} H_S \otimes \wedge^{j_2} W_S,$$

$$m^T(\partial^T(h_1 \otimes w_1) \otimes (h_2 \otimes w_2)) + (-1)^{i_1} m^T((h_1 \otimes w_1) \otimes \partial^T(h_2 \otimes w_2)) = \partial^T(m(h_1 \otimes w_1 \otimes h_2 \otimes w_2)).$$

This is a straightforward calculation. $\square$

5. Properties

5.1. **Relations from deletion and restriction.** The following theorem is a categorification of the deletion-restriction formula (13):

Theorem 5.1. *There is a short exact sequence of chain complexes of the form*

$$0 \to C^{i-1,j}_d(\mathcal{H}^{H_l}) \xrightarrow{\iota} C^{i,j}_d(\mathcal{H}) \xrightarrow{\pi} C^{i,j}_d(\mathcal{H} - H_l) \to 0.$$

This induces long exact sequence for $H^\bullet_d$:

$$0 \to \ldots \to H^{i-1}_d(\mathcal{H}^{H_l}) \to H^i_d(\mathcal{H}) \to H^i_d(\mathcal{H} - H_l) \to H^i_d(\mathcal{H}^{H_l}) \to \ldots \tag{19}$$

Proof. The proof is along the same lines as the proofs of the corresponding theorems in [HGR] and [JHR], and we recall the basic points here.

We want to define chain maps ι and π satisfying

$$0 \to \bigoplus_{S \subseteq [n]-l} \wedge^\bullet (H_S \cap H_l) \xrightarrow{\iota} \bigoplus_{T \subseteq [n]} \wedge^\bullet H_T \xrightarrow{\pi} \bigoplus_{U \subseteq [n]-l} \wedge^\bullet H_U \to 0. \tag{20}$$

Note that

$$\bigoplus_{T \subseteq [n]} \wedge^\bullet H_T = \bigoplus_{S \subseteq [n]-l} \wedge^\bullet H_{S \cup l} \oplus \bigoplus_{U \subseteq [n]-l} \wedge^\bullet H_U,$$

and $H_S \cap H_l = H_{S \cup l}$. The essential idea is to set ι to be the natural inclusion and π the natural projection map with respect to this decomposition. However, this is only correct up to sign: π commutes with the differential d, but the signs $\varepsilon_{S,r}$ cause a commutativity issue with ι. In order to fix this, we replace the inclusion ι by the map ι' by setting $\iota' = \bigoplus_{S \subseteq [n]-l} \iota'_S$, where ι'_S is the natural inclusion of the component if the number of elements $\{s \in S, s > l\}$ is even, and multiplication by (-1) on the component if this number is odd. $\square$

For the second choice of differentials, the short exact sequence of chain complexes induces a split long exact sequence, compatible with the dg-algebra structure. For a dg-algebra $\mathcal{A}$, let $\breve{\mathcal{A}}$ denote the dg-algebra isomorphic to $\mathcal{A}$ as a chain complex, but with trivial multiplication (that is, with all products equal to 0).

Theorem 5.2. *There is a split short exact sequence of bi-graded dg-algebras*

$$0 \to \breve{\mathcal{C}}_\partial(\mathcal{H}^{H_l})[1] \xrightarrow{\iota} \mathcal{C}_\partial(\mathcal{H}) \xrightarrow{\pi} \mathcal{C}_\partial(\mathcal{H} - H_l) \to 0.$$

Thus we have an isomorphism of bigraded algebras

$$H^\bullet_\partial(\mathcal{H}) \cong \breve{H}^\bullet_\partial(\mathcal{H}^{H_l})[1] \rtimes H^\bullet_\partial(\mathcal{H} - H_l), \tag{21}$$

Remark 4. In the above theorem, [1] denotes a shift by $(1, 0)$ in the bigrading. In the semidirect product above, the action of $H_\partial(\mathcal{H} - H_l)$ on $H_\partial(\mathcal{H}^{H_l})$ is induced from the action at the chain level given by choosing a representative cycle $h \in \wedge^\bullet H_S \subset \mathcal{C}(\mathcal{H} - H_l)$, projecting h to $H_S \cap H_l$, and letting h act by multiplication on $H^\bullet_\partial(\mathcal{H}^{H_l})$.

Proof of Theorem 5.2. It is easy to check that Equation (20) with the natural inclusion and projection maps is indeed a short exact sequence of chain complexes (not of dg-algebras) for $(\mathcal{C}, \partial)$. This implies that there is a long exact sequence (of vector spaces) for the homology $H^\bullet_\partial$. A straightforward check verifies that the transition maps $\gamma^j : H^j_\partial(\mathcal{H} - H_l) \to H^{j+1}_\partial(\mathcal{H}^{H_l})$ coming from the snake lemma satisfy $\gamma^j = 0$ for all j. Hence, the long exact sequence for homology splits into a direct sum of short exact sequences for each homology degree, which, together take the form of

$$0 \to H^\bullet_\partial(\mathcal{H}^{H_l}) \xrightarrow{\iota} H^\bullet_\partial(\mathcal{H}) \xrightarrow{\pi} H^\bullet_\partial(\mathcal{H} - H_l) \to 0.$$

Note that in the short exact sequence of chain complexes, π is a map of dg-algebras, but ι isn't, at least not with respect to the previously defined algebra structure on $\mathcal{C}(\mathcal{H}^{H_l})$. However, if the graded vector space $\mathcal{C}(\mathcal{H}^{H_l})$ is considered as a dg-algebra with trivial multiplication, then ι is a map of dg-algebras. Hence, we get a short exact sequence of graded algebras for homology if we replace $H^\bullet_\partial(\mathcal{H}^{H_l})$ by $\breve{H}^\bullet_\partial(\mathcal{H}^{H_l})[1]$.

Furthermore, this short exact sequence is split by a map $i : H^\bullet_\partial(\mathcal{H} - H_l) \to H^\bullet_\partial(\mathcal{H})$, which is induced by the natural inclusion $i : \mathcal{C}(\mathcal{H} - H_l) \to \mathcal{C}(\mathcal{H})$, a map of dg-algebras. It is simple to check that the action defined by this splitting is as described in the statement of the theorem. ∎

The proofs above relied crucially on the fact that for a subset $S \subseteq [n] - l$, the space at hypercube vertex S in the chain complex of $\mathcal{H}^{H_l}$ (denote this by $H_S^{H_l}$) is $H_S \cap H_l$. This vector space can be identified with $H_{S \cup l}$, which participates in the chain complex of $\mathcal{H}$ at vertex $S \cup l$. Thus we can include the hypercube of chain groups corresponding to $\mathcal{H}^{H_l}$ as a face of the bigger hypercube corresponding to $\mathcal{H}$.

Before we state the deletion-restriction theorems for the Poincare and Tutte cohomologies, let us determine what the analogous hypercube relationships are for the W and V spaces.

For $S \subseteq [n] - l$, let $W_S^{H_l}$ denote the "W_S-space" of the vector arrangement associated to $\mathcal{H}^{H_l}$. That is, $W_S^{H_l}$ is the space of linear dependencies between $\{P(\nu_s) : s \in S\}$, where P stands for the orthogonal projection onto H_l. Note that $w_{s_1} P(\nu_{s_1}) + ... + w_{s_p} P(\nu_{s_p}) = 0$ if and only if $w_{s_1}\nu_{s_1} + ... + w_{s_p}\nu_{s_p} = w_l \nu_l$. So the map ϕ sending the vector w with non-zero coordinates $w_{s_1}, ..., w_{s_p}$ to $w + (0, ..., 0, -w_l, 0, ..., 0)$ is a canonical isomorphism $W_S^{H_l} \cong W_{S \cup l}$.

For the V_S spaces, the analogous statement is somewhat different, since $\dim \wedge^\bullet V_S = 2\dim \wedge^\bullet V_S^{H_l}$. There is a natural inclusion $\iota_1 : \wedge^\bullet V_S^{H_l} \hookrightarrow \wedge^\bullet V_S$ as vector spaces. Recall that $V_S^{H_l}$ is spanned by $\{P(\nu_s) : s \in S\}$. Define $\iota_1(P(\nu_s)) = \nu_l \wedge \nu_s$. It is a simple exercise to check that ι_1 is well-defined, injective, and that $\deg \iota_1 = (1,1)$. Let $\nu_l^{\perp S} = \{v \in V_{S\cup l} : \langle v, \nu_l\rangle = 0\}$ be the orthogonal complement of ν_l in $V_{S\cup l}$. Let $p_S : V_{S\cup l} \to \nu_l^{\perp S}$ be the orthogonal projection, and define $\iota_2(\nu_s|_{H_l}) = p_S(\nu_s)$, extended multiplicatively to the exterior algebra. It is easy to see that ι_2 is well-defined and injective. Hence $\iota_1 \oplus \iota_2 : \wedge^\bullet V_S^{H_l} \oplus \wedge^\bullet V_S^{H_l} \to \wedge^\bullet V_S$ is an isomorphism.

Theorem 5.3. *When $\nu_l \neq 0$, there is a short exact sequence of chain complexes*

$$0 \to \mathcal{C}_d^P(\mathcal{H}^{H_l})\{1\}[1] \oplus \mathcal{C}_d^P(\mathcal{H}^{H_l})[1] \xrightarrow{\iota_1 \oplus \iota_2} \mathcal{C}_d^P(\mathcal{H}) \xrightarrow{\pi} \mathcal{C}_d^P(\mathcal{H} - H_l) \to 0,$$

where $[1]$ stands for shifting the first degree, and $\{1\}$ denotes shifting the second (exterior algebra) degree. This induces long exact sequence for $H_d^{P\bullet}$ of the form

$$0 \to ... \to H_d^{T,i-1}(\mathcal{H}^{H_l})\{1\} \oplus H_d^{T,i-1}(\mathcal{H}^{H_l}) \to H_d^{T,i}(\mathcal{H}) \to H_d^{T,i}(\mathcal{H} - H_l) \to H_d^{T,i}(\mathcal{H}^{H_l}) \to ..., \tag{22}$$

where $H_d^{T,i-1}(\mathcal{H}^{H_l})\{1\}$ denotes a shift of the second (exterior algebra) degree up by 1.

Proof. Identical to the proof of Theorem 5.1, but using $\iota_1 \oplus \iota_2$ in place of ι. Again, π commutes with $d_{S,r}$, but on account of the sign assignments $\varepsilon_{S,r}$, $\iota_1 \oplus \iota_2$ needs to be adjusted in the same way as in the proof of Theorem 5.1. □

Theorem 5.4. *When $\nu_l \neq 0$, there is a short exact sequence of chain complexes*

$$0 \to \mathcal{C}_d^T(\mathcal{H}^{H_l})[1] \xrightarrow{\iota} \mathcal{C}_d^T(\mathcal{H}) \xrightarrow{\pi} \mathcal{C}_d^T(\mathcal{H} - H_l) \to 0,$$

where $[1]$ denotes a shift of the first degree; inducing a long exact sequence for $H_d^{T\bullet}$ of the form

$$0 \to ... \to H_d^{T,i-1}(\mathcal{H}^{H_l}) \to H_d^{T,i}(\mathcal{H}) \to H_d^{T,i}(\mathcal{H} - H_l) \to H_d^{T,i}(\mathcal{H}^{H_l}) \to ... \tag{23}$$

Proof. The proof of Theorem 5.1 applies verbatim. □

Theorem 5.5. *There is a split short exact sequence (semidirect product) of (tri-graded) dg-algebras*

$$0 \to \breve{\mathcal{C}}_\partial^T(\mathcal{H}^{H_l})[1] \xrightarrow{\iota} \mathcal{C}_\partial^T(\mathcal{H}) \xrightarrow{\pi} \mathcal{C}_\partial^T(\mathcal{H} - H_l) \to 0,$$

where $[1]$ denotes a degree shift by $(1,0,0)$ in the trigrading. Thus there is an isomorphism of algebras

$$H_\partial^{T\bullet}(\mathcal{H}) = \breve{H}_\partial^{T\bullet}(\mathcal{H}^{H_l})[1] \rtimes H_\partial^{T\bullet}(\mathcal{H} - H_l). \tag{24}$$

Here, as before, $H^{T\bullet}(\mathcal{H} - H_l)$ acts on $H^{T\bullet}(\mathcal{H}^{H_l})$ in the natural way inherited from the chain level: a representative $h \otimes w \in \wedge^\bullet H_S \otimes \wedge^\bullet W_S \subset \mathcal{C}^T(\mathcal{H} - H_l)$, thought of as an element of $\mathcal{C}(\mathcal{H}^{H_l})$ via orthogonal projections, acts by multiplication (wedging).

Proof. Same as the proof of Theorem 5.2. □

5.2. **Factorization property.** For two vector arrangements $\mathcal{V} = \{V^k; \nu_1, ..., \nu_n\}$ and $\mathcal{V}' = \{V'^l; \nu'_1, ..., \nu'_m\}$, the *product arrangement* is defined to be the vector arrangement

$$\mathcal{V} \times \mathcal{V}' = \{V \times V'; \nu_1 \times 0, ..., \nu_m \times 0, 0 \times \nu'_1, ..., 0 \times \nu'_m\}.$$

For the associated hyperplane arrangements $\mathcal{H} = \{V^k; H_1, ..., H_n\}$ and $\mathcal{H}' = \{V'^l; H'_1, ..., H'_m\}$, the product arrangement is the hyperplane arrangement given by

$$\mathcal{H} \times \mathcal{H}' = \{V \times V';\ H_1 \times V', ..., H_n \times V', V \times H'_1, ..., V \times H'_m\}.$$

Theorem 5.6. *The characteristic homology of the product arrangement is the tensor product of the characteristic homologies of the factors:*

$$H_d^\bullet(\mathcal{H} \times \mathcal{H}') \cong H_d^\bullet(\mathcal{H}) \otimes H_d^\bullet(\mathcal{H}'),\ \ H_\partial^\bullet(\mathcal{H} \times \mathcal{H}') \cong H_\partial^\bullet(\mathcal{H}) \otimes H_\partial^\bullet(\mathcal{H}').$$

Proof. The product arrangement has $n+m$ hyperplanes, and subsets of $[n+m]$ are in one-to-one correspondence with pairs of subsets $S \subseteq [n]$ and $T \subseteq [m]$. Note that

$$\bigcap_{s \in S} H_s \times V' = H_S \times V', \ \text{ and } \ (H_S \times V') \cap (V \times H_T) = H_S \times H_T,$$

so the components of the chain complex for the product arrangement are products of those for $\mathcal{H}$ and $\mathcal{H}'$. In other words, the component in the product arrangement corresponding to the pair of subsets S, T is $H_{S,T} = H_S \times H_T$. Then

$$\mathcal{C}(\mathcal{H} \times \mathcal{H}') = \bigoplus_{S \in [n],\, T \in [m]} \wedge^\bullet H_{S,T} \cong \bigoplus_{S \in [n],\, T \in [m]} \wedge^\bullet H_S \otimes \wedge^\bullet H_T = \mathcal{C}(\mathcal{H}) \otimes \mathcal{C}(\mathcal{H}'),$$

as bi-graded vector spaces, with both grading conventions.

Regarding the differentials, for $h \in \wedge^j H_S$ and $h' \in \wedge^{j'} H_T$, we need to check that

$$d_{\mathcal{H}\times\mathcal{H}'}(h \wedge h') = d_{\mathcal{H}}(h) \wedge h' + (-1)^{|S|} h \wedge d_{\mathcal{H}'}(h'),$$

and that

$$\partial_{\mathcal{H}\times\mathcal{H}'}(h \wedge h') = \partial_{\mathcal{H}}(h) \wedge h' + (-1)^{j} h \wedge \partial_{\mathcal{H}'}(h').$$

This is verified by a direct computation in both cases.

It remains to show that the algebra structure on $H_\partial^\bullet(\mathcal{H} \times \mathcal{H}')$ is the (super) tensor product of the algebra structures on the factors: for $a, b \in \mathcal{C}(\mathcal{H})$ and $a', b' \in \mathcal{C}(\mathcal{H}')$ we want that

$$m_{\mathcal{H}\times\mathcal{H}'}(a \otimes a', b \otimes b') = (-1)^{\deg_2 a \deg_2 b} m(a,b) \otimes m(a', b'),$$

where $\deg_2$ denotes the second (exterior algebra) degree. This is straightforward from the definition of multiplication. $\square$

The essential ingredient of the proof above was that $H_{S,T} = H_S \times H_T$. It is straightforward from the definitions that this holds true for the $V_{S,T}$ and $W_{S,T}$ spaces associated to the product arrangement as well: $V_{S,T} = V_S \times V_T$ and $W_{S,T} = W_S \times W_T$. Hence the proof can be repeated without change to produce similar "Kunneth Theorems" for the Poincare and Tutte cohomologies.

Proposition 5.7. *For the Poincare and Tutte cohomologies of the product arrangement, we have*

$$H^{P\bullet}(\mathcal{H} \times \mathcal{H}') \cong H^{P\bullet}(\mathcal{H}) \otimes H^{P\bullet}(\mathcal{H}'),$$
$$H_d^{T\bullet}(\mathcal{H} \times \mathcal{H}') \cong H_d^{T\bullet}(\mathcal{H}) \otimes H_d^{T\bullet}(\mathcal{H}'),$$

$$H_\partial^{T\bullet}(\mathcal{H}\times\mathcal{H}')\cong H_\partial^{T\bullet}(\mathcal{H})\otimes H_\partial^{T\bullet}(\mathcal{H}'),$$

where the first two are isomorphisms of bi- and tri-graded vector spaces, and the last is an isomorphism of tri-graded algebras. □

5.3. **Gale duality and Tutte homology.** It is a well-known fact that Gale duality switches the variables of the Tutte polynomial. In other words, if $\mathcal{H}^\vee$ is the Gale dual arrangement to $\mathcal{H}$, then $\hat{T}(\mathcal{H}^\vee,x,y)=\hat{T}(\mathcal{H},y,x)$. This is a direct consequence of Remark 1. In this section we consider the relationship between the Tutte homology of an arrangement and its Gale dual.

Recall that for Tutte homology we had some freedom in choosing the differentials; in fact our defintion of boundary maps for the Tutte complex $\mathcal{C}^T=\bigoplus_{S\subseteq[n]}\wedge^\bullet H_S\otimes\wedge^\bullet W_S$ was one of four equally natural choices:

$$\partial_1=\bigoplus_{S\subseteq[n],s\in S} w_{S_s}\otimes b_{S,s},\quad \partial_2=\bigoplus_{S\subseteq[n],s\in S} b_{S_s}\otimes c_{S,s},\quad \partial_3=\bigoplus_{S\subseteq[n],r\notin S} d_{S_r}\otimes w_{S,r},\quad \partial_4=\bigoplus_{S\subseteq[n],r\notin S} c_{S_r}\otimes d_{S,r}.$$

(We used the map ∂_1 in our earlier definitions). With respect to the natural triple grading, these maps are of the following degrees:

$$\deg\partial_1=(-1,1,0),\quad \deg\partial_2=(-1,0,-1),\quad \deg\partial_3=(1,0,1),\quad\text{and}\quad \deg\partial_4=(1,-1,0).$$

To construct four chain complexes, we set $\deg\wedge^i H_S\otimes\wedge^j W_S$ to be

$$(|S|+i,i,j),\quad (|S|-j,i,j),\quad (|S|-j,i,j),\quad\text{and}\quad (|S|+i,i,j),$$

respectively. In these conventions, the differentials are of degrees

$$(0,1,0),\quad (0,0,-1),\quad (0,0,1),\quad\text{and}\quad (0,-1,0).$$

Of these differentials, ∂_1 and ∂_4 are linear duals of each other, as are ∂_2 and ∂_3. On the other hand, ∂_1 is related to ∂_3 by Gale duality, and similarly ∂_2 is Gale dual to ∂_4. Let us demonstrate what we mean by this on ∂_1 and ∂_3.

As explained in Remark 1, if $\mathcal{H}^\vee$ is the Gale dual arrangement to $\mathcal{H}$, then $H_S^\vee=W_{S^c}$ and $W_S^\vee=H_{S^c}$, where S^c is the complement of the set S in $[n]$.

So we have an isomorphism (as vector spaces) $\varphi:\mathcal{C}^T(\mathcal{H})\to\mathcal{C}^T(\mathcal{H}^\vee)$, where φ sends the component $\wedge^\bullet H_S\otimes\wedge^\bullet W_S$ isomorphically (by switching the tensor factors) to $\wedge^\bullet W_S\otimes\wedge^\bullet H_S$. The latter is the component of $\mathcal{C}^T(\mathcal{H}^\vee)$ corresponding to the subset $S^c\subseteq[n]$.

The isomorphism ϕ intertwines the differential ∂_1 on $C^T(\mathcal{H})$ with the differential ∂_3 on $C^T(\mathcal{H}^\vee)$, i.e. for $x\in\wedge^\bullet H_S\otimes\wedge^\bullet W_S$, $\varphi(\partial_1(x))=\partial_3(\varphi(x))$. So φ is an isomorphism of chain complexes $\mathcal{C}^T_{\partial_1}(\mathcal{H})\to\mathcal{C}^T_{\partial_3}(\mathcal{H}^\vee)$, except for the fact that it does not respect the grading: it sends the component of degree $(|S|+i,i,j)$ in $\mathcal{C}^T_{\partial_1}(\mathcal{H})$ to the component of degree $(n-(|S|+i),j,i)$ in $\mathcal{C}^T_{\partial_3}(\mathcal{H}^\vee)$.

There is a dg-algebra structure on all four chain complexes. For $\mathcal{C}^T_{\partial_1}$ this was defined in Section 4.3. For $\mathcal{C}^T_{\partial_2}$ the same definition of multiplication works. For $\mathcal{C}^T_{\partial_3}$ and $\mathcal{C}^T_{\partial_4}$ multiplication is defined in the following way. For subsets S and T of $[n]$, multiplication is a map

$$m:(\wedge^\bullet H_S\otimes\wedge^\bullet W_S)\otimes(\wedge^\bullet H_T\otimes\wedge^\bullet W_T)\to\wedge^\bullet H_{S\cap T}\otimes\wedge^\bullet W_{S\cap T}.$$

If $S\cup T\neq[n]$ then m is defined to be 0, otherwise m is "wedging": $m(h\otimes w\otimes h'\otimes w')=(h\wedge h')\otimes(w\wedge w')$, where we use the natural inclusions to interpret h and h' as elements of

$\wedge^\bullet H_{S\cap T}$, and orthogonal projections to interpret w and w' as elements of $\wedge^\bullet W_{S\cap T}$. We leave it to the reader to check that m is compatible with the differentials ∂_3 and ∂_4.

As $(S\cup T)^c = S^c \cap T^c$, and $S\cap T=\emptyset$ if and only if $S^c\cup T^c=[n]$, φ is not only compatible with the differentials but also an algebra homomorphism (by an easy verification). Hence φ descends to an algebra isomorphism on the homology.

When we consider signed arrangements in Section 7, we will see that the Gale duality statements for Tutte homology become even cleaner, though there will no longer be a dg-algebra structure on chain groups.

6. Examples

6.1. **Graphical Arrangements.** Important examples of central hyperplane arrangements are known as graphical arrangements, associated to a finite graph. We compute a few examples of homology groups $H_d^\bullet(\mathcal{H})$ and $H_\partial^\bullet$ associated to such arrangements in this section. The homology theories $H_d^\bullet$ and $H_d^{T,\bullet}$, when restricted to graphical arrangements, may be seen as an odd versions of the graph homologies considered in [HGR] and [JHR], respecitvely. In contrast, the restrictions of the homology theories $H_\partial^\bullet(\mathcal{H})$ and H_∂^T to graphs seem quite different from ones considered by those authors.

To a finite graph directed G with vertex set $V(G)=\{v_1,...,v_k\}$ and ordered edge set $E(G)$, we associate a vector arrangement $\mathcal{V}(G)=\{\Bbbk^k;\{\nu_e\}_{e\in E(G)}\}$, consisting of $|E(G)|$ vectors, as follows. If an edge $e\in E(G)$ starts at vertex v_i and ends at v_j, the corresponding vector in the arrangement $\mathcal{V}(G)$ is $\nu_e = x_i - x_j$. Let us denote the hyperplane arrangement arising from $\mathcal{V}(G)$ by $\mathcal{H}(G)$. For graphs with no loop edges, the arrangement $\mathcal{H}(G)$ characterizes the graph up to isomorphism, thus in a sense hyperplane arrangements generalize graphs. The hyperplane arrangements that arise from graphs via this construction are called graphical arrangements, and the characteristic polynomial of hyperplane arrangements specializes to the chromatic polynomial of graphs when restricted to graphical arrangements.

It is sometimes convenient to consider the hyperplane arrangement associated to a graph as living in a slightly smaller ambient vector space. Note that the line given by the equation $x_1 = ... = x_k$ is always included the intersection $H_{[n]}$ of all the hyperplanes in $\mathcal{H}(G)$. So we may consider a graphical vector or hyperplane arrangement modulo this subspace. We denote these arrangements by $\bar{\mathcal{V}}(G)$ and $\bar{\mathcal{H}}(G)$; the hyperplane arrangement then consists of $|E(G)|$ hyperplanes living in $\Bbbk^{k-1}$. One advantage of view the hyperplane arrangement associated to a graph as living in this smaller space is that planar graph duality corresponds to Gale duality: for a connected planar graph G and G^* its planar dual, $\bar{\mathcal{H}}(G^*) = \bar{\mathcal{H}}(G)^\vee$. Similarly, for any graph G the Tutte polynomial of this associated arrangement equals the Tutte polynomial of the graph: $T(\bar{\mathcal{H}}(G);x,y) = T(G;x,y)$.

Lemma 6.1. *For the empty arrangement $\mathcal{H}_0^k$ of no hyperplanes in $V=\Bbbk^k$,*

$$H_d^0(\mathcal{H}_0^k)\cong \wedge^\bullet \Bbbk^k, \text{ and } H_d^i(\mathcal{H}_0^k)=0 \text{ for } i\neq 0.$$

On the other hand,

$$H_\partial^\bullet(\mathcal{H}_0^k) = \wedge^\bullet \Bbbk^k$$

as bi-graded algebras.

Proof. This is straightforward from the definitions. $\square$

The following statement is the analogue of the computation for trees done in [HGR]. The proof is essentially the same, so we only provide a sketch. In the statement below, hyperplanes $\{H_i, \ldots, H_n\}$ are said to be linearly independent if their associated normal vectors $\{\nu_1, ..., \nu_n\}$ are linearly independent.

Proposition 6.2. *For a hyperplane arrangement with a maximal number of linearly independent hyperplanes $\mathcal{H}_n = \{\Bbbk^n; H_1, ..., H_n\}$, $H^0_d(\mathcal{H}) = \Bbbk\{n\}$, and all other homology groups are zero.*

Proof. We use induction. The case of $n = 0$ is trivial. Note that

$$\mathcal{H}_n^{H_n} = \mathcal{H}_{n-1} \text{ and } \mathcal{H}_n - H_n = \mathcal{H}_{n-1} \times \mathcal{H}^1_0,$$

so

$$H^i_d(\mathcal{H}_n - H_n) = H^i_d(\mathcal{H}_{n-1}) \otimes \wedge^\bullet \Bbbk = H^i_d(\mathcal{H}_{n-1}) \oplus H^i_d(\mathcal{H}_{n-1})\{1\}.$$

For each $i \geq 0$ we have

$$... \to H^i_d(\mathcal{H}_n) \to H^i_d(\mathcal{H}_{n-1}) \oplus H^i_d(\mathcal{H}_{n-1})\{1\} \xrightarrow{\gamma} H^i_d(\mathcal{H}_{n-1}) \to ...$$

where γ is the transition map arising from the snake lemma. Working through the snake lemma one can see that for $h \in H^i_d(\mathcal{H}_{n-1})$, $\gamma(h, 0) = h$. Hence γ is surjective and the long exact sequence falls apart to split short exact sequences, implying the result. □

For H_∂ a somewhat different statement is true:

Lemma 6.3. *The graded dimension of $H^i_\partial(\mathcal{H}_n)$ is*

$$q\dim H^i_\partial(\mathcal{H}_n) = \binom{n}{i}(1+q)^{n-i}.$$

Proof. By Theorem 5.2 we have

$$H^i_\partial(\mathcal{H}_n) = H^i_\partial(\mathcal{H}_{n-1})[1] + H^i(\mathcal{H}_{n-1} \times \mathcal{H}^1_0) = H^i_\partial(\mathcal{H}_{n-1})[1] \oplus H^i_\partial(\mathcal{H}_{n-1}) \oplus H^{i-1}_\partial(\mathcal{H}_{n-1}),$$

From this the statement follows by induction on n. □

Note that from these results one can compute the characteristic homology of all hyperplane arrangements with no dependencies amongst the hyperplanes, as these are products of some $\mathcal{H}_n$ with an empty arrangement.

Proposition 6.4. *If the arrangement $\mathcal{H}$ contains a degenerate hyperplane H (i.e., $H = V$), then $H^\bullet_d(\mathcal{H}) = 0$.*

Proof. The proof is the same as that of the corresponding theorem for loop edges in [HGR]. If H is degenerate, then $\mathcal{H}^H = \mathcal{H} - H$, and for each i the transition map $\gamma : H^i_d(\mathcal{H} - H) \to H^i_d(\mathcal{H}^H)$ is an isomorphism, which implies that then $H^\bullet_d(\mathcal{H}) = 0$. □

In contrast, $H^\bullet_\partial$ is not necessarily zero for arrangements that contain degenerate hyperplanes.

7. Signed hyperplane arrangements and Odd Khovanov homology

In this section we consider signed arrangements, that is, vector arrangements together with a sign associated to each vector. Just as graphs may be seen as a subset of vector arrangements (the graphical hyperplane arrangements), signed vector arrangements generalize signed graphs. The checkerboard coloring of a planar projection of a link in the three-sphere produces a signed graph, so in a sense signed hyperplane arrangements may be seen as a combinatorial generalization of such projections. Moveover, from the point of view of low dimensional topology, it is natural to consider planar link projections up to the equivalence relation generated by Reidemeister moves. This equivalence relation generalizes naturally from signed graphs to signed hyperplane arrangements, and via this generalization Reidemeister invariance questions may be posed for polynomials or chain complexes associated to signed arrangements.

In this section, we will define a version of Tutte homology for signed signed hyperplane arrangements, and prove hyperplane Reidemeister invariance for these homology groups. When restricted from signed hyperplane arrangements to planar link projections, this Tutte homology agrees with the reduced version of the odd Khovanov homology of Ozsvath-Rasmussen-Szabo [ORSz].

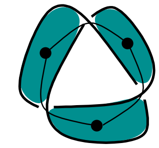
7.1. **Signed arrangements.** A signed vector arrangement is a vector arrangement $\mathcal{V} = \{V; \nu_1, ..., \nu_n\}$ together with an assignment of a sign ($+$ or $-$) to each vector ν_i. The hyperplane arrangement associated to a signed hyperplane arrangement is referred to as a signed hyperplane arrangement, since the sign associated to each vector can be thought of as a sign attached to the associated hyperplane. Since all constructions in this section will be carried out for signed arrangements, we use the same notation as we did for unsigned arrangements in previous sections; thus in the notation $\mathcal{V} = \{V; \nu_1, ..., \nu_n\}$, it is understood that each ν_i is really a vector together with a sign. Similarly, we denote by $\mathcal{H} = \{V; H_1, ..., H_n\}$ the signed hyperplane arrangement associated to $\mathcal{V}$.

Let $\mathcal{V} = \{V; \nu_1, ..., \nu_n\}$ be a signed vector arrangement. The sign assignment partitions the set $[n]$ into subsets $[n] = [n]_+ \sqcup [n]_-$, where $[n]_+$ is the set of vectors assigned $+$ and and $[n]_-$ is the set vectors assigned $-$. For a subset $S \subseteq [n]$ we write $S_+ = S \cap [n]_+$ and $S_- = S \cap [n]_-$.

Deletion and restriction of signed arrangements is defined just as for ordinary arrangements. We extend Gale duality from arrangements to signed arrangements as follows. If $\mathcal{V} = \{V; \nu_1, ..., \nu_n\}$ is a signed arrangement, the Gale dual $\mathcal{V}^\vee = \{W; \nu_1^\vee, ..., \nu_n^\vee\}$ is, as an unsigned arrangement, just the Gale dual of the unsigned arrangement $\mathcal{V}$. The sign assignment of $\mathcal{V}^\vee$ is given by declaring that the sign associated to $\nu_i^\vee$ is the opposite of the sign assigned to ν_i.

7.2. **Signed arrangements from planar link projections.** Let D be a planar projection of an oriented link in the three-sphere. To D we associate a signed hyperplane arrangement in a two-step process, as follows. We refer to the hyperplane arrangements constructed from planar link projections in this way as *link hyperplane arrangements*.

There is a well known way to associate a planar graph to an (oriented) link diagram D. Suppose D is connected as a planar graph. Choose a checkerboard coloring of D, put vertices in each shaded region, and draw an edge for each croossing, as illustrated in the picture on the right for a trefoil knot. The graphs corresponding to the two opposite checkerboard colorings are planar duals to each other.



When the graph D is not connected, the association of vertices is done component by component. Thus, if D is the disjoint union of two link diagrams, and we take the unbounded region to be shaded for both components, then this region will give rise to two vertices. Similarly, for planar graphs that are not connected we understand planar graph duality component-wise. In this way the double dual of a planar graph is the graph itself, and the planar graph associated to the opposite checkerboard coloring is the planar dual.

Of course, such a procedure does not distinguish between under and over crossings. In order to keep track of the under/over information, we associate to each crossing of a shaded oriented link diagram D a sign, as shown:

Thus, given any link diagram, we can encode it as a signed graph, i.e. a graph with signs assigned to the edges. Note that switching the checkerboard shading of the diagram switches the shaded signs of all crossings.

Now, the procedure from Section **??** — which was used there to associate an unsigned arrangement to an unsigned graph — extends in the obvious way to an assignment of signed arrangements to signed graph.

There are two slight ambiguities in the definition of a link hyperplane arrangement: firstly, the assignment of a hyperplane arrangement to a graph required an ordering of the edges. It is easy to see that different choices of edge ordering give rise to canonically isomorphic chain complexes in what follows. Secondly, we had to choose a checkerboard coloring of the link projection. For a fixed link projection, the signed arrangements corresponding to the two checkerboard colorings are Gale dual to each other. Thus, to obtain a link invariant from link hyperplane arrangements, it is natural to consider constructions which are invariant under Gale duality.

7.3. **Hyperplane Reidemeister moves.** Planar projections of isotopic links may be obtained from one another by a sequence of Reidemeister moves. For a detailed discussion and restatement of these moves in the language of signed graphs, we refer the reader to [BR] and references therein. We generalize these moves from signed graphs to signed arrangements as follows. (Note that each relation comes with a Gale dual pair):

- $R1$: If $\nu_l = 0$, then $\mathcal{V} \leftrightarrow \mathcal{V} - \nu_l$.
- $R1^\vee$: If $\nu_l^\vee = 0$ (i.e., ν_l is independent of $\mathcal{V} - \nu_l$), then $\mathcal{V} \leftrightarrow \mathcal{V}^{\nu_l}$.
- $R2$: If $\nu_l = \alpha\nu_m$, for some non-zero $\alpha \in \Bbbk$ and $l \neq m$ are of opposite signs, then $\mathcal{V} \leftrightarrow \mathcal{V} - \{\nu_l, \nu_m\}$.
- $R2^\vee$: If $\nu_l^\vee = \alpha\nu_m^\vee$, $\alpha \in \Bbbk$, for some $l \neq m$ of opposite signs, then $\mathcal{V} \leftrightarrow \mathcal{V}^{\nu_l,\nu_m}$.
- $R3$: Suppose there are three distinct vectors ν_l, ν_m and ν_p in $\mathcal{V}$ with $l, m \in [n]_+$ and $p \in [n]_-$, and a linear dependence $\nu_l^\vee = \alpha_m\nu_m^\vee + \alpha_p\nu_p^\vee$ with non-zero coefficients α_m, α_p. Then $\mathcal{V} \leftrightarrow \mathcal{V}'$, where $\mathcal{V}'$ is the arrangement obtained from $\mathcal{V}^{\nu_l}$ by adding an

extra vector $\nu_l' = \alpha_m \nu_m + \alpha_p \nu_p$. The signs in $\mathcal{V}'$ are the same as those in $\mathcal{V}$ except that the sign of l changes from positive to negative, i.e., $[n]_-' = [n]_- \cup l$.

- $R3^\vee$: The same statement as R^3, but with opposite sign assignments.

It is straightforward to verify the following proposition (for example, it follows from the description of graph Reidemeister moves in [BR]).

Proposition 7.1. *The hyperplane Reidemeister moves preserve the class of link hyperplane arrangements. When restricted to link hyperplane arrangements, the hyperplane Reidemeister moves agree with the usual link projection Reidemeister moves.*

7.4. **The Jones polynomial of a signed arrangement.** Given a signed arrangement $\mathcal{V} = \{V; \nu_1, ..., \nu_n\}$, we define the (unnormalized) Jones polynomial of $\mathcal{V}$ to be

$$J(\mathcal{V}) = \sum_{S \subseteq [n]} (-1)^{|S|+n_-} q^{|S|+n_+-2n_-} (q+q^{-1})^{\dim H_{S_+ \cup S_-^c} + \dim W_{S_+ \cup S_-^c}},$$

where n_+ and n_- denote the number of positive and negative signed vectors in $\mathcal{V}$, respectively.

We refer to the polynomial above as the Jones polynomial of the hyperplane arrangement, since on a link hyperplane arrangement it agrees with the Jones polynomial of the associated link.

Proposition 7.2. *The Jones polynomial of a hyperplane arrangement is a hyperplane Reidemeister invariant.*

Since this proposition follows by taking Euler characteristics in Theorem 7.10, we won't give an independent proof of it here.

7.5. **Odd Khovanov homology for signed arrangements.** To a signed vector arrangement $\mathcal{V}$ we associate a bigraded chain complex $\mathcal{T}(\mathcal{V})$ as follows.

The cube of chain groups associates to a subset $S \subseteq [n]$ the bi-graded vector space

$$T_S = \wedge^\bullet (H_{S_+ \cup S_-^c} \oplus W_{S_+ \cup S_-^c}) \cong \wedge^\bullet H_{S_+ \cup S_-^c} \otimes \wedge^\bullet W_{S_+ \cup S_-^c}.$$

Here S_-^c denotes the complement of S_- in $[n]_-$. Let us denote $\tilde{S} = S_+ \cup S_-^c$. Note that if $s \in [n]_+$ then $s \in \tilde{S}$ if and only if $s \in S$, while if $r \in [n]_-$ then $r \in \tilde{S}$ iff $r \notin S$.

We define the bi-grading on this complex (as in [ORSz]) by setting

$$\deg \wedge^i H_{\tilde{S}} \otimes \wedge^j W_{\tilde{S}} = (|S| - n_-, |S| + \dim H_{\tilde{S}} + \dim W_{\tilde{S}} - 2(i+j) + n_+ - 2n_-).$$

These global shifts will be needed for reidemeister invariance. We will refer to the first grading as the homological grading and the second as the q-grading.

For $r \notin S$ the map $\delta_{S,r} : T_S \to T_{S \cup r}$ is defined as

$$\delta_{S,r} = \begin{cases} d_{\tilde{S},r} \otimes d_{\tilde{S},r} & \text{if } r \in [n]_+ \text{ and } \nu_r \notin V_{\tilde{S}} \text{ (Type 1)} \\ d_{\tilde{S},r} \otimes w_{\tilde{S},r} & \text{if } r \in [n]_+ \text{ and } \nu_r \in V_{\tilde{S}} \text{ (Type 2)} \\ b_{\tilde{S},r} \otimes b_{\tilde{S},r} & \text{if } r \in [n]_- \text{ and } \nu_r \in V_{\tilde{S}-r} \text{ (Type 3)} \\ w_{\tilde{S},r} \otimes b_{\tilde{S},r} & \text{if } r \in [n]_- \text{ and } \nu_r \notin V_{\tilde{S}-r} \text{ (Type 4).} \end{cases} \tag{25}$$

Here $d_{\tilde{S},r}, w_{\tilde{S},r}, b_{\tilde{S},r}$ are the the same maps from Section 2 that have been used in the construction of the chain groups associated to unsigned hyperplane arrangements.

The differential of the chain complex is defined as $\delta = \bigoplus_{S\in[n],r\notin S} \epsilon_{S,r}\delta_{S,r}$ for some appropriate choices of scalars $\epsilon_{S,r}$ specified below. Note that with respect to the bi-grading defined above, $\deg\delta = (1,0)$.

Lemma 7.3. *In the cube of chain groups T_S and edge maps $\delta_{S,r}$, every square face commutes up to a scalar, i.e. for any $r, t \notin S$,*

$$\delta_{S\cup r,t} \circ \delta_{S,r} = \alpha_{S,r,t}\delta_{S\cup t,r} \circ \delta_{S,t},$$

for some scalars $\alpha_{S,r,t} \in \Bbbk$.

Proof. The proof amounts to checking several cases depending on which type of differential each side of the square belongs to. The first major case is when $r, t \in [n]_+$. This breaks down into four sub-cases, as follows:

- If $\nu_r \in V_{\tilde{S}}$ and $\nu_t \in V_{\tilde{S}}$, then all four edge maps of the square are of Type 2, so the square is anti-commutative.
- If $\nu_r \notin V_{\tilde{S}\cup t}$ and $\nu_t \notin V_{\tilde{S}\cup r}$, then all edge maps are of Type 1, and the square commutes.
- If $\nu_r \in V_{\tilde{S}}$ and $\nu_t \notin V_{\tilde{S}\cup r}$, then both r-edges are of Type 2, while both t-edges are of Type 1, so the square commutes.
- The most interesting case is when $\nu_r \notin V_{\tilde{S}}$ but $\nu_r \in V_{\tilde{S}\cup t}$, which implies that $\nu_t \notin V_{\tilde{S}}$ but $\nu_t \in V_{\tilde{S}\cup r}$. In this case there are two edges of Type 1 and two of Type 2, namely $\delta_{S,r} = d \otimes d$, $\delta_{S,t} = d \otimes d$, $\delta_{S\cup t,r} = d \otimes w$, and $\delta_{S\cup r,t} = d \otimes w$.

  Take $x \otimes y \in T_S = H_{\tilde{S}} \otimes W_{\tilde{S}}$. The two sides of the equality we need to check are:

  $$\delta_{S\cup r,t} \circ \delta_{S,r}(x \otimes y) = x \otimes \nu_t^\vee \wedge y \in H_{\tilde{S}\cup r\cup t} \otimes W_{\tilde{S}\cup r\cup t},$$

  and

  $$\delta_{S\cup t,r} \circ \delta_{S,t}(x \otimes y) = x \otimes \nu_r^\vee \wedge y \in H_{\tilde{S}\cup r\cup t} \otimes W_{\tilde{S}\cup r\cup t}.$$

  Recall that both $\nu_r^\vee$ and $\nu_t^\vee$ are interpreted in $W_{\tilde{S}\cup r\cup t}$ via orthogonal projections, and due to the condition that $\nu_r \notin V_{\tilde{S}}$ but $\nu_r \in V_{\tilde{S}\cup t}$, it follows that their projections onto $W_{\tilde{S}\cup r\cup t}$ only differ by a scalar $\alpha_{S,r,t}$.

The second big case to consider is when $r \in [n]_+$ and $t \in [n]_-$. This breaks into five sub-cases, four of which are trivial (the square either commutes or anti-commutes).

- The one interesting subcase is when $\nu_r \notin V_{\tilde{S}-t}$ but $\nu_r \in V_{\tilde{S}}$, which implies that $\nu_t \notin V_{\tilde{S}-t}$ but $\nu_t \in V_{\tilde{S}-t\cup r}$. Again, consider $x \otimes y \in T_S = H_{\tilde{S}} \otimes W_{\tilde{S}}$. The two sides of the eauality turn out to be

  $$\delta_{S\cup r,t} \circ \delta_{S,r}(x \otimes y) = x \otimes \nu_r^\vee \wedge y \in H_{\tilde{S}\cup r\cup t} \otimes W_{\tilde{S}\cup r-t},$$

  and

  $$\delta_{S\cup t,r} \circ \delta_{S,t}(x \otimes y) = \nu_t \wedge x \otimes y \in H_{\tilde{S}\cup r\cup t} \otimes W_{\tilde{S}\cup r-t}.$$

  The condition $\nu_r \notin V_{\tilde{S}-t}$, $\nu_r \in V_{\tilde{S}}$ implies that when projected orthogonally onto $H_{\tilde{S}\cup r\cup t}$ and $W_{\tilde{S}\cup r-t}$, respectively, both ν_t and $\nu_r^\vee$ map to zero, so the square commutes.

The third major case, when $r, t \in [n]_-$, is similar to the first, so we leave it to the reader to check. $\square$

Lemma 7.4. *There is a non-zero scalar assignment $\epsilon_{S,r} \in \Bbbk^\times$ to each edge to make the above cube anti-commutative, and hence the flattened cube is a chain complex. Furthermore, choosing a different scalar assignment does not change the homology of the complex.*

Proof. The proof is the same homological argument as the proof of the corresponding statements (Lemmas 1.2 and 2.2) in [ORSz], so we only give a brief outline here. Consider the cube $\mathcal{T}(\mathcal{V})$ as a cell complex, and define a 2-cochain $c \in C^2(\mathcal{T}(\mathcal{V}), \Bbbk^\times)$ by associating to each face the negative of the scalar which obstructs the commutativity of the square, described in Lemma 7.3 and its proof. (When the two compositions are both zero, associate -1.) It is easy to check that c is a cocycle. Since the cube is contractible, c must be a coboundary, and this provides the desired scalar assignment.

The uniqueness part of the lemma follows from the fact that the product of two such edge assignments is a 1-cocycle. So it is the coboundary of some zero-cochain $\gamma : \mathcal{T}(\mathcal{V}) \to \Bbbk^\times$, which associates non-zero scalars $\gamma(S)$ to each vertex T_S of the hypercube. The isomorphism of the chain complexes is the map which is given by multiplication with γ_S on T_S. $\square$

We have defined a bi-graded chain complex $\mathcal{T}(\mathcal{V})$. We will denote the resulting homology groups by $H^i_{Kh}(\mathcal{V})$, which, for each i, is a graded $\Bbbk$-vector space.

Proposition 7.5. *The cohomology $H^i_{Kh}(\mathcal{V})$ categorifies the Jones polynomial of hyperplane arrangements.*

Proof. Straigtforward from the definition of the grading and the chain groups. $\square$

Remark 5. Recall that when associating a signed vector arrangement to a link projection L, we first chose a checkerboard shading, used this to construct a signed planar graph $G(L)$, then chose an arbitrary ordering and of the edges and oriented them arbitrarily, and used this data to construct the vector arrangement $\mathcal{V}(L)$. However, these choices do not effect the cohomology $H^\bullet_{Kh}(\mathcal{V}(L))$. Chosing the opposite checkerboard shading would have led to the planar dual $G(L)^*$ of the graph $G(L)$, and ultimately to the Gale dual arrangement $\mathcal{V}(L)^\vee$. In the next section we will show that $H^\bullet_{Kh}$ is Gale-duality invariant. Choosing a different ordering of the edges amounts to permuting the vectors in $\mathcal{V}(L)$, while changing the orientation of an edge multiplies the corresponding vector of $\mathcal{V}(L)$ by -1. The following proposition states that $H^\bullet_{Kh}$ is also invariant under these operations, hence it is well-defined as an invariant of planar link projections.

Proposition 7.6. *The cohomology $H^i_{Kh}(\mathcal{V})$ is invariant under permuting the vectors of $\mathcal{V}$, and under multiplying a vector ν_i in $\mathcal{V}$ by -1.*

Proof. Let $\sigma(\mathcal{V})$ be the permuted vector arrangement, and $n_i(\mathcal{V})$ denote the arrangement where ν_i is replaced by $-\nu_i$. Note that $\sigma(\mathcal{V})$ has the same chain groups as $\mathcal{V}$, permuted, but due to the changed order the scalar assignment ϵ may be different. Consider the map $\phi : \mathcal{T}(\mathcal{V}) \to \mathcal{T}(\sigma(\mathcal{V}))$ which sends each chain group T_S isomorphically to the corresponding $T_{\sigma(S)}$ in $\mathcal{T}(\sigma(\mathcal{V}))$. This commutes with the differentials up to scalars, hence defines a 1-cocycle c_1 of the hypercube, just like in the proof of Lemma 7.4. Since the cube is contractible, c_1 is the boundary of some 0-cochain $c_0 : \{S \subseteq [n]\} \to \Bbbk$. This c_0 is the adjustment needed for ϕ to be a chain isomorphism.

The case of $n_i(\mathcal{V})$ is similar: now the chain goups are exactly the same, and some differentials may get multiplied by -1. The same homological argument works. $\square$

Remark 6. In the proof of Lemmas 7.3 and 7.4 above, we have worked over a field $\Bbbk$. In order to work over $\mathbb{Z}$, or another commutative ring, one must check that all the scalar obstructions to the commutativity of the hypercube are units. For signed graphical arrangements — and in particular for planar link projections — it is straightforward to check that these scalar obstructions are always ± 1. Thus in the graphical case, we may take $\mathbb{k} = \mathbb{Z}$. Then the proofs of Lemmas 7.3 and 7.4, as well as the rest of the proofs in this section, go through without change.

7.6. **Gale duality, Deletion-Restriction, and Kunneth Theorem.** The following proposition is almost immediate from the definitions.

Proposition 7.7. *If $\mathcal{V}$ and $\mathcal{V}^\vee$ are Gale dual signed arrangements, then there is an isomorphism of chain complexes*

$$\mathcal{T}(\mathcal{V}) \cong \mathcal{T}(\mathcal{V}^\vee).$$

Proof. For brevity, let us denote $T_S(\mathcal{V})$ by just T_S, and $T_S(\mathcal{V}^\vee)$ by $T^\vee_{S^\vee}$. $S^\vee$ is different from S in that the positive and negative signs are exchanged, hence $\widetilde{S^\vee} = \tilde{S}^c$. Observe that

$$T^\vee_{S^\vee} \cong \wedge^\bullet H^\vee_{\widetilde{S^\vee}} \otimes \wedge \bullet W^\vee_{\widetilde{S^\vee}} \cong \wedge^\bullet W_{\widetilde{S^\vee}^c} \otimes \wedge^\bullet H_{\widetilde{S^\vee}^c} \cong \wedge^\bullet W_{\tilde{S}} \otimes H_{\tilde{S}} \overset{\sigma}{\cong} T_S.$$

We claim that σ, which is the isomorphism of chain groups given by the switching of tensor factors, is actually an isomorphism of chain complexes.

Suppose $r \notin S$ and $r \in [n]_+$. Then $r \in [n]^\vee_-$ and $\nu_r \notin V_{\tilde{S}}$ if and only if $\nu^\vee_r \in V_{\widetilde{S^\vee}-r}$. So the Gale dual of a Type 1 differential is a Type 3 differential, and similarly the Gale dual of a Type 2 differential is a Type 4 differential. Thus, using the same scalar assignments in the complexes $\mathcal{T}(\mathcal{V})$ and $\mathcal{T}(\mathcal{V}^\vee)$, it follows that σ commutes with the differentials and that σ is an isomorphism of chain complexes. (If we used different scalar assignments in the differentials for $\mathcal{T}(\mathcal{V})$ and $\mathcal{T}(\mathcal{V}^\vee)$, we would have to modify σ accordingly in order to get a genuine chain map.) $\square$

Now suppose that we are given a deletion restriction triple $\{\mathcal{V}, \mathcal{V}^{\nu_l}, \mathcal{V} - \nu_l\}$. The following theorem is proven the same way as Theorem 5.4, the only difference is having to keep track of the sign of l which determines the direction of the maps.

Theorem 7.8. *There is long exact sequence of homology groups, depending on the sign of l. If $l \in [n]_+$, then*

$$... \to H^{i-1}_{Kh}(\mathcal{V}^{\nu_l}) \to H^i_{Kh}(\mathcal{V}) \to H^i_{Kh}(\mathcal{V} - \nu_l) \xrightarrow{\gamma_+} H^i_{Kh}(\mathcal{V}^{\nu_l}) \to ..., \tag{26}$$

where

$$\gamma_+(x \otimes y) = \begin{cases} x \otimes y & \text{if } l \notin V_{\tilde{S}} \\ x \otimes (\nu^\vee_l \wedge y) & \text{if } l \in V_{\tilde{S}}. \end{cases} \tag{27}$$

If $l \in [n]_-$, then

$$... \to H^{i-1}_{Kh}(\mathcal{V} - \nu_l) \to H^i_{Kh}(\mathcal{V}) \to H^i_{Kh}(\mathcal{V}^{\nu_l}) \xrightarrow{\gamma_-} H^i_{Kh}(\mathcal{V} - \nu_l) \to ..., \tag{28}$$

where

$$\gamma_-(x \otimes y) = \begin{cases} (\nu_l \wedge x) \otimes y & \text{if } l \notin V_{\tilde{S}} \\ x \otimes y & \text{if } l \in V_{\tilde{S}}. \end{cases} \tag{29}$$

$\square$

The proof of Theorem 5.6 also applies without any adjustment, so H_{Kh} satisfies a Kunneth Theorem:

Proposition 7.9. *For two signed vector arrangements $\mathcal{V}$ and $\mathcal{V}'$ and the product arrangement $\mathcal{V} \times \mathcal{V}'$, there is an isomorphism*

$$H^\bullet_{Kh}(\mathcal{V} \times \mathcal{V}') \cong H^\bullet_{Kh}(\mathcal{V}) \otimes H^\bullet_{Kh}(\mathcal{V}').$$

□

7.7. **Reidemeister Invariance.**

Theorem 7.10. *If $\mathcal{V}$ and $\mathcal{V}'$ are signed arrangements which differ by a Reidemeister move, then $Kh(\mathcal{V})$ and $Kh(\mathcal{V}')$ are chain homotopic.*

Proof. Since the Reidemeister moves come in Gale dual pairs, and Proposition 7.7 says that H_{Kh} is Gale duality–invariant, it is enough to show Reidemeister invariance for one member of each Gale dual pair. Like [ORSz], we essentially follow the method and exposition of [BN], Section 3.5.

For Reidemeister 1, we will prove $R1^\vee$ in the case where ν_l is positive. The negative case is very similar. The chain complex $\mathcal{T}(\mathcal{V})$ can be written as a direct sum of two faces of the cube, one for sets which contain l, and one for those which do not:

$$\mathcal{T}(\mathcal{V}) = \bigoplus_{l \notin S} T_S \oplus \bigoplus_{l \notin S} T_{S \cup l}.$$

If $l \notin S$, then $\nu_l \notin V_{\tilde{S}}$: this is implied by the assumption that $\nu^\vee = 0$. Hence $H_{\tilde{S}} \cong \Bbbk \nu_l \oplus H_{\tilde{S} \cup l}$. This means that

$$T_S = \wedge^\bullet H_{\tilde{S}} \otimes \wedge^\bullet W_{\tilde{S}} \cong (\wedge^\bullet H_{\tilde{S} \cup l} \otimes \wedge^\bullet W_{\tilde{S}}) \oplus (\nu_l \wedge (\wedge^\bullet H_{\tilde{S}_\cup l}) \otimes \wedge^\bullet W_{\tilde{S}}).$$

The differential $\delta_{S,l}$ is of Type 1, hence it is the identity on the first component, meaning that

$$\mathcal{T}' = \bigoplus_{l \notin S} \wedge^\bullet H_{\tilde{S} \cup l} \otimes \wedge^\bullet W_{\tilde{S}} \xrightarrow{\delta_{S,l}} \bigoplus T_{S \cup l}$$

is an acyclic subcomplex, so the homology of $\mathcal{T}(\mathcal{V})$ doesn't change if we factor out by $\mathcal{T}'$. After the factorization what remains is $\bigoplus_{l \notin S} \nu_l \wedge (\wedge^\bullet H_{\tilde{S}_\cup l}) \otimes \wedge^\bullet W_{\tilde{S}}$, which is isomorphic to $\mathcal{T}(\mathcal{V}^{\nu_l})$, as the degree shift gets canceled out by the global shift in the definition of the grading.

$$\begin{tikzcd}
\bigoplus T_{S \cup m} \arrow[r] & \bigoplus T_{S \cup l \cup m} \\
\bigoplus_{l,m \notin S} T_S \arrow[u] \arrow[r, "w \otimes b"] & \bigoplus T_{S \cup l} \arrow[u, "d \otimes d"']
\end{tikzcd}$$

For proving $R2^\vee$, suppose that $l \in [n]_-$ and $m \in [n]_+$. We write the cube $\mathcal{T}(\mathcal{V})$ as a direct sum of four faces, according to the incidence of l and m in S, as shown on the left. In the bottom right corner, $\nu_l^\vee = \alpha \nu_m^\vee$ implies that $\nu_l - \alpha \nu_m \in H_{\widetilde{S \cup l}}$. (Note that $\widetilde{S \cup l}$ includes neither l nor m.) So $H_{\widetilde{S \cup l}} \cong \Bbbk(\nu_l - \alpha \nu_m) \oplus \bar{H}_{\widetilde{S \cup l}}$, where $\bar{H}_{\widetilde{S \cup l}}$ denotes the orthogonal complement of $(\nu_l - \alpha \nu_m)$ in $H_{\widetilde{S \cup l}}$. Thus

$$T_{\widetilde{S \cup l}} \cong (\wedge^\bullet \bar{H}_{\widetilde{S \cup l}} \otimes W_{\widetilde{S \cup l}}) \oplus ((\nu_l - \alpha \nu_m) \wedge (\wedge^\bullet \bar{H}_{\widetilde{S \cup l}}) \otimes W_{\widetilde{S \cup l}}).$$

The differential $\delta_{S \cup l, m}$ is of Type 1, an isomorphism when restricted to the first component, hence there is an acyclic subcomplex

$$\mathcal{T}' = \bigoplus \bar{H}_{\widetilde{S \cup l}} \otimes W_{\widetilde{S \cup l}} \xrightarrow{\delta_{S \cup l, m}} \bigoplus T_{S \cup l \cup m}.$$

$$\begin{tikzcd}
\bigoplus T_{S\cup m} \arrow[rr] & & 0 \\
\bigoplus_{l,m\notin S} T_S \arrow[r] \arrow[u] & \bigoplus(\nu_l - \alpha\nu_m)\wedge(\wedge^\bullet \bar{H}_{\widetilde{S\cup l}})\otimes W_{\widetilde{S\cup l}} \arrow[ur, phantom] & \arrow[u]
\end{tikzcd}$$

Factoring out by $\mathcal{T}'$, we get the complex on the right. Note that the lower horizontal differential is now an isomorphism, so it can be inverted and composed with the differential going up to produce a map τ from the lower right corner to the upper left corner. Now consider the subcomplex $\mathcal{T}''$ given by all elements α in the upper left corner, and pairs

$$(\beta, \tau(\beta)) \in \bigoplus \left((\nu_l - \alpha\nu_m) \wedge (\wedge^\bullet \bar{H}_{\widetilde{S\cup l}}) \otimes W_{\widetilde{S\cup l}}\right) \oplus \bigoplus T_{S\cup m}.$$

This complex is acyclic due to the lower horizontal differential being an isomorphism.

Factoring out by $\mathcal{T}''$, all that is left is the top left corner, namely $\mathcal{T}''' = \bigoplus T_{S\cup m}$. This is isomorphic to $\mathcal{T}(\mathcal{V}^{\nu_l,\nu_m})$, taking the global degree shift into account.

To prove $R3^\vee$ we will consider the complexes for both $\mathcal{V}$ and $\mathcal{V}'$ and reduce each one until we get isomorphic complexes. Both $\mathcal{T}(\mathcal{V})$ and $\mathcal{T}(\mathcal{V}')$ can be written as three dimensional cubes according to the incidence of l, m and p in S. We will first deal with the top faces of these cubes, which include the sets S for which $l \in S$.

In the case of $\mathcal{T}(\mathcal{V})$, we play the same game as in the proof of $R2^\vee$ above, using that if $l \in S$, $m \in S$ and $p \notin S$, then $(\alpha_m\nu_m + \alpha_p\nu_p - \nu_l) \in H_{\tilde{S}}$. Using the same steps as in the $R2$ proof, we can reduce $\mathcal{T}(\mathcal{V})$ to the complex $\mathcal{T}'''(\mathcal{V}) = \bigoplus_{l\notin S} T_S \oplus \bigoplus_{l,p\in S, m\notin S} T_S$.

As for $\mathcal{T}(\mathcal{V}') = \bigoplus_S T'_S = H'_{\tilde{S}} \otimes W'_{\tilde{S}}$, do the same but now using the fact that when $l, p \in S, m \notin S$ we have $(\alpha_m\nu_m^\vee + \alpha_p\nu_p^\vee - \nu_l^\vee) \in W_{\tilde{S}}$. Again by the same process, we can reduce $\mathcal{T}(\mathcal{V}')$ to $\mathcal{T}'''(\mathcal{V}') = \bigoplus_{l\notin S} T'_S \oplus \bigoplus_{l,m\in S, p\notin S} T'_S$.

There is an isomorphism $\Phi : \mathcal{T}'''(\mathcal{V}) \to \mathcal{T}'''(\mathcal{V}')$, defined as follows: if $l \notin S$, then $\Phi : H_{\tilde{S}} \otimes W_{\tilde{S}} \to H'_{\tilde{S}} \otimes W'_{\tilde{S}}$ is the restriction to H_l, which in these cases is an isomorphism, as the reader can verify. It is also simple to check that if $l, p \in S, m \notin S$, then $T_S \cong T'_{S-p\cup m}$. Define the "top level" of Φ to be this isomorphism. $\square$

Corollary 7.11. *Let $\mathcal{V}$ be a link hyperplane arrangement corresponding to a planar projection of a link L in the three-sphere. Then $Kh(\mathcal{V})$ is a link invariant, isomorphic to the reduced Odd Khovanov homology of L.*

Proof. The fact that $Kh(\mathcal{V})$ is a link invariant follows immediately from Theorem 7.10. What is not completely obvious is that the resulting homology theory is the reduced Odd Khovanov homology of [ORSz]. In fact, the boundary maps we use differ slightly from those of [ORSz]; one can check, for example, that in the case of the Hopf link, the O-R-Sz boundary maps are not invariant under Gale duality. However, in [B], Bloom gives an alternative (slightly more symmetric) definition of a chain complex associated to a link projection, and he proves that the resulting homology is Odd Khovanov homology. It is a straightforward, if slightly tedious, combinatorial exercise (which we leave to the reader) to check that our boundary maps for a link hyperplane arrangement agree with Bloom's. $\square$

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Zsuzsanna Dancso, Fields Institute, 222 College St, Toronto Ontario M5T 3J1, Canada
E-mail address: `zsuzsi@math.toronto.edu`
URL: `http://www.math.toronto.edu/zsuzsi`

Anthony Licata, Institute for Advanced Study, Einstein Dr., Princeton, NJ, USA
E-mail address: `amlicata@math.ias.edu`
URL: `http://www.math.stanford.edu/amlicata`